15. A NUMERICAL METHOD OF INTEGRATION BY MEANS OF TAYLOR-STEFFENSEN SERIES AND ITS POSSIBLE USE IN THE STUDY OF THE MOTIONS OF COMETS AND MINOR PLANETS

V. F. MYACHIN and O. A. SIZOVA
Institute for Theoretical Astronomy, Leningrad, U.S.S.R.

Abstract. The Taylor formula is used directly in a method of numerical integration of the n-body problem of celestial mechanics; the derivatives in the expansion of the coordinates are calculated successively at each integration step according to the generalized Steffensen rule. The proposed method is the most precise of all numerical methods based on the predetermined part of the Taylor series. The method is used with a variable number of derivatives at each integration step and also with a variable step. The cumulative error in the coordinates increases more slowly in our method than in any other. We can apply the method to the study of the motion of a comet or minor planet, taking into account the perturbations by eight major planets; the method allows for the simultaneous integration of a great number of objects of zero mass.

All the methods of numerical integration used at present in celestial mechanics, whether they be ones utilizing differences or those of the Runge-Kutta type, are ultimately based on part of a Taylor series in which the derivatives of the right-hand sides of the integrated differential equations are replaced by linear combinations of the right-hand sides themselves, which greatly decreases the precision. We propose a method of numerical integration in which the Taylor series itself is used, with direct computation of all derivatives of the required coordinates by means of equations up to a specified order; the derivatives are computed successively according to the generalized Steffensen rule. We call this process the Taylor-Steffensen method. In so far as this method involves no interpolation, it is the most precise of all numerical methods based on Taylor series. Our method is applicable to any problem of n bodies in which only the forces of mutual Newtonian attraction are acting; all the equations of the system are integrated together in rectangular heliocentric coordinates. Unfortunately, in its present form the method cannot take care of nongravitational forces of a random or intermittent nature, since expansion in Taylor series of all functions included in the integrated system of equations is necessary.

The type of numerical method described here was first suggested by Deprit and Price (1965) for a problem of three bodies with a constant number of derivatives and invariable integration step; our method was developed independently and in a somewhat different form.

The principal features of the method are briefly described below.

(1) In all known numerical methods the coefficients of the differences diminish very slowly (they decrease the most rapidly in Cowell's method – in geometrical progression by factors of 1/4), which results, for each such method, in an optimum number of differences for obtaining (with a fixed step) the greatest accuracy possible.

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In the Taylor-Steffensen method the coefficients of the derivatives diminish as $1/n!$, which gives an unlimited increase of precision with the increase in the number of derivatives used. We therefore use the method with a variable number of derivatives at each step, care being taken that the last computed member of the Taylor series is smaller than a given number ε. In practice this is done very simply, since each member of the Taylor series is computed from the preceding ones in a recurrent manner. We may simultaneously provide for a variable step, which can be increased or diminished without additional difficulties any number of times and at any point in the integration interval. This allows us to maintain the required precision of the computation formula without excessive division into more steps, and we can thus decrease the cumulative error in the computed coordinates and velocities.

(2) It follows from the above, in particular, that when comparing the results of integration with the observations at any moment, it is not necessary to resort to interpolation; it will suffice to adjust the integration step to the moment of observation.

(3) Our method allows for greater precision in determining the initial velocities of objects for which two adjacent coordinate values are supplied. The velocities can be obtained without resorting to interpolation by using a special algorithm. This allows us to obtain velocities with the precision of the given coordinates, which is impossible in the conventional computation of initial velocities with the aid of interpolation formulae; and this is highly important, since errors in the initial velocities have an important effect on the total cumulative error in the computed coordinates.

(4) Because of the high precision of the method and the large number of significant figures used in the computer, the total cumulative error in the coordinates over large intervals of time is almost entirely determined by the errors in the initial coordinates; after n integration steps the expected magnitude of the error will be $O(n)$, instead of $O(n^{3/2})$ that follows from Brouwer's (1937) law. Thus the error in the coordinates accumulates much more slowly in our method than in other numerical methods.

We have a computer programme for applying the method to the motions of comets, with account taken of the perturbations by eight major planets, omitting Mercury or Pluto according as to the part of its orbit in which the comet is located. It is also possible to make a simultaneous integration of a large number (50 or more) of objects of zero mass.

Investigation of the motion of comets and minor planets by the Taylor-Steffensen method requires reliable initial data (coordinates and velocities), not only for the object studied but also for the perturbing bodies. Obtaining the latter with sufficient accuracy for any epoch t_0 is a very labour-consuming task. It was for this purpose that combined integration was undertaken of the equations of motion of eight major planets (except Mercury) for the period JD 2428000.5 to 2431820.5. Perturbations caused by Mercury were partially allowed for by the addition of its mass to that of the Sun. For the initial epoch we chose $t_0 = 2430000.5$, for which Schubart and Stumpff (1966) had obtained the values of coordinates and velocities with ten significant figures.

The principal arithmetical operations were made with double precision. The integration step was taken as a constant 10 days. With the accuracy adopted (10^{-15}) in the development of coordinates, we took into account most of the derivatives up to the twelfth order. Heliocentric coordinates and velocities were printed at each step and punched on cards after every other step; these can be used directly as initial data for the major planets in the solution of any cometary problem, as well as for a check, if necessary. The results of the integration by means of our programme were compared with the published ephemerides of the planets; for Venus the departure never exceeded 4×10^{-6}, this being mainly attributable to the perturbations by Mercury.

Integration over an interval of 100–150 yr is now in progress.

References

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