ON PRODUCT OF RADÓN MEASURES

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I. INTRODUCTION

Let X, Y be locally compact Hausdorff spaces and μ, ν be Radón outer measures on X and Y respectively. The classical product outer measure ϕ on $X \times Y$ generated by measurable rectangles, without direct reference to the topology, turns out to have some serious drawbacks. For example, one can only prove that closed $\mathcal{G}_\delta$ sets (and hence Baire sets) are ϕ-measurable. It is unknown, even when X and Y are compact, whether closed sets are ϕ-measurable. This has led to consideration of other product measures: the Radón product outer measure λ introduced through linear functionals and the Riesz representation theorem and the Bledsoe-Morse [1] product outer measure ψ generated by measurable rectangles and nil sets (sets N for which $\iint 1_N \, d\mu \, d\nu = 0 = \iint 1_N \, d\nu \, d\mu$), again without direct reference to the topology.

Our aim in this paper is to establish relations between these three product measures. Our key result is a Fubini theorem for ϕ over compact sets, even though we do not know whether such sets are ϕ-measurable. This enables us to check that the three outer measures agree on compact sets and that, in the σ-finite case, the restrictions of ψ, λ, ϕ to their respective measurable sets are extensions of each other. We also give a characterization of when closed sets are ϕ-measurable. Since the convolution of two measures on a topological group is intimately connected to the product measure, our results on ϕ, λ, ψ enable us to compare three corresponding definitions of convolution $\mu * \nu$. Our point of view also enables us to extend, in the Radón case, the validity of the formula

$$\iint f(x \cdot y^{-1}) \, d\mu(x) \, d\nu(y) = \int f \, d(\mu * \nu) = \iint f(x \cdot y^{-1}) \, d\nu(y) \, d\mu(x)$$

to the class of all absolutely measurable functions. This extends the results in Stromberg [8].

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II. NOTATION AND PRELIMINARIES

Throughout this paper we shall use the following notation and terminology:

1. General Notation.

1.1 The real numbers include $\pm\infty$ and $\infty \cdot 0 = 0 \cdot \infty = 0$.

1.2 ω is the set of non-negative integers.

1.3 0 is both the empty set and the number zero.

1.4 $A \sim B = \{x : x \in A \text{ and } x \notin B\}$.

1.5 1_A is the characteristic function of A .

1.6 If H is a family of sets, an H_σ is a countable union, and an H_δ a countable intersection, of sets from H . A $\mathcal{G}_\delta$ is a countable intersection of open sets.

2. Measures.

2.1 m is an outer measure on S if and only if m is a function on $\{A : A \subset S\}$ to the non-negative reals such that $m(0) = 0$ and if $A \subset \bigcup_{n\in\omega} B_n$ then $m(A) \leq \sum_{n\in\omega} m(B_n)$.

2.2 A is an m-measurable set if and only if m is an outer measure on S, $A \subset S$, and for every $T \subset S$,

$$m(T) = m(T \cap A) + m(T \sim A) .$$

$\mathfrak{M}_m$ denotes the set of all m-measurable subsets of S. It is a σ-field on which m is σ-additive.

2.3 f is an m-measurable function if and only if m is an outer measure on S, f is a function on S to the reals and for every real number t, $\{x : f(x) > t\}$ is an m-measurable set.

2.4 m is σ-finite on A if and only if there exists a sequence B such that $m(B_n) < \infty$ for each $n \in \omega$ and $A \subset \bigcup_{n\in\omega} B_n$.

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3. Integrals. The integral we use is the standard one obtained by a set theoretic definition (cf. Munroe [6], Halmos [4]). Thus, if m is an outer measure on S, f is an m-measurable function, and $f(x) \geq 0$ for all $x \in S$ then

$$\int f \; dm = \lim_{t \to 1^+} \sum_{i=-\infty}^{\infty} t^i \cdot m(A_i) + \infty \cdot m(B)$$

where for each $t > 1$,

$$A_i = \{x : t^i \leq f(x) < t^{i+1}\}$$

and

$$B = \{x : f(x) = \infty\} .$$

4. Radón Measures. Let S be a topological space.

4.1 m is a Radón outer measure on S if and only if m is an outer measure on S such that

(i) open sets are m-measurable;

(ii) for every open $\alpha \subset S$, $m(\alpha) = \sup\{m(K): K \text{ compact}, K \subset \alpha\}$;

(iii) for every $A \subset S$, $m(A) = \inf\{m(\alpha): \alpha \text{ open}, \alpha \supset A\}$;

(iv) for every compact $K \subset S$, $m(K) < \infty$.

The following theorem is well known.

4.2 THEOREM (Generation of Radón outer measures).

Let g *be a real-valued function such that for* B *and* C *compact*

$$0 \leq g(B) \leq g(B \cup C) \leq g(B) + g(C) < \infty ;$$

if $B \cap C = 0$ *then* $g(B \cup C) = g(B) + g(C)$, *and* $g(0) = 0$.

For any $A \subset S$, *let* $\mu(A) = \inf_{\substack{\alpha \text{ open} \\ \alpha \supset A}} \sup_{\substack{K \text{ compact} \\ K \subset \alpha}} g(K)$.

Then if S is a locally compact Hausdorff space, μ is a Radón outer measure.

5. Linear Functionals. Let S be a locally compact Hausdorff space.

5.1 $C(S)$ is the set of all continuous functions on S to the finite reals vanishing outside a compact set.

5.2 L is a positive linear functional on $C(S)$ if and only if

(i) L is a function on $C(S)$ to the finite reals;

(ii) for every $f, g \in C(S)$ and finite real numbers a and b, $L(a \cdot f + b \cdot g) = a \cdot L(f) + b \cdot L(g)$;

(iii) if $f \geq 0$ then $L(f) \geq 0$.

Note that if L is a positive linear functional on $C(S)$, L is continuous in the sense that for each compact $K \subset S$ there exists $a_K < \infty$ such that $|L(f)| \leq a_K \sup_{x \in S} |f(x)|$ whenever f vanishes outside K.

5.3 For L a positive linear functional on $C(S)$ the outer measure associated with L is the outer measure m on S such that for every $A \subset S$

$$m(A) = \inf_{\substack{\alpha \text{ open} \\ A \subset \alpha}} \sup_{\substack{f \in C(S) \\ 0 \leq f \leq 1_\alpha}} L(f) .$$

Note that also

$$m(A) = \inf_{\substack{\alpha \text{ open} \\ A \subset \alpha}} \sup_{\substack{K \text{ compact} \\ K \subset \alpha}} \inf_{\substack{f \in C(S) \\ f \geq 1_K}} L(f) .$$

Reformulation of the standard Riesz Representation Theorem yields the following:

5.4 THEOREM. If L is a positive linear functional on $C(S)$ then the outer measure associated with L is the unique Radón outer measure m on S such that

$$L(f) = \int f \, dm \quad \text{for every } f \in C(S) .$$

III. PRODUCT OUTER MEASURES

Throughout this part X, Y are locally compact Hausdorff spaces and μ, ν are Radón outer measures on X and Y respectively. Our aim here is to discuss and compare three outer measures on the Cartesian product $X \times Y$.

6. Definitions.

6.1 For $A \subset X \times Y$, $\hat{A}_x = \{y : (x, y) \in A\}$.

6.2 $\mathcal{R} = \{a \times b : a \in \mathfrak{M}_\mu, \; b \in \mathfrak{M}_\nu, \; \mu(a) + \nu(b) < \infty\}$.

6.3 N is a nil set if and only if $\iint 1_N \, d\mu \, d\nu = 0 = \iint 1_N \, d\nu \, d\mu$.

6.4 L_μ and L_ν are the positive linear functionals on $C(X)$ and $C(Y)$ respectively given by:

$$L_\mu(f) = \int f \, d\mu$$

$$L_\nu(f) = \int f \, d\nu .$$

6.5 $L_\mu \otimes L_\nu$ is the positive linear functional on $C(X \times Y)$ such that for every $f \in C(X)$, $g \in C(Y)$, if $h(x, y) = f(x) \cdot g(y)$ then

$$L_\mu \otimes L_\nu(h) = L_\mu(f) \cdot L_\nu(g) .$$

For the existence and uniqueness of $L_\mu \otimes L_\nu$ see, e.g., Bourbaki [2, Theorème 1, page 89].

6.6 The classical product outer measure ϕ is defined, for $A \subset X \times Y$, by

$$\phi(A) = \inf \{ \sum_{a \times b \in F} \mu(a) \cdot \nu(b) : F \text{ is countable}, \; F \subset \mathcal{R},$$

$$A \subset \bigcup_{\alpha \in F} \alpha \} .$$

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6.7 The Bledsoe-Morse product outer measure ψ is defined, for $A \subset X \times Y$, by

$$\psi(A) = \inf \{ \sum_{a \times b \in F} \mu(a) \cdot \nu(b) : F \text{ is countable, } F \subset \mathfrak{R},$$

$$A \sim \bigcup_{\alpha \in F} \alpha \text{ is a nil set} \} .$$

6.8 The Radón product outer measure λ is the Radón outer measure associated with $L_\mu \otimes L_\nu$ by the Riesz Representation Theorem 5.4.

Remark. The definition of λ guarantees that open sets are λ-measurable. One of the main results in Bledsoe-Morse [1, Theorem 7.7] states that open sets are also ψ-measurable. It is unknown if open sets are ϕ-measurable even when X and Y are compact.

7. Elementary Properties. The following are immediate consequences of the definitions or well-known facts.

7.1 If $a \in \mathfrak{M}_\mu$ and $b \in \mathfrak{M}_\nu$ then $a \times b \in \mathfrak{M}_\phi \cap \mathfrak{M}_\psi \cap \mathfrak{M}_\lambda$ and if $\mu(a) + \nu(b) < \infty$ $(a \times b \in \mathfrak{R})$ then $\phi(a \times b) = \lambda(a \times b) = \psi(a \times b) = \mu(a) \cdot \nu(b)$. Standard Fubini theorems for ϕ, ψ, and λ follow from 7.1 by using well-known lines of argument. In particular, we have:

7.2 If A belongs to the σ-ring generated by $\mathfrak{R}$ then $A \in \mathfrak{M}_\phi \cap \mathfrak{M}_\lambda \cap \mathfrak{M}_\psi$ and $\phi(A) = \lambda(A) = \psi(A)$.

7.3 $\psi \leq \phi$.

7.4 If $\alpha \subset X \times Y$, α open, then $\alpha \in \mathfrak{M}_\psi$ and

$$\psi(\alpha) = \sup \{\psi(K) : K \text{ is compact, } K \subset \alpha\} .$$

This follows from Theorems 7.5 and 7.7 of Bledsoe-Morse [1]. In the definitions of ϕ and ψ we could replace $\mathfrak{R}$ by the family of open rectangles. Thus

7.5 For every $A \subset X \times Y$,

$$\phi(A) = \inf \{\phi(\alpha) : \alpha \text{ is open and } \phi\text{-measurable, } A \subset \alpha\} .$$

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7.6 Let $m = \phi, \psi, \lambda$. If $m(A) < \infty$ and $\varepsilon > 0$, then for some compact rectangle $K \in \mathfrak{R}$ (not necessarily contained in A)

$$m(A \sim K) < \varepsilon .$$

Every closed $\mathcal{G}_\delta$ in $X \times Y$ is ϕ-measurable. This follows from 7.6, since every compact $\mathcal{G}_\delta$ in $X \times Y$ is ϕ-measurable (Halmos [4, Theorem 51*]).

7.8 Every continuous function on $X \times Y$ to the reals is ϕ-measurable (by 7.7).

8. <u>Comparison of the Product Outer Measures.</u> A key result for comparing the three product measures is the Fubini theorem 8.1 below. It is obtained in spite of the fact that we do not know whether compact sets are ϕ-measurable.

8.1 THEOREM. <u>If</u> $C \subset X \times Y$ <u>and</u> C <u>is compact,</u> <u>then</u>

$$\phi(C) = \iint 1_C \, d\nu \, d\mu = \iint 1_C \, d\mu \, d\nu .$$

The proof of this theorem is given after proving Lemmas A and B below.

LEMMA A. <u>If</u> $C \subset X \times Y$ <u>is compact,</u> $t > 0$, $T = \{x : \nu(\hat{C}_x) < t\}$, <u>then</u> T <u>is open and for any</u> $A \subset T$

$$\phi(C \cap (A \times Y)) \leq t \cdot \mu(A) .$$

<u>Proof</u>. For each $x \in T$ there exists open $\beta_x \supset \hat{C}_x$ with $\nu(\beta_x) < t$. Then $X \times \beta_x$ is open and $C \sim (X \times \beta_x)$ is compact, whence its projection π_x onto X is compact and thus $\alpha_x = X \sim \pi_x$ is open. Now $x \in \alpha_x$,

$$\alpha_x = \{u : \hat{C}_u \subset \beta_x\} \subset T$$

and thus T is open.

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The remainder of the conclusion is trivial if $\mu(A) = \infty$. Hence let $A \subset T$, $\mu(A) < \infty$, $\varepsilon > 0$. Let B be open, $A \subset B$,

$$\mu(B) < \mu(A) + \frac{\varepsilon}{t} .$$

Since $\bigcup_{x \in A} \alpha_x \supset A$, by 4.1 there exists a countable subset of A, say $\{a_n : n \in \omega\} \subset A$, such that

$$\mu\left(A \sim \bigcup_{n \in \omega} \alpha_{a_n}\right) = 0 .$$

Let

$$\alpha'' = A \sim \bigcup_{n \in \omega} \alpha_{a_n}$$

$$\alpha'_n = B \cap \alpha_{a_n} \sim \bigcup_{m < n} \alpha_{a_m} .$$

Then the α'_n and α'' are μ-measurable and disjoint. Since C is compact, $C \subset D \times E$, for some D and E compact. For each $x \in \alpha_{a_n}$, $\hat{C}_x \subset \beta_{a_n}$ so

$$C \cap (A \times Y) \subset \bigcup_{n \in \omega} (\alpha'_n \times \beta_{a_n}) \cup (\alpha'' \times E) .$$

Hence

$$\begin{aligned} \phi(C \cap (A \times Y)) &\leq \sum_{n \in \omega} \mu(\alpha'_n) \cdot \nu(\beta_{a_n}) \\ &\leq t \cdot \sum_{n \in \omega} \mu(\alpha'_n) \\ &\leq t \cdot \mu(B) \\ &\leq t \cdot \mu(A) + \varepsilon . \end{aligned}$$

Thus

$$\phi(C \cap (A \times Y)) \leq t \cdot \mu(A).$$

LEMMA B. If $C \subset X \times Y$, $A \subset X$ and for every $x \in A$, $\nu(\hat{C}_x) \geq t$ then

$$\phi(C \cap (A \times Y)) \geq t \cdot \mu(A).$$

Proof. Let $\phi(C \cap (A \times Y)) < \infty$, $\varepsilon > 0$. Choose $a_i \times b_i \in \Re$ for $i \in \omega$ such that

$$C \cap (A \times Y) \subset \bigcup_{i \in \omega} (a_i \times b_i)$$

and

$$\sum_{i \in \omega} \mu(a_i) \cdot \nu(b_i) \leq \phi(C \cap (A \times Y)) + \varepsilon .$$

Let $B = \bigcup_{i \in \omega} (a_i \times b_i)$.

Then

$$\begin{aligned}\phi(C \cap (A \times Y)) + \varepsilon &\geq \phi(B) = \int \nu(\hat{B}_x)\, d\mu(x)\\ &\geq \int_A \nu(\hat{B}_x)\, d\mu(x)\\ &\geq \int_A \nu(\hat{C}_x)\, d\mu(x)\\ &\geq t \cdot \mu(A).\end{aligned}$$

Proof of Theorem 8.1. Let $1 < t < \infty$ and for each integer n let

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$$A_n = \{x : t^n \le \nu(\hat{C}_x) < t^{n+1}\}$$

$$B = \{x : \hat{C}_x \ne 0 , \ \nu(C_x) = 0\} .$$

Then by Lemmas A and B,

$$t^n \cdot \mu(A_n) \le \phi(C \cap (A_n \times Y)) \le t^{n+1} \cdot \mu(A_n)$$

$$\text{and} \quad \phi(C \cap (B \times Y)) = 0 .$$

Thus

$$\sum_{n=-\infty}^{\infty} t^n \cdot \mu(A_n) \le \sum_{n=-\infty}^{\infty} \phi(C \cap (A_n \times Y))$$

$$= \phi(C) \le \sum_{n=-\infty}^{\infty} t^{n+1} \cdot \mu(A_n)$$

$$= t \cdot \sum_{n=-\infty}^{\infty} t^n \cdot \mu(A_n) .$$

Since by Lemma A, $\{x : \nu(\hat{C}_x) < u\}$ is open for each u, $\int \nu(\hat{C}_x)\, d\mu(x) < \infty$ and (see 3)

$$\int \nu(\hat{C}_x)\, d\mu(x) = \lim_{t \to 1+} \sum_{n=-\infty}^{\infty} t^n \cdot \mu(A_n) .$$

Hence

$$\phi(C) = \int \nu(\hat{C}_x)\, d\mu(x) = \iint 1_C(x, y)\, d\nu(y)\, d\mu(x) .$$

Similarly

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$$\phi(C) = \iint 1_C(x, y)\, d\mu(x)\, d\nu(y) .$$

8.2 THEOREM. Let $C \subset X \times Y$ be compact. Then

$$\psi(C) = \lambda(C) = \phi(C) .$$

Proof. This follows immediately from 8.1 and the Fubini theorem for ψ and λ (remark following 7.1).

8.3 COROLLARY. For open $\alpha \subset C \times Y$,

$$\psi(\alpha) = \lambda(\alpha) \leq \phi(\alpha) .$$

Proof. This follows from 8.2, 7.4 and 4.1 (ii).

8.4 THEOREM. (i) $\psi \leq \lambda \leq \phi$.

(ii) $\mathfrak{M}_\phi \subset \mathfrak{M}_\lambda \subset \mathfrak{M}_\psi$.

(iii) If $A \in \mathfrak{M}_\phi$ and $\phi(A) < \infty$ then $\psi(A) = \lambda(A) = \phi(A)$.

(iv) If $A \in \mathfrak{M}_\lambda$ and $\lambda(A) < \infty$ then $\psi(A) = \lambda(A)$.

Thus, if μ and ν are σ-finite on X and Y respectively then the measure $\psi/\mathfrak{M}_\psi$ is an extension of the measure $\lambda/\mathfrak{M}_\lambda$ which in turn is an extension of the measure $\phi/\mathfrak{M}_\phi$.

Proof. (i) Follows immediately from 8.3, 4.1 (iii) and 7.5. To check part of (ii) and (iii), let $A \in \mathfrak{M}_\phi$. If $\phi(A) < \infty$ then there exists an $(\mathfrak{R}_\sigma)_\delta$ set B such that $A \subset B$ and $\phi(B \sim A) = 0$. Then, by 7.2, $B \in \mathfrak{M}_\lambda \cap \mathfrak{M}_\psi$ and, by (i), $\lambda(B \sim A) = \psi(B \sim A) = 0$, so that $A \in \mathfrak{M}_\lambda \cap \mathfrak{M}_\psi$ and $\psi(A) = \lambda(A) = \phi(A)$. To see that $A \in \mathfrak{M}_\lambda$ even when $\phi(A) = \infty$, let $T \subset X \times Y$, $\lambda(T) < \infty$ and $\varepsilon > 0$. Then, by 7.6 there exists a compact $K \in \mathfrak{R}$ such that $\lambda(T \sim K) < \varepsilon$. Since $\phi(A \cap K) < \infty$, we have $A \cap K \in \mathfrak{M}_\lambda$ and therefore

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$$\lambda(T \cap A) + \lambda(T \sim A) \leq \lambda(T \cap K \cap A) + \lambda(T \sim (K \cap A)) + 2\varepsilon$$

$$= \lambda(T) + 2\varepsilon .$$

A similar argument yields the other part of (ii) and (iv), if we replace '$(\mathcal{R}_\sigma)_\delta$' by '$\mathcal{G}_\delta$' and use 7.4.

We conclude this section by giving a characterization of when open sets are ϕ-measurable.

8.5 THEOREM. All open sets in $X \times Y$ are ϕ-measurable if and only if for every open α with $\phi(\alpha) < \infty$,

$$\phi(\alpha) = \sup_{\substack{K \text{ compact} \\ K \subset \alpha}} \phi(K) .$$

We first prove Lemma D.

LEMMA D. All open sets in $X \times Y$ are ϕ-measurable if and only if all open α with $\phi(\alpha) < \infty$ are ϕ-measurable.

Proof. Suppose for every open α with $\phi(\alpha) < \infty$, $\alpha \in \mathfrak{M}_\phi$.
Let γ be open, $\phi(T) < \infty$.
Then there exists open $\beta \supset T$ with $\phi(\beta) < \infty$.
Then $\phi(\beta \cap \gamma) < \infty$ and hence $\beta \cap \gamma$ is ϕ-measurable, so

$$\phi(T) = \phi(T \cap \beta \cap \gamma) + \phi(T \sim (\beta \cap \gamma))$$

$$= \phi(T \cap \gamma) + \phi(T \sim \gamma) .$$

Proof of Theorem 8.5. Suppose all open sets are ϕ-measurable. Given open α with $\phi(\alpha) < \infty$ and $\varepsilon > 0$ there exist by 7.5 and 7.6, compact K with $\phi(\alpha \sim K) < \varepsilon$, and open $\gamma \supset K \sim \alpha$ with $\phi(\gamma) < \phi(K \sim \alpha) + \varepsilon$. Let

$$C = K \sim \gamma .$$

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Then C is compact, $C \subset \alpha$, and

$$\begin{aligned} \phi(C) &\geq \phi(K) - \phi(\gamma) \\ &\geq \phi(K) - \phi(K \sim \alpha) - \varepsilon \\ &= \phi(\alpha \cap K) - \varepsilon \\ &\geq \phi(\alpha \cap K) + \phi(\alpha \sim K) - 2\varepsilon \\ &\geq \phi(\alpha) - 2\varepsilon . \end{aligned}$$

Conversely, if α is open, $\phi(\alpha) < \infty$ and

$$\phi(\alpha) = \sup_{\substack{K \text{ compact} \\ K \subset \alpha}} \phi(K),$$

then there exists $\beta \subset \alpha$, a countable union of open rectangles, with $\phi(\beta) = \phi(\alpha)$. Then since β is ϕ-measurable, $\phi(\alpha \sim \beta) = \phi(\alpha) - \phi(\beta) = 0$ and thus α is ϕ-measurable. Application of Lemma D completes the proof.

8.6 COROLLARY. <u>If open sets in</u> $X \times Y$ <u>are</u> ϕ-<u>measurable and</u> μ, ν <u>are</u> σ-<u>finite on</u> X, Y <u>respectively then</u> $\phi = \lambda$.

<u>Proof.</u> This follows from 8.2, 8.5, and 7.5.

9. <u>Examples and Remarks</u>. The examples below show certain directions in which the theorems of Section 8, particularly 8.4, cannot be extended.

9.1 <u>Remarks</u>. Let $\mathfrak{R}' = \{a \times b : a \in \mathfrak{M}_\mu, \ b \in \mathfrak{M}_\nu\}$ and for each $A \subset X \times Y$ let

$$\theta(A) = \inf \{ \sum_{a \times b \in F} \mu(a) \cdot \nu(b) : F \text{ is countable,}$$

$$F \subset \mathfrak{R}', \ A \subset \bigcup_{\alpha \in F} \alpha \} .$$

The essential difference between ϕ and θ is that if $0 \neq a \subset X$, $\mu(a) = 0$, $b \subset Y$ and ν is not σ-finite on b then $\theta(a \times b) = \mu(a) \cdot \nu(b) = 0$, but $\phi(a \times b) = \infty$.

However, if $\phi(A) < \infty$ then $\phi(A) = \theta(A)$ and thus many of the results in this paper will hold as well for θ as for ϕ. Further comparisons are made in the examples below.

9.2 We may have $\phi = \lambda$ but $\psi \neq \lambda$, even in the σ-finite case.

Proof. Let X and Y be the real line with the ordinary topology, μ and ν be Lebesgue measure on X and Y respectively. Then ϕ and λ are equal to Lebesgue measure on $X \times Y$.

Let A be the set constructed by Sierpinski [7] which is not ϕ-measurable but intersects any straight line in the plane in at most two points.

Then

$$\iint 1_A \, d\mu \, d\nu = 0 = \iint 1_A \, d\nu \, d\mu ,$$

i.e. A is a nil set, and hence $\psi(A) = 0$. Since A is not ϕ-measurable, we have $\phi(A) \neq 0$.

9.3 We may have A λ-measurable, $\lambda(A) = \infty$ and $\psi(A) = 0$.

Proof. Let μ be Lebesgue measure on X, the real line with the ordinary topology, and let ν be counting measure on Y, an uncountable set with the discrete topology.

Then if α is open in $X \times Y$, we easily check that

$$\phi(\alpha) = \lambda(\alpha) = \sum_{y \in Y} \mu\{x : (x, y) \in \alpha\}$$

and hence $\phi = \lambda$.

Let $A = \{0\} \times Y$. If α is open, $\alpha \supset A$, then $\lambda(\alpha) = \infty$, and hence $\phi(A) = \lambda(A) = \infty$, but A is a nil set and hence $\psi(A) = 0$. Note that in this case the σ-algebra generated in $X \times Y$ by the open rectangles does not include all the open sets of $X \times Y$.

Note also that $0 = \theta(A) \neq \inf_{\substack{\alpha \text{ open} \\ \alpha \supset A}} \theta(\alpha) = \infty$ and hence the outer measure θ (of 9.1) does not have the property of approximation from above by open sets.

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9.4 For some X and Y all open sets in $X \times Y$ are ϕ-measurable and $\infty = \phi(X \times Y) \neq \sup_{\substack{K \text{ compact} \\ K \subset X \times Y}} \phi(K) = 0$

and hence $\phi(X \times Y) \neq \lambda(X \times Y)$.

Proof. Let $X \neq 0$, $\mu(X) = 0$, Y be an uncountable set with the discrete topology, ν = counting measure on Y. For any $A \subset X \times Y$, if the projection of A onto Y is countable, then $\phi(A) = 0$, and if the projection of A onto Y is uncountable, then $\phi(A) = \infty$. Thus every subset of $X \times Y$ is ϕ-measurable.

IV. CONVOLUTION OF MEASURES

Throughout the next two sections, X is a locally compact, Hausdorff topological group, μ and ν are Radón outer measures on X, and ϕ, λ, ψ denote the product outer measures of μ and ν introduced in Section 6.

There are many ways of defining a convolution of μ and ν. The ultimate goal is to find a measure ξ such that the equation

(1) $$\iint f(x \cdot y^{-1})\, d\mu(x)\, d\nu(y) = \int f\, d\xi = \iint f(x \cdot y^{-1})\, d\nu(y)\, d\mu(x)$$

holds for as large a class of real-valued functions f on X as possible including the continuous functions with compact support.

10. Convolution and Absolute Measurability.

10.1 Preliminary Definitions

1. For any $A \subset X$, $A^* = \{(x, y) : x \cdot y^{-1} \in A\}$.

2. For any function f on X and $x, y \in X$

$$f^*(x, y) = f(x \cdot y^{-1})$$

3. For any topological space S and $A \subset S$, A is absolutely measurable if and only if $A \in \bigcap_{m \in M} \mathfrak{M}_m$ where M is the family of outer measures m on S such that open sets are m-measurable. A real-valued function f on S is absolutely measurable if and only if $\{x : f(x) < t\}$ is absolutely measurable for every real t.

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We note that the family of absolutely measurable sets is a σ-field which contains the Borel sets as well as the analytic sets and their complements. The following lemma is known and follows readily from the definitions.

10.2 LEMMA. Let h be a function on S to S', m be an outer measure on S and $m'(A) = m(h^{-1}(A))$ for any $A \subset S'$. Then

.1. m' is an outer measure on S' :

.2. If $A \subset S'$ and $h^{-1}(A)$ is m-measurable then A is m'-measurable.

.3. If S and S' are topological spaces, h is continuous, $A \subset S'$ and A is absolutely measurable, then $h^{-1}(A)$ is absolutely measurable.

Now, we also easily check that, for any $A \subset X$, $(1_A)^* = 1_{A^*}$ so that if we formally substitute 1_A for f in equation (1) above, we get

$$\iint 1_{A^*}\, d\mu\, d\nu = \xi(A) = \iint 1_{A^*}\, d\nu\, d\mu .$$

The Fubini theorem therefore leads us to consider the following definitions of convolution.

10.3 Definition. For any $A \subset X$,

1. $(\mu * \nu)_\phi (A) = \phi(A^*)$

2. $(\mu * \nu)_\lambda (A) = \lambda(A^*)$

3. $(\mu * \nu)_\psi (A) = \psi(A^*)$.

In view of Lemma 10.2, for $m = \phi, \lambda, \psi$, we see that $(\mu * \nu)_m$ is an outer measure on X that, for any $A \subset X$, A is $(\mu * \nu)_m$-measurable whenever A^* is m-measurable, and that, for any function f, $\int f\, d(\mu * \nu)_m = \int f^*\, dm$.

11. Properties of the Convolutions. The properties of $(\mu * \nu)_m$, for $m = \phi, \lambda, \psi$, follow from properties of m and the applicability of the Fubini theorem.

11.1 THEOREM. Let μ and ν be σ-finite and $m = \phi, \lambda, \psi$. If f is a real-valued function on X such that f^* is m-measurable, in particular if f is continuous, and $\int f \, d(\mu * \nu)_m$ exists then

$$\iint f(x \cdot y^{-1}) \, d\mu(x) \, d\nu(y) = \int f \, d(\mu * \nu)_m = \iint f(x \cdot y^{-1}) \, d\nu(y) \, d\mu(x) .$$

Proof. This follows immediately from the Fubini theorem applied to $\int f^* \, dm$ and 7.8.

11.2 COROLLARY. Let μ and ν be σ-finite and $m = \lambda, \psi$. Then for any absolutely measurable function f on X,

$$\iint f(x \cdot y^{-1}) \, d\mu(x) \, d\nu(y) = \int f \, d(\mu * \nu)_m = \iint f(x \cdot y^{-1}) \, d\nu(y) \, d\mu(x) .$$

Proof. This follows immediately from 11.1, 10.2.

11.3 THEOREM. If μ and ν are σ-finite then, for any compact $K \subset X$,

$$(\mu * \nu)_\phi(K) = (\mu * \nu)_\lambda(K) = (\mu * \nu)_\psi(K) .$$

Proof. Let C be an ascending sequence of compact rectangles such that $m(X \times X \sim \bigcup_{n \in \omega} C_n) = 0$ for $m = \phi, \lambda, \psi$.

Since K^* is closed in $X \times X$ then by 8.3 we have for every $n \in \omega$,

$$\phi(K^* \cap C_n) = \lambda(K^* \cap C_n) = \psi(K^* \cap C_n) .$$

Passing to the limit, we get the desired result.

11.4 THEOREM. If $\mu(X) + \nu(X) < \infty$ then $(\mu * \nu)_\lambda$ is a Radón outer measure on X.

Proof. This follows from the fact that λ is a Radón outer measure on $X \times X$ and straightforward computations using the continuity of the map $h(x,y) = x \cdot y^{-1}$.

Note that the condition $\mu(X) + \nu(X) < \infty$ is essential in 11.4 as can be seen by taking Lebesgue measure on the line for μ and ν.

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Then λ is 2-dimensional Lebesque measure and $(\mu * \nu)_\lambda(A) = \infty$ for any interval A of positive length.

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