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HOLOMORPHIC CURVES IN THE ORTHOGONAL TWISTOR SPACE

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A complete description of holomorphic curves in the Hermitian symmetric space $SO(6)/U(3)$ is given in terms of orthogonal differential invariants.

0. INTRODUCTION

The Hermitian symmetric space $SO(2n)/U(n)$ is naturally identified with the twistor space $\mathcal{T}(\mathbb{R}^{2n})$, the space of orthogonal complex structures on $\mathbb{R}^{2n}$. Thus a curve in $SO(2n)/U(n)$ can be thought of as a 1-parameter family of complex structures on $\mathbb{R}^{2n}$, and the study of holomorphic curves in $SO(2n)/U(n)$ pertains to the deformation problem of complex structures.

The case of $SO(6)/U(3)$ is particularly appealing as this space is symmetric space isomorphic to $\mathbb{C}P^3$, albeit via a complicated isomorphism. (The space $SO(4)/U(2)$ is symmetric space isomorphic to $\mathbb{C}P^1$, and the geometry of curves in $SO(4)/U(2)$ is trivial.) The study of holomorphic curves in $SO(6)/U(3)$ yields a new perspective on the study of holomorphic curves, and more generally minimal surfaces, in $\mathbb{C}P^3$ (see [2]).

In the present paper we give a complete description of holomorphic curves in $SO(6)/U(3)$ in terms of orthogonal differential invariants. Given a Riemann surface M we derive a system of partial differential equations with 3 unknown functions, (τ_i), on M. These partial differential equations (Section 4 (I1 – 3)) are the integrability conditions in the following sense: given a solution (τ_i) one can manufacture a holomorphic curve by integrating a Frobenius system. To put it another way, a solution to the integrability conditions determines a holomorphic curve constructively up to integration involving ordinary differential equations only. Moreover, every holomorphic curve in $SO(6)/U(3)$ arises in this manner.

1. THE SPACE OF ORTHOGONAL COMPLEX STRUCTURES ON $\mathbb{R}^{2n}$

Let $\mathcal{T}(\mathbb{R}^{2n})$ denote the space of orientation preserving orthogonal complex structures on $\mathbb{R}^{2n}$. More precisely,

$$\mathcal{T}(\mathbb{R}^{2n}) = \{J \in \mathrm{Aut}^+(\mathbb{R}^{2n}) \colon J^2 = -id,\ \langle Jv, Jw\rangle = \langle v, w\rangle,\ v, w \in \mathbb{R}^{2n}\},$$

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where $\mathrm{Aut}^+(\mathbb{R}^{2n})$ denotes the set of orientation preserving automorphisms of $\mathbb{R}^{2n}$. The space $\mathcal{T}(\mathbb{R}^{2n})$ can be identified with the Hermitian symmetric space $SO(2n)/U(n)$ as we shall see below.

Let $i\colon U(n) \to SO(2n)$ be the Lie group monomorphism induced by the identification

$$\mathbb{R}^{2n} = \mathbb{C}^n,\ (x^\alpha) \leftrightarrow (x^1 + ix^2, \cdots, x^{2n-1} + ix^{2n}).$$

More explicitly, if one writes

$$Z = X + iY \in U(n),\ X, Y \text{ real matrices,}$$

then

$$i(Z) = S \begin{bmatrix} X & -Y \\ Y & X \end{bmatrix} {}^tS,$$

where ${}^tS = (\varepsilon_1, \varepsilon_{n+1}, \varepsilon_2, \varepsilon_{n+2}, \cdots, \varepsilon_n, \varepsilon_{2n})$ and ε_i denotes the column vector with 1 at the ith entry and zeros elsewhere.

We put $H = i(U(n)) \subset SO(2n)$; we shall occasionally confuse H with $U(n)$. Let $A = (A_\alpha) = (A_1, \ldots, A_{2n}) \in SO(2n)$, and consider the assignment

$$A \mapsto J_A \in \mathcal{T}(\mathbb{R}^{2n})$$

given by the prescription

$$J_A(A_{2i-1}) = A_{2i},\ J_A(A_{2i}) = -A_{2i-1},\ 1 \leqslant i \leqslant n,$$

that is, the matrix of J_A with respect to the basis (A_α) is

$$j_n = \begin{bmatrix} 0 & -1 & & & & \\ 1 & 0 & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & 0 & -1 \\ & & & & 1 & 0 \end{bmatrix}.$$

This assignment induces a bijection

$$SO(2n)/H \to \mathcal{T}(\mathbb{R}^{2n})$$

as it is easily seen that $J_A = J_B$ if and only if the cosets AH and BH coincide.

In what follows we recall the root space decomposition of the Lie algebra $\mathfrak{o}(2n)$. First define ε^α_β to be the $2n \times 2n$ with 1 at the (α, β)-entry and zeros elsewhere. We then define the following matrices in $\mathfrak{o}(2n)$ ($\mathfrak{o}(2n)$ is thought of as the set of all $2n \times 2n$ real skew-symmetric matrices):

$$\begin{aligned}
F_i &= \varepsilon_{2i-1}^{2i} - \varepsilon_{2i}^{2i-1}, \\
E_{ij} &= \varepsilon_{2j-1}^{2i-1} + \varepsilon_{2j}^{2i} - \varepsilon_{2i-1}^{2j-1} - \varepsilon_{2i}^{2j}, \\
F_{ij} &= \varepsilon_{2j-1}^{2i} + \varepsilon_{2i-1}^{2j} - \varepsilon_{2i}^{2j-1} - \varepsilon_{2j}^{2i-1}, \\
E'_{ij} &= \varepsilon_{2j-1}^{2i-1} + \varepsilon_{2i}^{2j} - \varepsilon_{2i-1}^{2j-1} - \varepsilon_{2j}^{2i}, \\
F'_{ij} &= \varepsilon_{2j}^{2i-1} + \varepsilon_{2j-1}^{2i} - \varepsilon_{2i-1}^{2j} - \varepsilon_{2i}^{2j-1},
\end{aligned}$$

where $1 \leqslant i < j \leqslant n$. Now put

$$V_i = \mathbb{R} - \operatorname{span}\{F_i\}, \quad V_{ij} = \mathbb{R} - \operatorname{span}\{E_{ij}, F_{ij}\}, \quad V'_{ij} = \mathbb{R} - \operatorname{span}\{E'_{ij}, F'_{ij}\}.$$

The root spaces of $\sigma(2n)$ relative to the standard maximal torus

$$T = SO(2)^n \subset SO(2n)$$

($SO(2)^n$ is diagonally included in $SO(2n)$) are precisely

$$\oplus \sum V_i = t,\ V_{ij},\ V'_{ij},$$

where t is the Lie algebra of T, the trivial root space.

The Lie algebra of $H \subset SO(2n)$ is given by

$$\mathfrak{h} = \{X \in \mathfrak{o}(2n) \colon {}^tXj_n + j_nX = 0\}$$

so that

$$\mathfrak{h} = t \oplus \sum V_{ij}.$$

Consequently

$$m = \oplus \sum V'_{ij}$$

is the orthogonal complement to $\mathfrak{h}$ relative to the Killing form. Via π_{*e}, where π is the projection $SO(2n) \to SO(2n)/H$, $\mathfrak{m}$ gets identified with the tangent space at the identity coset of $SO(2n)/H$:

$$\mathfrak{m} = T_0(SO(2n)/H), \quad 0 = H.$$

Let $\Omega = \left(\Omega_\beta^\alpha\right)$ denote the $\mathfrak{o}(2n)$-valued Maurer–Cartan form of $SO(2n)$. We then have the decomposition

$$\Omega = \Omega_{\mathfrak{h}} + \Omega_{\mathfrak{m}}, \quad \Omega_{\mathfrak{m}} = \sum \Omega_{V'_{ij}},$$

where

$$\Omega_{V'_{ij}} = \frac{1}{2}[(\Omega^{2i-1}_{2j-1} - \Omega^{2i}_{2j}) \otimes E'_{ij} + (\Omega^{2i}_{2j-1} + \Omega^{2i-1}_{2j}) \otimes F'_{ij}] \text{(no sum)}.$$

We also put

$$\Theta'^{ij} = \frac{1}{2}[(\Omega^{2i-1}_{2j-1} - \Omega^{2i}_{2j}) + \sqrt{-1}(\Omega^{2i}_{2j-1} + \Omega^{2i-1}_{2j})], \quad 1 \leqslant i < j \leqslant n.$$

Recall that the space $SO(2n)/U(n)$ has, up to conjugacy, exactly one integrable almost complex structure; this complex structure is characterised by letting the pull-backs of (Θ'^{ij}) (by a local section of $SO(2n) \to SO(2n)/H$) span the space of type $(1, 0)$ forms on $SO(2n)/H$.

Any invariant metric on $SO(2n)/H$ is given by the pullback of the symmetric product

$$c \cdot \sum \Theta'^{ij} \cdot \overline{\Theta}'^{ij}, \; c > 0.$$

We shall use the metric coming from $c = 1$.

2. The linear isotropy representation

In this section we explicitly compute the linear isotropy representation

$$\rho : U(n) \to GL(\mathfrak{m}) = GL(T_0(SO(2n)/H)).$$

For $Z \in U(n)$ one has the inner automorphism

$$\text{Inn}_z : SO(2n) \to SO(2n), \; g \mapsto zyz^{-1},$$

where $z = i(Z) \in H$. The map Inn_z fixes the origin $0 = H \in SO(2n)/H$ and we obtain the automorphism

$$\text{Inn}_{z*0} : \mathfrak{m} \to \mathfrak{m}, \; \mathfrak{m} = T_0(SO(2n)/H).$$

The assignment

$$Z \in U(n) \mapsto \text{Ad}(Z)|_{\mathfrak{m}} = \text{Inn}_{z*0} \in GL(\mathfrak{m})$$

is nothing but the adjoint representation of $U(n)$ restricted to the invariant subspace $\mathfrak{m} \subset \mathfrak{o}(2n)$ $(\text{Ad}(H)\mathfrak{m} \subset \mathfrak{m})$. Then $\rho = \text{Ad}|_{\mathfrak{m}}$.

In what follows we take $n = 3$. Recall that a basis of $\mathfrak{m}$ is given by

$$E'_{12}, E'_{13}, E'_{23}, F'_{12}, F'_{13}, F'_{23} \in \mathfrak{o}(6).$$

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$\mathrm{Ad}_Z(E'_{ij})$ is computed from the matrix multiplication

$$i(Z)\cdot E'_{ij}\cdot i(Z)^{-1}.$$

Written out more fully,

$$\mathrm{Ad}_Z\,(E'_{ij})=\left(S\cdot\begin{bmatrix}X & -Y\\ Y & X\end{bmatrix}\cdot{}^tS\right)E'_{ij}\left(S\cdot\begin{bmatrix}{}^tX & {}^tY\\ -{}^tY & {}^tX\end{bmatrix}\cdot{}^tS\right).$$

Similarly one computes $\mathrm{Ad}_Z\,(F'_{ij})$. We want to write $\mathrm{Ad}_Z\,(E'_{ij})$ and $\mathrm{Ad}_Z\,(F'_{ij})$ as linear combinations in $(E'_{ij},\,F'_{ij})$. For this we define 3×3 matrices J_{ij} by the formula:

$$J_{ij}=\varepsilon^i_j-\varepsilon^j_i\in\mathfrak{o}(3),\quad 1\leqslant i<j\leqslant 3.$$

Note that

$${}^tS\cdot E'_{ij}\cdot S=\begin{bmatrix}J_{ij} & 0\\ 0 & -J_{ij}\end{bmatrix},\quad {}^tS\cdot F'_{ij}\cdot S=\begin{bmatrix}0 & J_{ij}\\ J_{ij} & 0\end{bmatrix}.$$

Computations reveal that

$$\begin{aligned}\mathrm{Ad}\,(Z)\colon E'_{ij}&\mapsto\{XJ_{ij}{}^tX-YJ_{ij}{}^tY\}_1+\{XJ_{ij}{}^tY+YJ_{ij}{}^tX\}_2,\\ F'_{ij}&\mapsto\{-XJ_{ij}{}^tY-YJ_{ij}{}^tX\}_1+\{XJ_{ij}{}^tX-YJ_{ij}{}^tY\}_2,\end{aligned}$$

where $\{\cdot\}_1$ means identify $J_{k\ell}$ with $E'_{k\ell}$, and $\{\cdot\}_2$ means identify $J_{k\ell}$ with $F'_{k\ell}$.

It is more convenient to rewrite the above using complex notation. So we write

$$Z=\left(x^i_j+\sqrt{-1}y^i_j\right)=\left(z^i_j\right)\in U(3),$$

and compute $\rho(Z)$ with respect to the complex basis

$$E'_{12}+iF'_{12},\,E'_{13}+iF'_{13},\,E'_{23}+iF'_{23},$$

so that $\rho(Z)\in GL(3,\,\mathbb{C})$. (In other words, we have chosen an identification of $\mathfrak{m}$ with $\mathbb{C}^3$.) We find that

$$\rho(Z)=\begin{bmatrix}Z_{33} & Z_{32} & Z_{31}\\ Z_{23} & Z_{22} & Z_{21}\\ Z_{13} & Z_{12} & Z_{11}\end{bmatrix}\in GL(3,\,\mathbb{C}),$$

where Z_{ij} denotes the $(i,\,j)$-minor of Z. We see that $\rho(Z)$ is related to the *adjoint matrix* of Z in a simple way. (The adjoint matrix of Z is the transpose of the cofactor matrix of Z so that Z times its adjoint matrix is the determinant of Z.) More precisely, we let

$$\delta=\begin{bmatrix} & & 1\\ & -1 & \\ 1 & & \end{bmatrix}.$$

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Then

$$(*) \qquad \rho(Z) = \det(Z)\delta \cdot \overline{Z} \cdot \delta \in GL(3, \mathbb{C}).$$

REMARK. More generally, consider the linear isotropy representation

$$\rho\colon U(n) \to GL(N, \mathbb{C}),$$

where $\mathfrak{m}$ is identified with $\mathbb{C}^N$, $N = n(n-1)/2$, via the lexicographical ordering of the root basis as in the $SO(6)/U(3)$ case. Identify $\mathbb{C}^N$ with $\Lambda^2(\mathbb{C}^n)$, the space of 2-vectors in $\mathbb{C}^n$. Calculations then show that

$$\rho(Z)(\varepsilon_i \wedge \varepsilon_j) = Z_i \wedge Z_j,$$

where $Z = (Z_i) \in U(n)$, and (ε_i) are the canonical basis vectors of $\mathbb{C}^n$. Extending linearly over the basis $\{\varepsilon_i \wedge \varepsilon_j \colon 1 \leqslant i < j \leqslant n\}$ of $\mathbb{C}^N$ one obtains the matrix representation ρ. However, the orbit structure of this action is very complicated for large n.

3. The Frenet frame along a holomorphic curve

Let M be a Riemann surface and consider a holomorphic map

$$f\colon M \to SO(6)/H.$$

We let $e\colon U \subset M \to SO(6)$ denote a smooth local section of the $U(3)$-principal bundle

$$f^{-1}SO(6) \to M.$$

The holomorphy of f is reflected by the fact that the forms

$$e^*\Theta'^{ij}, \quad 1 \leqslant i < j \leqslant 3,$$

are all of type $(1, 0)$ on M.

INDEX CONVENTION. $(12) = 1$, $(13) = 2$, $(23) = 3$; $1 \leqslant i, j \leqslant 3$, $1 \leqslant \alpha, \beta \leqslant 6$.

Fix a Riemannian metric on M from its conformal class, say ds^2_M. This means that we can locally write

$$ds^2_M = \varphi \cdot \overline{\varphi}$$

for some type $(1, 0)$ nonvanishing form φ. The 1-form φ is called a unitary coframe.

We define complex valued local functions (Z^i) on M by

$$e^*\Theta'^i = Z^i\varphi,$$

where φ is a fixed unitary coframe.

Put $\tau_1 = \sum |Z^i|^2$. It is routinely verified that τ_1 is a globally defined smooth function on M. Let $\sum_1$ denote the zero set of τ_1; we shall mostly work away from the set $\sum_1$.

Since f is nonconstant, the function τ_1 is not identically zero: since the pullback of the standard metric on $SO(6)/U(3)$ is given by

$$ds_1^2 = \tau_1 \varphi \cdot \overline{\varphi},$$

we see that $\tau_1(x) = 0$ if and only if $x \in M$ is a non-immersion point. In fact, we can say a lot more. But first we need to recall the notion of an analytic type function.

DEFINITION: Let U be a domain in the Riemann surface M. A $\mathbb{C}^n$-valued smooth function $h = (h^i)$ on U is said to be of analytic type if for each point $x \in U$, if z is a local holomorphic coordinate centred at x, then

$$h = z^b \widetilde{h},$$

where b is a positive integer and $\widetilde{h}$ is a smooth $\mathbb{C}^n$-valued function with $\widetilde{h}(x) \neq 0$.

So if h is a function of analytic type on U, then h is either identically zero or its zeros are isolated and of finite multiplicity (the integer b in the above definition).

It is known [1] that the functions of analytic type are exactly solutions of exterior equation

$$dh = \Phi h \ (\operatorname{mod} \varphi),$$

where Φ is an $n \times n$ matrix of complex valued 1-forms on U and φ is a nowhere zero type $(1, 0)$ form on U.

PROPOSITION. *The function $\tau_1 : M \to \mathbb{R}$ is an analytic type function on M.*

PROOF: We will show that the local function

$$(Z^i) : U \subset M \to \mathbb{C}^3$$

is of analytic type. Since $\tau_1 = \sum |Z^i|^2$, the rest follows. Exterior differentiation of both sides of the equations

$$e^* \Theta'^i = Z^i \varphi$$

leads to

$$dZ^i \equiv \Psi^i_j Z^j \ (\operatorname{mod} \varphi):$$

one uses the Maurer-Cartan structure equations of $SO(6)$ and the equation

$$d\varphi = -\theta_C \wedge \varphi,$$

where θ_C is the complex connection form. For example,

$$dZ^1 \equiv \left(\theta_C + i\omega_2^1 + i\omega_4^3\right)Z^1 + \left(i\omega_6^3 - \omega_6^4\right)Z^2 + \left(\omega_6^2 - i\omega_6^1\right)Z^3 \ (\text{mod } \varphi),$$

where $\omega = e^*\Omega$. ∎

Note that $r_1 = \sqrt{\tau_1}$ is a continuous function on M smooth away from its zeros.

Suppose $\tilde{e}\colon \tilde{U} \to SO(6)$ is another lifting of f. This means that

$$\tilde{e} = e \cdot k,$$

for some smooth function $k\colon U \cap \tilde{U} \to U(3)$. Define $\left(\tilde{Z}^i\right)$ by setting

$$\tilde{e}^*\Theta'^i = \tilde{Z}^i\varphi.$$

We then obtain the following

TRANSFORMATION RULE. $\left(\tilde{Z}^i\right) = \rho\left(k^{-1}\right)\left(Z^i\right)$.

The above rule follows from the fact that

$$(e \cdot k)^*\Omega_{\mathfrak{m}} = \operatorname{Ad}\left(k^{-1}\right)e^*\Omega_{\mathfrak{m}}.$$

Consulting (*) in the preceding section it is clear that near a point in $M \setminus \sum_1$ we can make

$$Z^1 = r_1, \quad Z^2 = Z^3 = 0,$$

where $r_1 = \sqrt{\tau_1}$. Any lifting e achieving this will be called a first order frame.

Let e be any first order frame along f, and write $\omega = e^*\Omega$. We then have

(E1) $$\frac{1}{2}\left[\left(\omega_3^1 - \omega_4^2\right) + i\left(\omega_3^2 + \omega_4^1\right)\right] = r_1\varphi,$$

(E2) $$\omega_5^1 = \omega_6^2, \omega_5^2 = -\omega_6^1,$$

(E3) $$\omega_5^3 = \omega_6^4, \omega_5^4 = \omega_6^3.$$

Define a subgroup of $U(3)$ by

$$G_1 = \{k \in U(3)\colon \rho(k)^t(1, 0, 0) = {}^t(1, 0, 0)\}.$$

We find that

$$G_1 = \left\{\begin{bmatrix} Z & 0 \\ 0 & \exp(i\theta)\end{bmatrix} : Z \in SU(2)\right\} \cong SU(2) \times U(1).$$

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Let e be a first order frame along f. Then any other first order frame along f is given by $e \cdot k$, where k is a G_1-valued local function on M.

The first order frames are constructed near a point $x \in M \setminus \sum_1$. Near a point in $\sum_1$ we can find a *generalised first order frame* e such that

$$e^*\Theta'^2 = e^*\Theta'^3 = 0, \quad e^*\Theta'^1 = Z_1\varphi,\, Z_1 \text{ complex valued,}$$

where $|Z_1|^2 = \tau_1$. To see this observe that (Z^i) can be written as

$$z^b\left(\widehat{Z}^i\right),$$

where z is a local holomorphic coordinate centred at x and $\left(\widehat{Z}^i(x)\right) \neq 0$. Thus we can use the H-action to bring about

$$\widehat{Z}^1 = \widehat{r}_1 > 0,\, \widehat{Z}^2 = \widehat{Z}^3 = 0.$$

Then $Z_1 = z^b \widehat{r}^1$ does the trick.

By way of notation we put

$$\psi^1 = \frac{1}{2}\left(\omega_3^1 - \omega_4^2\right),\, \psi^2 = \frac{1}{2}\left(\omega_3^2 + \omega_4^1\right),\, \psi = \psi^1 + i\psi^2.$$

Note that ψ is of type $(1, 0)$, and that

$$\psi = Z_1\varphi,$$

and near a point in $M \setminus \sum_1$

$$\psi = r_1\varphi.$$

We now exterior differentiate the relations (E1 – 3) and construct the *second order frames*. We will use the Maurer–Cartan structure equations of $SO(6)$. We will also need the structure equations for $(M,\, \varphi \cdot \overline{\varphi})$:

$$d\varphi = -\theta_C \wedge \varphi.$$

The purely imaginary 1-form θ_C is the complex connection form with respect to φ, and is nothing but $-i$ times the Levi–Civita connection form of $(M,\, (\varphi^i))$. Differentiating this equation one more time we obtain

$$d\theta_C = \frac{K}{2}\varphi \wedge \overline{\varphi}, \quad K = \text{ the Gaussian curvature.}$$

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We now exterior differentiate the left hand side of (E1):

$$d(\psi^1 + i\psi^2) = i(\omega_2^1 + \omega_4^3) \wedge (\psi^1 + i\psi^2) = i(\omega_2^1 + \omega_4^3) \wedge r_1\varphi.$$

On the other hand,

$$d(r_1\varphi) = dr_1 \wedge \varphi - r_1\theta_C \wedge \varphi = (dr_1 - r_1\theta_C) \wedge \varphi.$$

It follows that

$$[d\log r_1 - i(\omega_2^1 + \omega_4^3) - \theta_C] \wedge \varphi = 0.$$

Since $d\log r_1$ is real and $i(\omega_2^1 + \omega_4^3) + \theta_C$ is purely imaginary, we then must have

(F1) $$*d\log r_1 = i[\theta_C + i(\omega_2^1 + \omega_4^3)],$$

where $*$ is the Hodge operator of (M, ds^2).

Exterior differentiation of the first equation in (E2) leads to

(†) $$\omega_5^3 \wedge \varphi^1 + \omega_5^4 \wedge \varphi^2 = 0;$$

the second equation yields

(‡) $$\omega_5^4 \wedge \varphi^1 - \omega_5^3 \wedge \varphi^2 = 0.$$

It follows from (†, ‡) that

$$\omega_5^4 = - * \omega_5^3, \quad \omega_5^2 = - * \omega_5^1,$$

and we may set

$$\omega_5^1 - i\omega_5^2 = Z^1\varphi, \quad \omega_5^3 - i\omega_5^4 = Z^2\varphi$$

for some local complex valued functions (Z^i) on M.

Put $\tau_2 = \sum |Z^i|^2$. It is easily verified that τ_2 is a well-defined smooth function on M. A consideration similar to that given for τ_1 shows that τ_2 is an analytic type function on M. Let $\sum_2$ denote the zero set of τ_2.

Define tilded quantities $\widetilde{Z}^1$, $\widetilde{Z}^2$ by

$$\widetilde{\omega}_5^1 - i\widetilde{\omega}_5^2 = \widetilde{Z}^1\varphi, \quad \widetilde{\omega}_5^3 - i\widetilde{\omega}_5^4 = \widetilde{Z}^2\varphi, \quad \widetilde{\omega} = \widetilde{e}^*\Omega,$$

where $\widetilde{e} = e \cdot k$, $k \in G_1$-valued, is another first order frame. Write

$$k = (Z, e^{i\theta}),$$

where Z is $SU(2)$-valued. We can write Z as

$$Z = \begin{bmatrix} z_1^1 & z_2^1 \\ -\overline{Z}_2^1 & \overline{z}_1^1 \end{bmatrix}, \quad z_j^i = x_j^i + iy_j^i.$$

From the formula $\widetilde{\omega} = i(k^{-1}) \cdot \omega \cdot i(k)$ we compute that

$${}^t\big(\widetilde{Z}^i\big) = e^{-i\theta} \cdot {}^tZ \cdot {}^t(Z^i).$$

We see from this that we can make $Z^1 = r_2 > 0$, and $Z^2 = 0$.

Summarising the preceding computation, we have

PROPOSITION. *Let $f\colon M \to SO(6)/U(3)$ be a nonconstant holomorphic map. Near any point $x \in M \setminus \{\sum_1 \cup \sum_2\}$ there exists a local lifting e into $SO(6)$ such that in addition to (E1 – 3) we have*

(E4) $$\omega_5^1 - i\omega_5^2 = r_2\varphi, \quad \omega_5^3 = \omega_5^4 = 0,$$

where, as usual, $\omega = e^*\Omega$.

The totality of such frames, called the second order frames, is determined up to the structure group

$$\begin{aligned} G_2 &= \{(Z, e^{i\theta}) \in SU(2) \times U(1)\colon e^{-i\theta} \cdot {}^tZ \cdot {}^t(1,0) = {}^t(1,0)\} \\ &= \{(e^{i\theta}, e^{-i\theta}, e^{i\theta}) \in U(3)\} \cong U(1). \end{aligned}$$

THEOREM. *Suppose $\tau_2(f) \equiv 0$. Then $f(M)$ is congruent to an open submanifold of $SO(4)/U(2) \cong \mathbb{C}P^1$.*

PROOF: $\tau_2(f) \equiv 0$ means that the bundle of first order frames along f, denoted by L_1, is an integral manifold of the exterior system

$$\Omega_B^A = 0, \quad 1 \leqslant A \leqslant 4, \quad 5 \leqslant B \leqslant 6,$$

on $SO(6)$. (So a first order frame along f is a local section of $L_1 \to M$.) It follows that L_1 is a translate of $SO(4) \times SO(2) \subset SO(6)$. Then $f(M)$ is congruent to a submanifold of

$$SO(4) \times SO(2)/(U(3) \cap SO(4) \times SO(2)) \cong SO(4)/U(2) \cong \mathbb{C}P^1.$$

∎

Hereafter we assume that τ_2 is not identically zero.

We now exterior differentiate both sides of the equations in (E4), and construct the third order frames.

We obtain from the first equation of (E4)

$$d(\omega_5^1 - i\omega_5^2) = i(\omega_6^5 - \omega_2^1) \wedge r_2\varphi,$$
$$d(r_2\varphi) = dr_2 \wedge \varphi - r_2\theta_C \wedge \varphi.$$

Consequently,

$$\{d\log r_2 - \theta_C - i(\omega_6^5 - \omega_2^1)\} \wedge \varphi = 0.$$

Therefore

$$\text{(F2)} \qquad *d\log r_2 = i(\theta_C + i(\omega_6^5 - \omega_2^1)).$$

The remaining two equations in (E4), upon exterior differentiation, yield

$$\omega_3^1 \wedge \varphi^1 - \omega_3^2 \wedge \varphi^2 = 0, \quad \omega_4^1 \wedge \varphi^1 - \omega_4^2 \wedge \varphi^2 = 0.$$

Consequently, we can write

$$\text{(1)} \qquad \omega_3^1 = a\varphi^1 + b\varphi^2, \quad \omega_4^1 = b\varphi^1 + c\varphi^2,$$
$$\text{(2)} \qquad \omega_3^2 = *\omega_3^1, \quad \omega_4^2 = *\omega_4^1,$$

where a, b, c are some local functions on M with $a + c = 2r_1$.

Define tilded quantities $\widetilde{a}$, $\widetilde{b}$, $\widetilde{c}$ using another second order frame $\widetilde{e} = e \cdot k$, where

$$k = (e^{i\theta}, e^{-i\theta}, e^{i\theta}), \quad \theta \text{ a local function on } M.$$

We want to know how $(\widetilde{a}, \widetilde{b}, \widetilde{c})$ are related to (a, b, c).

Again using the formula

$$\widetilde{\omega} = i(k^{-1}) \cdot \omega \cdot i(k), \omega = e^*\Omega, \widetilde{\omega} = \widetilde{e}^*\Omega,$$

we compute that

$$\widetilde{a} = a \cdot \cos^2\theta + c \cdot \sin^2\theta - 2b \cdot \cos\theta\sin\theta,$$
$$\widetilde{b} = b \cdot \cos 2\theta + (a - c) \cdot \cos\theta\sin\theta,$$
$$\widetilde{c} = a \cdot \sin^2\theta + c \cdot \cos^2\theta + b \cdot \sin 2\theta.$$

If b does not vanish, then we can smoothly choose θ so that

$$\operatorname{cotan} 2\theta = (c - a)/2b$$

making $\widetilde{b} = 0$. All this leads to the third, and final, normal form

$$\text{(E5)} \qquad \omega_3^1 + i\omega_3^2 = a\varphi, \quad \omega_4^1 + i\omega_4^2 = -ic\varphi,$$

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where $a + c = 2r_1$. Put $r_3 = (a - c)/2$.

The function $\tau_3 = r_3^2$ is an analytic type function on M; we let $\sum_3$ denote the zero set of τ_3.

The isotropy group G_3 is given by

$$G_3 = \{(e^{i\theta}, e^{-i\theta}, e^{i\theta}) \in G_2 \colon \theta = n\pi/2,\, n \in \mathbb{Z}\} \cong \mathbb{Z}_4.$$

It follows that near a point $x \in M \setminus \{\sum_1 \cup \sum_2 \cup \sum_3\}$ there is a more or less unique lifting

$$e_f \colon U \subset M \to SO(6)$$

achieving the normal forms (E1) through (E5). Such a lifting will be called a *Frenet frame* along f.

Exterior differentiation of both sides of the equations in (E5) leads to

$$\text{(F1)} \qquad *d\log r_1 = i\big(\theta_C + i(\omega_2^1 + \omega_4^3)\big),$$

$$\text{(F3)} \qquad *d\log r_3 = i\big(\theta_C + i(\omega_2^1 - \omega_4^3)\big).$$

Remark. Suppose $\tau_3(f)$ is identically zero. Then one can show that $f(M)$ lies in the image of a $\mathbb{C}P^2 \subset \mathbb{C}P^3$ under the symmetric space isomorphism

$$\mathbb{C}P^3 \cong SO(6)/U(3).$$

Theorem. *Let $f\colon M \to SO(6)/U(3)$ be a nonconstant holomorphic map. Fix a conformal metric $ds^2 = \varphi \cdot \overline{\varphi}$, $\varphi \in$ type $(1,0)$, and define the differential invariants (r_i) as in the above. We then have*

$$\text{(I1)} \qquad \Delta \log r_1 = K - 4\tau_1 + 2\tau_2,$$

$$\text{(I2)} \qquad \Delta \log r_2 = K + 2(\tau_1 + \tau_3) - 4\tau_2,$$

$$\text{(I3)} \qquad \Delta \log r_3 = K + 2\tau_2 - 4\tau_3,$$

away from the singular locus $\sum_1 \cup \sum_2 \cup \sum_3 \subset M$.

Proof: Exterior differentiate both sides of the equations in (F1 – 3) using

$$d * d\log r_i = \frac{i}{2}\Delta \log r_i\, \varphi \wedge \overline{\varphi},$$

$$d\omega_2^1 = \frac{i}{2}(a^2 + c^2 - 2r_2^2)\varphi \wedge \overline{\varphi},$$

$$d\omega_4^3 = iac\,\varphi \wedge \overline{\varphi},$$

$$d\omega_6^5 = ir_2^2\,\varphi \wedge \overline{\varphi}.$$

∎

We give an application.

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COROLLARY. *Suppose $f: M \to SO(6)/U(3)$ is a holomorphic isometric immersion from a compact M. Further suppose that $K \geqslant 4/3$, where K is the Gaussian curvature of (M, ds^2). Then we must have $K = 4/3$.*

PROOF: $K \geqslant 4/3$ implies that

$$\Delta \log (r_2^2 r_3) \geqslant 0.$$

Thus $\log (r_2^2 r_3)$ is a subharmonic function with singularties at the zeros of r_2 and r_3 where it goes to $-\infty$. In particular, this function attains a maximum on M. Now the maximum principle for subharmonic functions says that it must be a constant. □

4. THE INTEGRABILITY CONDITIONS AND THE ASSOCIATED PDE SYSTEM

In this section we summarise the frame construction by setting up a bijective correspondence between the holomorphic curves in $SO(6)/U(3)$ and the solutions to the PDE system coming from (I1 – 3).

DEFINITION: We shall say that $f: M \to SO(6)/U(3)$ is a nondegenerate curve if none of the r_i's are identically zero. The map f will be called a regular curve if $r_1 \cdot r_2 \cdot r_3$ is never zero.

Observe that the regularity assumption is a global assumption.

Consider the following exterior differential system, denoted by S, defined on $M \times SO(6)$ with independence condition $\varphi \wedge \overline{\varphi} \neq 0$:

$$\Omega_3^1 + i\Omega_3^2 = (r_1 + r_3)\varphi, \; -\Omega_4^2 + i\Omega_4^1 = (r_1 - r_3)\varphi,$$
$$\Omega_5^1 - i\Omega_5^2 = \Omega_6^2 - i\Omega_6^1 = r_2\varphi,$$
$$\Omega_5^3 = \Omega_5^4 = \Omega_6^4 = \Omega_6^3 = 0.$$
$$\Omega_2^1 + \Omega_4^3 = i\theta_C - *d\log r_1,$$
$$\Omega_6^5 - \Omega_2^1 = i\theta_C - *d\log r_2,$$
$$\Omega_2^1 - \Omega_4^3 = i\theta_C - *d\log r_3,$$

where θ_C is the complex connection form of $(M, \varphi \cdot \overline{\varphi})$, and the r_i's are any positive functions on M solving the PDE system

(I1) $$\Delta \log r_1 = K - 4r_1^2 + 2r_2^2,$$

(I2) $$\Delta \log r_2 = K + 2(r_1^2 + r_3^2) - 4r_2^2,$$

(I3) $$\Delta \log r_3 = K + 2r_2^2 - 2r_3^2.$$

THEOREM. *The set of regular holomorphic curves $M \to SO(6)/U(3)$ is in bijective correspondence with the set of all solutions (r_1, r_2, r_3) to the integrability conditions (I1 –3).*

PROOF: Any regular curve certainly gives rise to such a solution: this is the content of the frame construction given in the preceding section. Conversely suppose we are given such a solution (r_i). Counting the number of independent equations in $\mathcal{S}$ we see that $\mathcal{S}$ defines a two-dimensional distribution on $M \times SO(6)$. Moreover, this distribution is completely integrable and, hence, defines a foliation on $M \times SO(6)$. The independence condition $\varphi \wedge \overline{\varphi} \neq 0$ implies that a leaf of this foliation can be written locally as

$$U \to U \times SO(6), \quad z \mapsto (z, e(z)).$$

It is straightforward to verify that $e(z)$ is a Frenet frame along $f = \pi \circ e$, where π denotes the projection $SO(6) \to SO(6)/U(3)$. ∎

5. Compact curves

In this section we give the integrated version of the integrability conditions (I1 – 3) assuming that M is compact.

DEFINITION: Let M be a Riemann surface. A singular Hermitian metric on M is given locally as

$$ds^2 = \psi \cdot \overline{\psi},$$

where ψ is a type $(1,0)$ smooth form of analytic type, that is, ψ can be written as the product of an analytic type function and a nowhere vanishing type $(1,0)$ form. We can rewrite ds^2 as

$$ds^2 = h(z)dz \cdot d\overline{z},$$

where $h(z) \geqslant 0$ and z is a holomorphic coordinate. Moreover, we have

$$h(z) = |z|^{2n}\widetilde{h}(z),$$

where $\widetilde{h}(z)$ is never zero and n is a nonnegative integer. The integer n is the order of ψ at $z = 0$ and we write $\mathrm{ord}_0\psi = n$. The singular divisor of ds^2, denoted by D_ψ, is defined to be the zero divisor of ψ. So

$$D_\psi = \sum \mathrm{ord}_p(\varphi)p,\ p \in M.$$

It is easy to see that D_ψ depends only on the singular metric, not on the particular choice of ψ. The degree of D_ψ is locally finite, and is the total number of zeros of ψ counted with multiplicity.

Given a singular metric ds^2 on M we have the usual Hermitian structure equations away from the support of the singular divisor:

$$d\psi = -\theta_C \wedge \psi,\ d\theta_C = \frac{K}{2}\psi \wedge \overline{\psi} = (-iK) \cdot \text{ the Kähler form.}$$

There is the

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Generalised Gauss–Bonnet–Chern Theorem. *Let M be a compact Riemann surface of genus g equipped with a singular metric $\psi\overline{\psi}$. Then*

$$\frac{i}{2\pi}\int_M d\theta_C = 2 - 2g + \deg D_\psi.$$

Proof: This follows from the usual Gauss–Bonnet–Chern theorem combined with the argument principle: one notes that $d\theta_C$ is a multiple of $\Delta \log h(dz \wedge d\overline{z})$. □

Given a nondegenerate holomorphic curve $f\colon M \to SO(6)/U(3)$ we define the ith osculating metric to be

$$ds_i^2 = \tau_i \varphi \cdot \overline{\varphi}.$$

These metrics are singular metrics. (Note that ds_1^2 is just the induced metric.) We put

$$\varphi_i = \tau_i \varphi,$$
$$\Lambda_i = \frac{i}{2}\varphi_i \wedge \overline{\varphi}_i = \text{ the Kähler form of } (M, ds_i^2),$$
$$\theta_{i,C} = \text{ the complex connection form of } (M, ds_i^2),$$
$$K_i = \text{ the Gaussian curvature of } (M, ds_i^2)$$

so that
$$d\varphi_i = -\theta_{i,C} \wedge \varphi_i,\ d\theta_{i,C} = -iK_i\Lambda_i.$$

Let e_f be a Frenet frame along f, and put $\omega = e^*\Omega$. Consulting the normal forms (E1 – 5) in Section 3 we compute that

$$\theta_{1,C} = i(\omega_1^2 + \omega_3^4),$$
$$\theta_{2,C} = i(\omega_5^6 - \omega_1^2),$$
$$\theta_{3,C} = i(\omega_1^2 - \omega_3^4).$$

(For example, the first and third relations follow upon exterior differentiating the first two equations in (E5).) Exterior differentiation of these equations leads to

$$d\theta_{1,C} = 2i(-2\Lambda_1 + \Lambda_2),$$
$$d\theta_{2,C} = 2i(\Lambda_1 - 2\Lambda_2 + \Lambda_3),$$
$$d\theta_{3,C} = 2i(\Lambda_2 - 2\Lambda_3).$$

Theorem. *Let $M = M_g$ denote a compact Riemann surface of genus g, and consider a nondegenerate curve $f\colon M \to SO(6)/U(3)$. Then*

(P) $$2g - 2 - \#_i = d_{i-1} - 2d_i + d_{i+1},$$

where $$\#_i = \deg D_{\varphi_i} = \text{ the total number of zeros of } \tau_i,$$
$$d_i = \frac{1}{\pi}\cdot \text{ (the area of } (M, ds_i^2)),$$
$$d_{-1} = d_4 = 0.$$

PROOF: We have

$$\frac{i}{2\pi}\int_M d\theta_{i,C} = 2 - 2g + \#_i$$

from the generalised Gauss–Bonnet–Chern theorem. From the Wirtinger theorem

$$\int_M \Lambda_i = \text{ the area of } \left(M,\, ds_i^2\right),$$

and the result follows. □

REMARK. The relations in (P) correspond to the Plucker relations for algebraic curves in $\mathbb{C}P^3$ [2] p.86–95).

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