Glasgow Math. J. **46** (2004) 217–225. © 2004 Glasgow Mathematical Journal Trust.
DOI: 10.1017/S0017089504001715. Printed in the United Kingdom

ON (a, b)-DICHOTOMY FOR EVOLUTIONARY PROCESSES ON A HALF-LINE

PETRE PREDA, ALIN POGAN and CIPRIAN PREDA

Department of Mathematics, West University of Timişoara, Bd. V. Pârvan, No. 4, 300223 - Timişoara, Romania
e-mail:preda@math.uvt.ro, e-mail: ciprian.preda@fse.uvt.ro

(Received 25 November, 2002; accepted 21 October 2003)

Abstract. In this paper we investigate the most general dichotomy concept of evolutionary processes. This dichotomy concept includes many interesting situations, among them we note the nonuniform dichotomy. We characterize the (a, b)-dichotomy in terms of the admissibility of the pair (L^1_a, L^∞_b). Also, generalizations of the results of **[20]**, **[23]** are obtained.

2000 *Mathematics Subject Classification.* 34D05, 47D06, 93D20.

1. Introduction. In his famous paper **[17]**, Perron gave a characterization of exponential dichotomy of the solutions to the linear differential equations

$$\frac{dx}{dt} = A(t)x, \quad t \in [0, +\infty), \quad x \in \mathbb{R}^n,$$

where $A(t)$ is a matrix bounded continuous function, in terms of the existence of bounded solutions of the equations $\frac{dx}{dt} = A(t)x + f(t)$, where f is a continuous bounded function on $\mathbb{R}_+$.

This result serves as a starting point for numerous works on the qualitative theory of solutions of differential equations. We refer the reader to the book of Massera-Schäffer **[11]** and Daleckij-Krein **[4]**.

The more general case of evolutionary processes has been studied in **[5]** by R. Datko for exponential stability and by D. L. Lovelady **[9]**, P. Preda and M. Megan in **[12, 21]** for exponential dichotomy. In last few years, several results about exponential stability and exponential dichotomy for the case of exponentially bounded and strongly continuous evolution families were obtained by N. van Minh **[13, 14]**, F. Räbiger **[13]**, Y. Latushkin **[2, 6, 7, 8]**, M. Pinto **[1, 18]**, T. Randolph **[7, 8]**, R.Rau **[22]**, R. Sacker **[24]**, G. Sell **[24]** and R. Schnaubelt **[8, 13, 25]**, W. Zhang **[26]**. All these results are given for the uniform case.

For the first time, J. L. Massera and J. J. Schäffer have obtained in **[10]** results for the behaviour of solutions of the homogenous differential equations imitating a nonuniform exponential dichotomy, and later this case has been studied by M. Reghiş **[23]**, J. S. Muldowney **[15]**, V. A. Pliss **[19]**, P. Preda **[20, 21]**, M. Pinto **[16]** and others.

The concept of (a, b)-dichotomy is very general one and it is a natural extension of stability. This concept was introduced with the intention of obtaining results concerning stability for a weakly stable system (at least, weaker than those given by exponential stability). It covers many interesting situations in the asymptotic behavior of evolutionary processes, including the most general concept of uniform dichotomy, the so-called ordinary dichotomy.

The first aim of this paper is to establish the connections between (a, b)-dichotomy and the admissibility of the (a, b)- weighted spaces (L_a^1), L_b^∞, for the general abstract evolutionary processes. Also, the results obtained extend the theorems proved in [**20**], [**23**].

2. Preliminaries. Let X be a real or complex Banach space and $B(X)$ the Banach algebra of all bounded linear operators from X into itself.

Also, we denote by $\mathcal{M}(I, X)$ the space of all strongly measurable functions $f : I \to X$, where I is a real interval, and by

$$L_{loc}^1(\mathbb{R}_+, X) = \left\{ f \in \mathcal{M}(\mathbb{R}_+, X) : \int_0^\alpha \|f(s)\|\, ds < \infty, \text{ for all } \alpha \geq 0 \right\};$$

$$L^1([t, \infty), X) = \left\{ f \in \mathcal{M}([t, \infty), X) : \int_t^\infty \|f(s)\|\, ds < \infty \right\};$$

$$L_a^1(\mathbb{R}_+, X) = \left\{ f \in L_{loc}^1(\mathbb{R}_+, X) : \int_0^\infty a(s)\|f(s)\|\, ds < \infty \right\};$$

$$L_b^\infty(\mathbb{R}_+, X) = \left\{ f \in L_{loc}^1(\mathbb{R}_+, X) : \operatorname*{ess\,sup}_{s \geq 0} b(s)\|f(s)\| < \infty \right\},$$

where $a, b : \mathbb{R}_+ \to \mathbb{R}_+^*$ are two continuous functions and $t \geq 0$.

We note that $L^1([t, \infty), X)$, $L_a^1(\mathbb{R}_+, X)$, $L_b^\infty(\mathbb{R}_+, X)$ are Banach spaces endowed with the norms

$$\|f\|_1 = \int_t^\infty \|f(s)\|\, ds,$$

$$\|f\|_{1,a} = \int_0^\infty a(s)\|f(s)\|\, ds,$$

$$\|f\|_{\infty,b} = \operatorname*{ess\,sup}_{s \geq 0} b(s)\|f(s)\|,$$

respectively.

DEFINITION 2.1. A family of bounded linear operators acting on X, denoted by $\mathcal{U} = \{U(t, s)\}_{t \geq s \geq 0}$ is called an evolutionary process if

u_1) $U(t, t) = I$ (where I is the identity operator on X), for all $t \geq 0$;

u_2) $U(t, s)U(s, r) = U(t, r)$, for all $t \geq s \geq r \geq 0$;

u_3) $U(\cdot, s)x$ is continuous on $[s, \infty)$ for all $(s, x) \in \mathbb{R}_+ \times X$;
$U(t, \cdot)x$ is continuous on $[0, t]$ for all $(t, x) \in \mathbb{R}_+ \times X$;

u_4) there exist $M, \omega > 0$ such that

$$\|U(t, s)\| \leq M e^{\omega(t-s)}, \quad \text{for all } t \geq s \geq 0$$

DEFINITION 2.2. An application $P : \mathbb{R}_+ \to \mathcal{B}(X)$ is said to be a *dichotomy projection family* if

i) $P^2(t) = P(t)$, for all $t \geq 0$;

ii) $P(\cdot)x$ is a bounded continuous function for all $x \in X$.

We set $Q(t) = I - P(t), \ t \geq 0$.

REMARK 2.1. The family $\{Q(t)\}_{t\geq 0}$ is also a dichotomy projection family. For continuous functions $a, b : \mathbb{R}_+ \to \mathbb{R}_+^*$, we give the following definition.

DEFINITION 2.3. An evolutionary process $\mathcal{U}$ is said to be (a, b)-dichotomic if there exist P a dichotomy projection family and $N_1, N_2 > 0$ such that

d_1) $U(t, s)P(s) = P(t)U(t, s)$, for all $t \geq s \geq 0$;

d_2) $U(t, s) : \operatorname{Ker} P(s) \to \operatorname{Ker} P(t)$ is an isomorphism for all $t \geq s \geq 0$;

d_3) $\|U(t, s)x\| \leq N_1 \frac{a(s)}{b(t)} \|x\|$, for all $t \geq s \geq 0$ and all $x \in \operatorname{Im} P(s)$;

d_4) $\|U(t, s)x\| \geq N_2 \frac{b(s)}{a(t)} \|x\|$, for all $t \geq s \geq 0$ and all $x \in \operatorname{Ker} P(s)$.

In what follows we will consider the evolutionary processes $\mathcal{U}$ for which there exists a dichotomy projection family P such that the conditions d_1) and d_2) are satisfied. In that case we will use the notation

$$U_1(t, s) = U(t, s)_{|\operatorname{Im} P(s)}, \quad U_2(t, s) = U(t, s)_{|\operatorname{Ker} P(s)}.$$

We note that, in the area of differential equations, there is an extensive literature concerning the subject related to the dichotomy of evolutionary processes, provided by differential systems or otherwise. Most of the concepts, presented some decades ago, (see for instance [**3**, **4**, **11**], or more recently [**2**]) are of the uniform type. It is well-known that the most general concept of uniform dichotomy is the so-called ordinary dichotomy, which goes back to the book due to J. L. Massera and J. J. Schäffer [**11**]. In our case it can be obtained by taking $a(t) = b(t) = 1$ in Definition 2.3.

In order to show the consistency of our results and the connection with the cited papers we will present an example of evolutionary process, provided by a differential system, which is (a, b)-dichotomic, but is not ordinary dichotomic.

EXAMPLE 2.1. Let $X = \mathbb{R}$ and consider the differential equation

$$x'(t) = A(t)x(t)$$

where

$$A(t) = e^{-\pi} + 1 + \sin\ln(t+1) - \cos\ln(t+1).$$

We can associate the evolutionary process:

$$U(t, s) = V(t)V^{-1}(s)$$

where

$$V(t)x = e^{e^{-\pi} - (t+1)(e^{-\pi} + 1 - \cos\ln(t+1))}x$$

A simple computation shows that:

$$\|U(t, t_0)\| \leq e^{-e^{-\pi}t + 2t_0 + 2 + e^{-\pi}t_0}, \qquad \text{for all} \quad t \geq t_0 \geq 0$$

and hence U is (a, b)-dichotomic where:

$$a(t) = e^{(2+e^{-\pi})t+2}, \quad b(t) = e^{e^{-\pi}t}, \quad P(t)x = x.$$

We claim that U is not ordinary dichotomic. Assuming for a contradiction that U is ordinary dichotomic we have that there exist $\{Q(t)\}_{t\geq 0}$ a dichotomy projection family

https://doi.org/10.1017/S0017089504001715 Published online by Cambridge University Press

and two constants N_1, N_2 such that the conditions $d_1) - d_4)$ (from Definition 2.3) holds. Having in mind that $X = \mathbb{R}$ it follows that there exists $q : \mathbb{R}_+ \to \mathbb{R}$, a bounded and continuous function such that $Q(t)x = q(t)x$. It is easy to observe that

$$q(t) \in \{0, 1\}, \quad \text{for all} \quad t \geq 0 \tag{$*$}$$

On the other hand, it is easy to check that

$$\|U(t, 0)\| = \|V(t)\| \leq e^{-e^{-\pi}t}, \quad \text{for all} \quad t \geq 0,$$

and so it results that $\operatorname{Im} Q(0) = \mathbb{R}$ which implies that $q(0) = 1$. Using $(*)$ and the fact that q is continuous we obtain that $q(t) = 1$, for all $t \geq 0$. It follows that

$$\|U(t, s)\| \leq N_1, \quad \text{for all} \quad t \geq s \geq 0.$$

Also one can easily verify that

$$\lim_{k\to\infty} \|U(e^{2k\pi} - 1, e^{2k\pi-\pi} - 1)\| = \infty,$$

which is a contradiction.

Definition 2.4. The pair $(L^1_a(\mathbb{R}_+, X), L^\infty_b(\mathbb{R}_+, X))$ is said to be admissible for $\mathcal{U}$ if for all $f \in L^1_a(\mathbb{R}_+, X)$ we have:

i) $U_2^{-1}(\cdot, t)Q(\cdot)f \in L^1([t, \infty), X)$, for all $t \geq 0$;

ii) $x_f : \mathbb{R}_+ \to X$, $x_f(t) = \int_0^t U_1(t, s)P(s)f(s)\,ds - \int_t^\infty U_2^{-1}(s, t)Q(s)f(s)\,ds$ lies in $L^\infty_b(\mathbb{R}_+, X)$.

Lemma 2.1. *With our assumption we have that $U_2^{-1}(\cdot, t_0)Q(\cdot)x$ is continuous on $[t_0, \infty)$, for all $(t_0, x) \in \mathbb{R}_+ \times X$.*

Proof. Let $t \geq t_0 \geq 0$, $h \in (0, 1)$, $x \in X$. Then

$$\begin{aligned} U_2(t+1, t_0) &= U_2(t+1, r)U_2(r, t_0), \text{ for all } r \in [t, t+1] \text{ and so} \\ U_2^{-1}(t+h, t_0) &= U_2^{-1}(t+1, t_0)U_2(t+1, t+h) \\ U_2^{-1}(t, t_0) &= U_2^{-1}(t+1, t_0)U_2(t+1, t) \end{aligned}$$

It follows that

$$\begin{aligned} &\left\|U_2^{-1}(t+h, t_0)Q(t+h)x - U_2^{-1}(t, t_0)Q(t)x\right\| \\ &\quad = \left\|U_2^{-1}(t+1, t_0)[U_2(t+1, t+h)Q(t+h)x - U_2(t+1, t)Q(t)x]\right\| \\ &\quad \leq \left\|U_2^{-1}(t+1, t_0)\right\| \|U_2(t+1, t+h)Q(t+h)x - U_2(t+1, t)Q(t)x\| \\ &\quad = \left\|U_2^{-1}(t+1, t_0)\right\| \|U(t+1, t+h)Q(t+h)x - U_2(t+1, t)Q(t)x\| \\ &\quad \leq \left\|U_2^{-1}(t+1, t_0)\right\| \|U(t+1, t+h)(Q(t+h)x - Q(t)x)\| \\ &\qquad + \left\|U_2^{-1}(t+1, t_0)\right\| \|U(t+1, t+h)Q(t)x - U(t+1, t)Q(t)x\| \\ &\quad \leq \left\|U_2^{-1}(t+1, t_0)\right\| M e^{\omega(1-h)} \|Q(t+h)x - Q(t)x\| \\ &\qquad + \left\|U_2^{-1}(t+1, t_0)\right\| \|U(t+1, t+h)Q(t)x - U(t+1, t)Q(t)x\| \end{aligned}$$

It is easy to see that $U_2^{-1}(\cdot, t_0)Q(\cdot)x$ is right-handed continuous on $[t_0, \infty)$, for all x in X.

https://doi.org/10.1017/S0017089504001715 Published online by Cambridge University Press

Consider now $t > t_0 \geq 0$, $h \in (0, t - t_0)$, $x \in X$. Then

$$U_2(t, t_0) = U_2(t, t - h)U_2(t - h, t_0)$$

and so

$$U_2^{-1}(t - h, t_0) = U_2^{-1}(t, t_0)U_2(t, t - h)$$

It follows that

$$\begin{aligned}
&\left\| U_2^{-1}(t - h, t_0)Q(t - h)x - U_2^{-1}(t, t_0)Q(t)x \right\| \\
&\quad = \left\| U_2^{-1}(t, t_0)U_2(t, t - h)Q(t - h)x - U_2^{-1}(t, t_0)Q(t)x \right\| \\
&\quad \leq \left\| U_2^{-1}(t, t_0) \right\| \, \| U_2(t, t - h)Q(t - h)x - Q(t)x \| \\
&\quad = \left\| U_2^{-1}(t, t_0) \right\| \, \| U(t, t - h)Q(t - h)x - Q(t)x \| \\
&\quad \leq \left\| U_2^{-1}(t, t_0) \right\| \, [\| U(t, t - h)Q(t - h)x - U(t, t - h)Q(t)x \| \\
&\qquad + \| U(t, t - h)Q(t)x - Q(t)x \|] \\
&\quad \leq \left\| U_2^{-1}(t, t_0) \right\| \, [\| U(t, t - h) \| \, \| Q(t - h)x - Q(t)x \| \\
&\qquad + \| U(t, t - h)Q(t)x - Q(t)x \|] \\
&\quad \leq \left\| U_2^{-1}(t, t_0) \right\| \, [Me^{\omega h} \| Q(t - h)x - Q(t)x \| + \| U(t, t - h)Q(t)x - Q(t)x \|]
\end{aligned}$$

It then follows easily that $U_2^{-1}(\cdot, t_0)Q(\cdot)x$ is left-handed continuous on $[t_0, \infty)$ and finally we obtain that $U_2^{-1}(\cdot, t_0)Q(\cdot)x$ is continuous on $[t_0, \infty)$, for all x in X. □

3. The main results.

THEOREM 3.1. *If the pair* $(L_a^1(\mathbb{R}_+, X), L_b^\infty(\mathbb{R}_+, X))$ *is admissible to* $\mathcal{U}$ *then there exists* $K > 0$ *such that*

$$\| x_f \|_{\infty, b} \leq K \| f \|_{1,a}, \text{ for all } f \in L_a^1(\mathbb{R}_+, X).$$

Proof. Let us define $\wedge_t : L_a^1(\mathbb{R}_+, X) \to L^1([t, \infty), X)$ by

$$\wedge_t f = U_2^{-1}(\cdot, t)Q(\cdot)f$$

for any $t \geq 0$. It is obvious that $\wedge_t$ is a linear operator for all $t \geq 0$.

Consider $t \geq 0$, $\{f_n\}_{n \geq 1} \subset L_a^1(\mathbb{R}_+, X), f \in L_a^1(\mathbb{R}_+, X)$, $g \in L^1([t, \infty), X)$ such that

$$f_n \xrightarrow{L_a^1} f, \quad \wedge_t f_n \xrightarrow{L^1} g.$$

Then there exists a subsequence $\{f_{n_k}\}_{k \geq 1}$ of $\{f_n\}_{n \geq 1}$ such that

$$f_{n_k} \to f, \quad \wedge_t f_{n_k} \to g \text{ a.p.t.}$$

But

$$\| (\wedge_t f_{n_k})(s) - (\wedge_t f)(s) \| \leq \left\| U_2^{-1}(s, t)Q(s) \right\| \, \| f_{n_k}(s) - f(s) \|$$

for all $k \geq 1$ and all $s \geq t$.

It follows easily that $\wedge_t$ is a bounded operator for any $t \geq 0$.

https://doi.org/10.1017/S0017089504001715 Published online by Cambridge University Press

Let $T : L^1_a(\mathbb{R}_+, X) \to L^\infty_b(\mathbb{R}_+, X)$ the linear operator defined by

$$(Tf)(t) = \int_0^t U_1(t,s)P(s)f(s)\,ds - \int_t^\infty U_2^{-1}(s,t)Q(s)f(s)\,ds.$$

If $\{g_n\}_{n\geq 1} \subset L^1_a(\mathbb{R}_+, X)$, $g \in L^1_a(\mathbb{R}_+, X)$, $h \in L^\infty_b(\mathbb{R}_+, X)$ and $g_n \xrightarrow{L^1_a} g$, $Tg_n \xrightarrow{L^\infty_b} h$ then,

$$\begin{aligned}
\|(Tg_n)(t) - (Tg)(t)\| &\leq \left\| \int_0^t U_1(t,s)P(s)(g_n(s) - g(s))\,ds \right\| \\
&\quad + \left\| \int_t^\infty U_2^{-1}(s,t)Q(s)(g_n(s) - g(s))\,ds \right\| \\
&\leq \int_0^t \|U_1(t,s)P(s)(g_n(s) - g(s))\|\,ds + \| \wedge_t (g_n - g)\|_1 \\
&\leq tMe^{\omega t} \sup_{s\geq 0} \|P(s)\| \int_0^t \|g_n(s) - g(s)\|\,ds + \| \wedge_t (g_n - g)\| \\
&= tMe^{\omega t} \sup_{s\geq 0} \|P(s)\| \int_0^t \frac{1}{a(s)} a(s)\|g_n(s) - g(s)\|\,ds + \| \wedge_t (g_n - g)\|_1 \\
&\leq tMe^{\omega t} \sup_{s\geq 0} \|P(s)\| \sup_{s\in[0,t]} \frac{1}{a(s)} \|g_n - g\|_{1,a} + \| \wedge_t (g_n - g)\|_1,
\end{aligned}$$

for all $t \geq 0$ and all $n \in N$.

It follows that $Tg = h$, hence T is bounded.

So

$$\|x_f\|_{\infty,b} = \|Tf\|_{\infty,b} \leq \|T\|\,\|f\|_{1,a}, \quad \text{for all } f \in L^1_a(\mathbb{R}_+, X). \qquad \square$$

THEOREM 3.2. *The evolutionary process $\mathcal{U}$ is (a,b)-dichotomic if and only if the pair $(L^1_a(\mathbb{R}_+, X), L^\infty_b(\mathbb{R}_+, X))$ is admissible to $\mathcal{U}$.*

Proof. Necessity. It follows easily from Lemma 2.1 that $U_2^{-1}(\cdot,t)Q(\cdot)f$ is strongly measurable on $[t,\infty)$, for all $f \in L^1_a(\mathbb{R}_+, X)$ and all $t \geq 0$.

By Definition 2.3 we have that

$$\left\| U_2^{-1}(s,t)x \right\| \leq \frac{1}{N_2} \frac{a(s)}{b(t)} \|x\|,$$

for all $s \geq t \geq 0$ and all $x \in \operatorname{Ker} P(s)$, which implies that

$$\left\| U_2^{-1}(s,t)Q(s)f(s) \right\| \leq \frac{1}{N_2} \sup_{s\geq 0} \|Q(s)\| \frac{1}{b(t)} a(s)\|f(s)\|,$$

for all $s \geq t \geq 0$ and all $f \in L^1_a(\mathbb{R}_+, X)$ and hence $U_2^{-1}(\cdot,t)Q(\cdot)f \in L^1([t,\infty), X)$, for all $t \geq 0$ and all $f \in L^1_a(\mathbb{R}_+, X)$

$$\begin{aligned}
\|x_f(t)\| &\leq \int_0^t \|U_1(t,s)P(s)f(s)\|\,ds + \int_t^\infty \left\| U_2^{-1}(s,t)Q(s)f(s) \right\| ds \\
&\leq \int_0^t N_1 \frac{a(s)}{b(t)} \|P(s)f(s)\|\,ds + \int_t^\infty \frac{1}{N_2} \sup_{s\geq 0} \|Q(s)\| \frac{a(s)}{b(t)} \|f(s)\|\,ds
\end{aligned}$$

$$\leq \frac{N_1}{b(t)} \sup_{s\geq 0} \|P(s)\| \int_0^t a(s)\|f(s)\|\,ds + \frac{1}{N_2 b(t)} \sup_{s\geq 0} \|Q(s)\| \int_t^\infty a(s)\|f(s)\|\,ds$$
$$\leq \frac{1}{b(t)} \left(N_1 \sup_{s\geq 0} \|P(s)\| + \frac{1}{N_2} \sup_{s\geq 0} \|Q(s)\| \right) \|f\|_{1,a},$$

for all $t \geq 0$ and all $f \in L_a^1(\mathbb{R}_+, X)$. Now it is clear that $x_f \in L_b^\infty(\mathbb{R}_+, X)$, for all $f \in L_a^1(\mathbb{R}_+, X)$.

Sufficiency. Consider $t_0 \geq 0$, $x \in X$, $\delta > 0$ and $f : \mathbb{R}_+ \to X$

$$f(t) = \begin{cases} U(t, t_0)x, & t \in [t_0, t_0 + \delta] \\ 0, & t \notin [t_0, t_0 + \delta]. \end{cases}$$

We observe that $f \in L_a^1(\mathbb{R}_+, X)$ and

$$\|f\|_{1,a} = \int_{t_0}^{t_0+\delta} a(s)\|U(s, t_0)x\|\,ds,$$

and

$$x_f(t) = \int_0^t U_1(t, s)P(s)f(s)\,ds - \int_t^\infty U_2^{-1}(s, t)Q(s)f(s)\,ds$$
$$= \begin{cases} \delta U_1(t, t_0)P(t_0)x, & t \geq t_0 + \delta \\ -\delta U_2^{-1}(t_0, t)Q(t_0)x, & t \leq t_0. \end{cases}$$

It follows that

$$\begin{aligned} \delta b(t)\|U_1(t, t_0)P(t_0)x\| &\leq b(t)\|x_f(t)\| \\ &\leq \|x_f\|_{\infty,b} \leq K\|f\|_{1,a} \\ &\leq K \int_{t_0}^{t_0+\delta} a(s)\|U(s, t_0)x\|\,ds, \end{aligned}$$

for all $t_0 \geq 0$, $\delta > 0$, $x \in X$, $t \geq t_0 + \delta$.

If we make $\delta \to 0$ we obtain that

$$b(t)\|U_1(t, t_0)x\| \leq Ka(t_0)\|x\|,$$

for all $t \geq t_0 \geq 0$ and all $x \in \operatorname{Im} P(t_0)$ and hence d_3) is satisfied.

On the other hand

$$\begin{aligned} \delta b(t)\left\|U_2^{-1}(t_0, t)Q(t_0)x\right\| &\leq b(t)\|x_f(t)\| \\ &\leq \|x_f\|_{\infty,b} \leq K\|f\|_{1,a} \\ &\leq K \int_{t_0}^{t_0+\delta} a(s)\|U(s, t_0)x\|\,ds, \end{aligned}$$

for all $t_0 \geq 0$, $\delta > 0$, $x \in X$, $t \in [0, t_0]$.

Again by making $\delta \to 0$ we have that

$$b(t)\left\|U_2^{-1}(t_0, t)x\right\| \leq Ka(t_0)\|x\|,$$

https://doi.org/10.1017/S0017089504001715 Published online by Cambridge University Press

for all $t_0 \geq t \geq 0$ and all $x \in \text{Ker}\, P(t_0)$. Now it is clear that the condition d_4) holds too. $\square$

ACKNOWLEDGEMENT. The authors are indebted to the referee, whose comments and suggestions greatly improved the quality of the paper.

REFERENCES

1. S. Castillo and M. Pinto, Asymptotic integration of ordinary different systems, *J. Math. Anal. Appl.* **218** (1998), 1–12.

2. C. Chicone and Y. Latushkin, *Evolution semigroups in dynamical systems and differential equations*, Mathematical Surveys and Monographs, Vol. 70 (Amer. Math. Soc., 1999).

3. W. A. Coppel, *Dichotomies in stability theory*, Lecture Notes in Mathematics No 629 (Springer-Verlag, 1978).

4. J. L. Daleckij and M. G. Krein, *Stability of differential equations in Banach spaces* (Amer. Math. Soc., 1974.

5. R. Datko, Uniform asymptotic stability of evolution processes in Banach spaces, *SIAM. J. Math. Anal.* **3** (1973), 428–445.

6. Y. Latushkin and S. Montgomery-Smith, Evolutionary semigroups and Lyapunov theorems in Banach spaces, *J. Funct. Anal.* **127** (1995), 173–197.

7. Y. Latushkin and T. Randolph, Dichotomy of differential equations on Banach spaces and an algebra of weighted composition operators, *Integral Equations Operator Theory* **23** (1995), 472–500.

8. Y. Latushkin, T. Randolph and R. Schnaubelt, Exponential dichotomy and mild solution of nonautonomous equations in Banach spaces, *J. Dynam. Differential Equations* **3** (1998), 489–510.

9. D. L. Lovelady, A sufficient condition for an exponential dichotomy, *Atti Accad. Na. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8)* **57** (1974), 525–530.

10. J. L. Massera and J. J. Schäffer, Linear differential equations and functional analysis *I Ann. of Math.* **67** (1958), 517–573.

11. J. L. Massera and J. J. Schäffer, *Linear differential equations and function spaces*, (Academic Press, 1966).

12. M. Megan and P. Preda, Admissibility and uniform dichotomy for evolutionary processes in Banach spaces, *Ricerche Mat.* **XXXVII** (1988), 227–240.

13. N. van Minh, F. Räbiger and R. Schnaubelt, Exponential stability, exponential expansiveness and exponential dichotomy of evolution equations on the half-line, *Integral Equations Operator Theory* **32** (1998), 332–353.

14. N. van Minh, On the proof of characterizations of the exponential dichotomy, *Proc. Amer. Math. Soc.* **127** (1999), 779–782.

15. J. S. Muldowney, Dichotomies and asymptotic behaviour for linear differential systems, *Trans. Amer. Math. Soc.* **283** (1984), 465–484.

16. R. Nauín and M. Pinto, Roughness of (h, k)-dichotomies, *J. Differential Equations* **118** (1995), 20–35.

17. O. Perron, Die stabilitätsfrage bei differentialgeighungen, *Math.* Z. **32** (1930), 703–728.

18. M. Pinto, Dichotomies and asymptotic formulae of solutions of differential equations, *J. Math. Anal. Appl.* **195** (1995), 16–31.

19. V. A. Pliss, A reduction principle in the theory of stability of motion, *Izv. Acad. Nauk, SSSR, Mat. Ser.* **28** (1964), 1297–1324.

20. P. Preda, *a*, *b*- Dichotomie des Équations Differentiales Linéaires, *An. Univ. Timişoara, Ser. St. Mat.* **17** (1979), 65–71.

21. P. Preda and M. Megan, Non-uniform dichotomy of evolutionary processes in Banach space, *Bull. Austral. Math. Soc.* **27** (1983), 31–52.

22. R. Rau, Hyperbolic evolution groups and dichotomic of evolution families, *J. Dynam. Differential Equations* **6** (1994), 107–118.

23. M. Reghiş, On nonuniform asymptotic stability, *Prikl. Mat. Meh.* **27** (1963), 231–243 (Russian). English translation, *J. Appl. Math. Mech* **27** (1963), 344–362.

24. R. Sacker and G. Sell, Dichotomies for linear evolutionary equations in Banach spaces, *J. Differential Equations* **113** (1994), 17–67.

25. R. Schnaubelt, Sufficient conditions for exponential stability and dichotomy of evolution equations, *Forum Math.* **11** (1999), 543–566.

26. W. Zhang, The Fredholm alternative and exponential dichotomies for parabolic equations, *J. Math. Anal. Appl.* **191** (1995), 180–201.