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NECESSARY CONDITIONS OF OPTIMAL IMPULSE CONTROLS FOR DISTRIBUTED PARAMETER SYSTEMS

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Optimal control problem of semilinear evolutionary distributed parameter systems with impulse controls is considered. Necessary conditions of optimal controls are derived. The result generalises the usual Pontryagin's maximum principle.

1. INTRODUCTION

Control problems for systems with impulse controls are important and interesting because of their wide applications. We refer the interested readers to [4] and the references cited therein for substantial discussions. We note that most of the existing literature treats the problem via the dynamic programming approach [2, 3, 18, 19]. Recently, in [17], some sort of Pontryagin type maximum principle was derived for the finite dimensional case with the cost functional being linear in the impulse control. The result was extended to general Volterra-Stieltjes systems in infinite dimensions in [22] (see [21] also). In [13], for finite dimensional stochastic and deterministic systems with the cost functional being not necessarily linear in the impulse control, a similar result was proved. Some other approaches were used to treat the similar problem in [6, 7]. The purpose of this paper is to derive the Pontryagin type necessary conditions for optimal controls of semilinear evolutionary distributed parameter systems with impulse controls and with the cost functional being not necessarily linear in the impulse control.

The approach we will use is the variational method combining the Ekeland's variational principle. It should be pointed out that if we applied the usual variational technique as in [15] to our problem, we would immediately see that the variational system along the optimal solution of the problem with respect to any given admissible control is not clearly defined since it will involve the δ-function. As a consequence, the adjoint equation is hard to determine and the final result is not clear (even formally). To overcome this difficulty, we do not derive the exact variational system. Instead, we work with the approximating variational system along the approximating optimal solution, which is well-defined, and derive the approximating maximum principle. It

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is very important that the approximating adjoint system and the approximating maximum condition do have limits. Thus, our final results are obtained by taking the limits. In proving the nontriviality of the costate, we use the finite codimensionality of certain sets in the state space. The essence of our approach is to avoid deriving the exact variational system along the optimal solution with respect to any admissible controls, which is *not necessary* in stating the final result. This method was introduced by one of the authors in [**20**] for nonsmooth problems and later was used in [**13**] for impulse control problems. Some of the ideas can even be traced back to [**1, 11, 12**] for some other problems. We refer the reader to [**5, 14, 16**] for related problems.

2. Control problem

Let us start with the following hypotheses.

(H1) X is a Banach space with the dual X^* being strictly convex. K is a closed and convex cone in X, Ω is a convex and closed subset of $X \times X$, U is a metric space and $T > 0$ is a constant.

(H2) $\{e^{At}, t \geqslant 0\}$ is a C_0-semigroup on X with generator $A\colon \mathcal{D}(A) \subset X \to X$.

(H3) Maps $f\colon [0, T] \times X \times U \to X$ and $f^0\colon [0, T] \times X \times U \to \mathbb{R}$ satisfy the following:

(i) For any $(t, u) \in [0, T] \times U$, $f(t, \cdot, u)$ and $f^0(t, \cdot, u)$ are Fréchet differentiable with bounded Fréchet derivatives and $f(t, \cdot, u)$, $f_x(t, \cdot, u)$, $f^0(t, \cdot, u)$ and $f_x^0(t, \cdot, u)$ are continuous;

(ii) For any $(x, u) \in X \times U$, $f(\cdot, x, u)$, $f_x(\cdot, x, u)$, $f^0(\cdot, x, u)$ and $f_x^0(\cdot, x, u)$ are measurable and the limits

$$\lim_{x \to t \pm 0} f_x(s, x, u) = f_x(t \pm 0, x, u), \qquad t \in [0, T], \tag{2.1}$$

exist (of course for $t = 0$, T, the above is understood as suitable one-side limits).

(iii) For any $(t, x) \in [0, T] \times X$, $f(t, x, \cdot)$, $f_x(t, x, \cdot)$, $f^0(t, x, \cdot)$ and $f_x^0(t, x, \cdot)$ are (strongly) continuous.

(H4) The map $\ell\colon [0, T] \times K \to \mathbb{R}^+ \equiv [0, \infty)$ is C^1 and satisfies the following conditions:

$$\inf_{t \in [0, T],\, \xi \in K} \ell(t, \xi) \equiv \ell_0 > 0, \tag{2.2}$$

$$\lim_{\xi \in K,\, |\xi| \to \infty} \inf_{t \in [0, T]} \ell(t, \xi) = \infty, \tag{2.3}$$

$$\ell\left(t, \xi + \widehat{\xi}\right) < \ell(t, \xi) + \ell\left(t, \widehat{\xi}\right), \qquad \forall t \in [0, T], \xi, \widehat{\xi} \in K. \tag{2.4}$$

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We define our control sets as follows:

$$\mathcal{U} = \{u(\cdot) : [0, T] \to U \mid u(\cdot) \text{ measurable }\},$$

$$\mathcal{K} = \left\{ \xi(\cdot) = \sum_{j\geqslant 1} \xi_j \chi_{[\tau_j, T]}(\cdot) : [0, T] \to K \mid \tau_j \in [0, T],\ \tau_j \uparrow,\ \sum_{j\geqslant 1} \ell(\tau_j, \xi_j) < \infty \right\}.$$

An element $u(\cdot) \in \mathcal{U}$ or $\xi(\cdot) \in \mathcal{K}$ is called a continuous control or impulse control, respectively. Now, for any pair of controls $u(\cdot) \in \mathcal{U}$ and $\xi(\cdot) = \sum_{j\geqslant 1} \xi_j \chi_{[\tau_j, T]}(\cdot) \in \mathcal{K}$, we consider the following controlled system: (formally)

$$\dot{x}(t) = Ax(t) + f(t, x(t), u(t)) + \dot{\xi}(t), \qquad t \in [0, T]. \tag{2.5}$$

A right continuous function $x(\cdot) : [0, T] \to X$ is called a trajectory of the system (2.5) corresponding to control $(u(\cdot), \xi(\cdot)) \in \mathcal{U} \times \mathcal{K}$, if it is a solution of the following integral equation:

$$\begin{aligned} x(t) = e^{At}x(0^-) &+ \int_0^t e^{A(t-s)} f(s, x(s), u(x))dx \\ &+ \sum_{j\geqslant 1} e^{A(t-\tau_j)} \xi_j \chi_{[\tau_j, T]}(t), \qquad t \in [0, T]. \end{aligned} \tag{2.6}$$

Sometimes, we call $x(\cdot)$ satisfying (2.6) a mild solution of (2.5). Hereafter, we will not distinguish (2.5) and (2.6). It is clear that under our assumptions, for any given $(x(0^-), u(\cdot), \xi(\cdot)) \in X \times \mathcal{U} \times \mathcal{K}$, there exists a unique $x(\cdot)$ satisfying (2.6). We let $\mathcal{A}$ be the set of all triplets $(x(\cdot), u(\cdot), \xi(\cdot))$ with $(u(\cdot), \xi(\cdot)) \in \mathcal{U} \times \mathcal{K}$ and $x(\cdot)$ is a trajectory of (2.6) corresponding to $(u(\cdot), \xi(\cdot))$ satisfying the following conditions

$$\big(x(0^-), x(T)\big) \in \Omega, \tag{2.7}$$

and

$$f^0(\cdot, x(\cdot), u(\cdot)) \in L^1(0, T). \tag{2.8}$$

Here, we should note that from (2.6), for the given $(x(\cdot), u(\cdot), \xi(\cdot))$ satisfying (2.6), we have

$$x(0^-) = x(0) - \xi(0). \tag{2.9}$$

Next, for any $(x(\cdot), u(\cdot), \xi(\cdot)) \in \mathcal{A}$, we define the associated cost functional to be

$$J(x(\cdot), u(\cdot), \xi(\cdot)) = \int_0^T f^0(s, x(s), u(s))ds + \sum_{j\geqslant 1} \ell(\tau_j, \xi_j). \tag{2.10}$$

Then, our optimal control problem can be stated as follows:

PROBLEM CI: Find a triplet $(\bar{x}(\cdot), \bar{u}(\cdot), \bar{\xi}(\cdot)) \in \mathcal{A}$, such that

$$J(\bar{x}(\cdot), \bar{u}(\cdot), \bar{\xi}(\cdot)) = \inf_{\mathcal{A}} J(x(\cdot), u(\cdot), \xi(\cdot)). \tag{2.11}$$

The goal of this paper is to give Pontraygin's type of necessary conditions for optimal solutions of Problem CI. Thus, let us hereafter assume that $(\bar{x}(\cdot), \bar{u}(\cdot), \bar{\xi}(\cdot)) \in \mathcal{A}$ is an optimal triplet of Problem CI.

From the definition of $\mathcal{K}$ and (2.2)–(2.3), we see that there exists an $m \geqslant 0$, such that

$$\bar{\xi}(\cdot) = \sum_{j=1}^{m} \bar{\xi}_j \chi_{[\bar{\tau}_j, T]}(\cdot). \tag{2.12}$$

On the other hand from (2.4), we see that

$$\bar{\tau}_j < \bar{\tau}_{j+1}, \qquad j \leqslant m-1. \tag{2.13}$$

Next, we let $G(\cdot, \cdot)$ be the evolution operator generated by $A + f_x(t, \bar{x}(t), \bar{u}(t))$, that is,

$$G(t, s)x = e^{A(t-s)}x + \int_s^t e^{A(t-r)} f_x(\tau, \bar{x}(\tau), \bar{u}(\tau))G(\tau, s)x d\tau, \qquad \forall x \in X,\ 0 \leqslant s \leqslant t \leqslant T. \tag{2.14}$$

Then, we define

$$\mathcal{R} = \left\{ \int_0^T G(T, s)[f(s, \bar{x}(s), u(s)) - f(s, \bar{x}(s), \bar{u}(s))]ds \mid u(\cdot) \in \mathcal{U} \right\}, \tag{2.15}$$

$$Q = \{x_1 - G(T, 0)x_0 \mid (x_0, x_1) \in \Omega\}. \tag{2.16}$$

We introduce the Hamiltonian:

$$H(t, x, u, \psi^0, \psi) = \psi^0 f^0(t, x, u) + \langle \psi, f(t, x, u) \rangle, \qquad \forall (t, x, u, \psi^0, \psi) \in [0, T] \times X \times U \times \mathbb{R} \times X^*. \tag{2.17}$$

Our main result of this paper is the following:

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Theorem 2.1. (Maximum Principle) *Let (H1)–(H4) hold. Let* $(\overline{x}(\cdot), \overline{u}(\cdot), \overline{\xi}(\cdot)) \in \mathcal{A}$ *be an optimal solution of Problem CI with* $\overline{\xi}(\cdot) = \sum_{j=1}^{m} \overline{\xi}_j \chi_{[\overline{\tau}_j, T]}(\cdot)$. *Let the set*

$$\mathcal{R} + \sum_{j\geqslant 1} G(T, \overline{\tau}_j)K - Q$$

be finite codimensional in X. *Then, there exists a pair* $(\psi^0, \psi(\cdot)) \neq 0$, *such that*

$$\psi^0 \leqslant 0, \tag{2.18}$$

$$\psi(t) = e^{A^*(T-t)}\psi(T) + \int_t^T e^{A^*(s-t)} f_x(s, \overline{x}(s), \overline{u}(s))^* \psi(s)\,ds \tag{2.19}$$
$$+ \psi^0 \int_t^T e^{A^*(s-t)} f_x^0(s, \overline{x}(s), \overline{u}(s))^* ds, \qquad t \in [0, T],$$

$$H(t, \overline{x}(t), \overline{u}(t), \psi^0, \psi(t)) = \max_{u\in U} H(t, \overline{x}(t), u, \psi^0, \psi(t)), \tag{2.20}$$
$$\text{almost everywhere } t \in [0, T].$$

$$\langle \psi(\overline{\tau}_j) + \psi^0 \ell_\xi(\overline{\tau}_j, \overline{\xi}_j), \xi\rangle \leqslant 0, \qquad \forall \xi \in K,\ j \geqslant 1. \tag{2.21}$$

$$\langle \psi(0), x_0 - \overline{x}(0^-)\rangle - \langle \psi(T), x_1 - \overline{x}(T)\rangle \leqslant 0, \qquad \forall (x_0, x_1) \in \Omega. \tag{2.22}$$

Moreover, if $\overline{\xi}_j \in \mathcal{D}(A)$ *then, for the case* $\overline{u}(\overline{\tau}_j + 0)$ *exists,*

$$\int_0^1 H_x(\overline{\tau}_j + 0, \overline{x}(\overline{\tau}_j) - \sigma\overline{\xi}_j, \overline{u}(\overline{\tau}_j + 0), \psi^0, \psi(\overline{\tau}_j))\,d\sigma \tag{2.23}$$
$$+ [\langle \psi(\overline{\tau}_j), A\overline{\xi}_j\rangle - \psi^0 \ell_\tau(\overline{\tau}_j, \overline{\xi}_j)] \geqslant 0,$$

and for the case $\overline{u}(\overline{\tau}_j - 0)$ *exists,*

$$\int_0^1 H_x(\overline{\tau}_j - 0, \overline{x}(\overline{\tau}_j) - \sigma\overline{\xi}_j, \overline{u}(\overline{\tau}_j - 0), \psi^0, \psi(\overline{\tau}_j))\,d\sigma \tag{2.24}$$
$$+ [\langle \psi(\overline{\tau}_j), A\overline{\xi}_j\rangle - \psi^0 \ell_\tau(\overline{\tau}_j, \overline{\xi}_j)] \leqslant 0.$$

Remark 2.2. In the case $K = \{0\}$ and $\ell \equiv 0$, our problem is reduced to the optimal control problem for usual evolutionary distributed parameter systems and our result coincides with that of [15]. In the above, (2.21) and (2.23)–(2.24) are new. Also, it is natural that due to the appearance of the impulse control, the usual finite codimensional condition is easier to satisfy because one of $\mathcal{R}$, $\sum_{j\geqslant 1} G(T, \overline{\tau}_j)K$ and Q being finite codimensional in X, implies $\mathcal{R} + \sum_{j\geqslant 1} G(T, \overline{\tau}_j)K - Q$ is [15].

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REMARK 2.3. Let us set

$$y(t) = x(t) - \xi(t), \qquad t \in [0, T]. \tag{2.25}$$

Then, $y(\cdot) \in C([0, T]; X)$,

$$y(0) = x(0^-), \tag{2.26}$$

and

$$\begin{aligned} y(t) = e^{At}y(0) + \int_0^t e^{A(t-s)} f(s, y(s) - \xi(s), u(s))ds \\ + \sum_{j \geqslant 1} e^{A(t-\tau_j)} \xi_j \chi_{[\tau_j, T]}(t) - \xi(t), \qquad t \in [0, T]. \end{aligned} \tag{2.27}$$

The cost functional can also be written as

$$J(y(\cdot), u(\cdot), \xi(\cdot)) = \int_0^T f^0(s, y(s) + \xi(s), u(s))ds + \sum_{j \geqslant 1} \ell(\tau_j, \xi_j). \tag{2.28}$$

It is possible to discuss our problem under this setting. But we find that these two settings have the same level of complexity and the one we choose seems a little more convenient.

3. PROOF OF THE MAXIMUM PRINCIPLE

In this section, we present a proof of our main result, Theorem 2.1. The proof is long and technical. Thus, we split it into several lemmas. First of all, for any $(x_0, u(\cdot), \xi(\cdot)) \in X \times \mathcal{U} \times \mathcal{K}$, we let $x(\cdot; x_0, u(\cdot), \xi(\cdot))$ be the corresponding unique solution of (2.6) with $x(0^-) = x_0$ and let

$$\begin{aligned} x^0(x_0, u(\cdot), \xi(\cdot)) &= J(x(\cdot; x_0, u(\cdot), \xi(\cdot)), u(\cdot), \xi(\cdot)) \\ &= \int_0^T f^0(s, x(s; x_0, u(\cdot), \xi(\cdot)), u(s))ds + \sum_{j \geqslant 1} \ell(\tau_j, \xi_j), \end{aligned} \tag{3.1}$$

where $\{\tau_j, \xi_j\}$ is associated with $\xi(\cdot)$ in an obvious way. Now, for the given optimal triplet $(\overline{x}(\cdot), \overline{u}(\cdot), \overline{\xi}(\cdot))$, we let

$$\begin{cases} \overline{x}_0 = \overline{x}(0^-), \\ \overline{x}^0 = J(\overline{x}(\cdot), \overline{u}(\cdot), \overline{\xi}(\cdot)). \end{cases} \tag{3.2}$$

And, we assume $\overline{\xi}_j \in \mathcal{D}(A)$ for all $1 \leqslant j \leqslant m$ (the other case can be treated similarly). Next, we define

$$\mathcal{K}_m = \left\{ \xi(\cdot) = \sum_{j=1}^{m} \xi_j \chi_{[\tau_j, T]}(\cdot) \in \mathcal{K} \right\}, \tag{3.3}$$

and

$$\begin{aligned} &\overline{d}\Big((x_0, u(\cdot), \xi(\cdot)), \Big(\widehat{x}_0, \widehat{u}(\cdot), \widehat{\xi}(\cdot)\Big)\Big) = |x_0 - \widehat{x}_0| \\ &\quad + \operatorname{meas}\{t \in [0, T] : u(t) \neq \widehat{u}(t)\} + \sum_{j=1}^{m} \Big(|\tau_j - \widehat{\tau}_j| + \Big|\xi_j - \widehat{\xi}_j\Big|\Big), \\ &\quad \forall (x_0, u(\cdot), \xi(\cdot)), \Big(\widehat{x}_0, \widehat{u}(\cdot), \widehat{\xi}(\cdot)\Big) \in X \times \mathcal{U} \times \mathcal{K}_m. \end{aligned} \tag{3.4}$$

It is not hard to see that $\overline{d}$ is a metric under which $X \times \mathcal{U} \times \mathcal{K}_m$ is complete [9]. The following lemma shows that the trajectory $x(\cdot\,; x_0, u(\cdot), \xi(\cdot))$ and the cost $x^0(x_0, u(\cdot), \xi(\cdot))$ are continuous in $(x_0, u(\cdot), \xi(\cdot))$ under this metric.

Lemma 3.1. *Let $x_0 \in X$, $u(\cdot) \in \mathcal{U}$ and $\xi(\cdot) = \sum_{j=1}^{m} \xi_j \chi_{[\tau_j, T]}(\cdot) \in \mathcal{K}_m$ be given and let $x(\cdot)$ be the corresponding solution of (2.6) with $x(0^-) = x_0$. Then, there exist nondecreasing continuous functions C_0, $\widehat{\omega}\colon \mathbb{R}^+ \to \mathbb{R}^+$, with $\widehat{\omega}(0) = 0$, such that for any $\widehat{x}_0 \in X$, $\widehat{u}(\cdot) \in \mathcal{U}$ and $\widehat{\xi}(\cdot) = \sum_{j=1}^{m} \widehat{\xi}_j \chi_{[\widehat{\tau}_j, T]}(\cdot) \in \mathcal{K}_m$, with*

$$|\widehat{x}_0|, \Big|\widehat{\xi}_j\Big| \leqslant r, \qquad 1 \leqslant j \leqslant m, \tag{3.5}$$

if $\widehat{x}(\cdot) = x\Big(\cdot\,; \widehat{x}_0, \widehat{u}(\cdot), \widehat{\xi}(\cdot)\Big)$, then

$$\begin{aligned} &|x(t) - \widehat{x}(t)| \leqslant C_0(r) \left[\widehat{\omega}\Big(\overline{d}\Big((x_0, u(\cdot), \xi(\cdot)), \Big(\widehat{x}_0, \widehat{u}(\cdot), \widehat{\xi}(\cdot)\Big)\Big)\Big) + \sum_{j=1}^{m} \chi_{[\tau_j \wedge \widehat{\tau}_j, \tau_j \vee \widehat{\tau}_j)}(t)\right], \\ &\qquad 0 \leqslant t \leqslant T. \end{aligned} \tag{3.6}$$

In particular,

$$|x(T) - \widehat{x}(T)| \leqslant C_0(r)\widehat{\omega}\Big(\overline{d}\Big((x_0, u(\cdot), \xi(\cdot)), \Big(\widehat{x}_0, \widehat{u}(\cdot), \widehat{\xi}(\cdot)\Big)\Big)\Big). \tag{3.7}$$

Also,

$$\begin{aligned} &\Big|x^0(x_0, u(\cdot), \xi(\cdot)) - x^0\Big(\widehat{x}_0, \widehat{u}(\cdot), \widehat{\xi}(\cdot)\Big)\Big| \\ &\qquad \leqslant C_0(r)\widehat{\omega}\Big(\overline{d}\Big((x_0, u(\cdot), \xi(\cdot)), \Big(\widehat{x}_0, \widehat{u}(\cdot), \widehat{\xi}\Big)\Big)\Big). \end{aligned} \tag{3.8}$$

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PROOF: Without loss of generality, we assume

$$\tau_j \leqslant \widehat{\tau}_j, \qquad 1 \leqslant j \leqslant m. \tag{3.9}$$

Then, we have

$$\begin{aligned}\widehat{x}(t) - x(t) = e^{At}(\widehat{x}_0 - x_0) &+ \int_0^t e^{A(t-s)}[f(s, \widehat{x}(s), \widehat{u}(s)) - f(s, x(s), \widehat{u}(s))]ds \\ &+ \int_0^t e^{A(t-s)}[f(s, x(s), \widehat{u}(s)) - f(s, x(s), u(s))]ds \\ &+ \sum_{j=1}^m \left(e^{A(t-\widehat{\tau}_j)}\widehat{\xi}_j - e^{A(t-\tau_j)}\xi_j\right)\chi_{[\widehat{\tau}_j, T]}(t) \\ &- \sum_{j=1}^m e^{A(t-\tau_j)}\xi_j\chi_{[\tau_j, \widehat{\tau}_j)}(t).\end{aligned} \tag{3.10}$$

Thus, (3.6) follows from (H3) and Gronwall's inequality. Then (3.7) follows immediately and (3.8) can be proved similarly. ∎

The next result is a key step in our proof of the main result.

LEMMA 3.2. *For any $\varepsilon > 0$, there exist $\varphi^\varepsilon, \psi^\varepsilon \in X^*$ and $\psi^{0,\varepsilon} \in [0, 1]$ satisfying*

$$|\varphi^\varepsilon|_{X^*}^2 + |\psi^\varepsilon|_{X^*}^2 + |\psi^{0,\varepsilon}|^2 = 1, \tag{3.11}$$

$$\langle \varphi^\varepsilon, \widehat{x}_0 - \overline{x}(0^-)\rangle + \langle \psi^\varepsilon, \widehat{x}_1 - \overline{x}(T)\rangle \leqslant \sigma_\varepsilon, \qquad \forall (\widehat{x}_0, \widehat{x}_1) \in \Omega, \tag{3.12}$$

with $\sigma_\varepsilon \to 0$, independent of $(\widehat{x}_0, \widehat{x}_1) \in \Omega$, such that for any $x_0 \in X$, $u(\cdot) \in \mathcal{U}$, $\xi_j \in K$ and $r_j \in \mathbb{R}$ $(1 \leqslant j \leqslant m)$,

$$-\sqrt{\varepsilon}[|x_0| + T + \sum_{j=1}^m (|r_j| + |\xi_j|)] \leqslant \langle \varphi^\varepsilon, x_0\rangle + \langle \psi^\varepsilon, y_\rho^\varepsilon(T)\rangle + \psi^{0,\varepsilon} y_\rho^{0,\varepsilon} + o(1). \tag{3.13}$$

Here, $o(1) \to 0$, as $\rho \to 0$ and $y_\rho^\varepsilon(\cdot)$ and $y_\rho^{0,\varepsilon}$ satisfy the following:

$$\begin{aligned}y_\rho^\varepsilon(t) = e^{At}x_0 &+ \int_0^t e^{A(t-s)}\Delta f_\varepsilon(s)ds \\ &+ \int_0^t e^{A(t-s)}g_\rho^\varepsilon(s)y_\rho^\varepsilon(s)ds + \sum_{j=1}^m e^{A(t-\tau_j^\varepsilon-\rho r_j)}\xi_j\chi_{[\tau_j^\varepsilon+\rho r_j, T]}(t)\end{aligned}$$

$$(3.14)\qquad -\frac{1}{\rho}\sum_{j=1}^{m}(\operatorname{sgn}\tau_j)e^{A\left(t-\tau_j^\varepsilon\wedge(\tau_j^\varepsilon+\rho\tau_j)\right)}\xi_j^\varepsilon\chi_{\left[\tau_j^\varepsilon\wedge\left(\tau_j^\varepsilon+\rho\tau_j\right),\,\tau_j^\varepsilon\vee\left(\tau_j^\varepsilon+\rho\tau_j\right)\right]}(t)$$
$$-\sum_{j=1}^{m}\tau_j e^{A\left(t-\tau_j^\varepsilon\vee(\tau_j^\varepsilon+\rho\tau_j)\right)}\frac{e^{A\rho|\tau_j|}-I}{\rho|\tau_j|}\xi_j^\varepsilon\chi_{[\tau_j^\varepsilon\vee\left(\tau_j^\varepsilon+\rho\tau_j\right),\,T]}(t)+o(1),$$
$$\forall t\in[0,T],$$

$$(3.15)\qquad y_\rho^{0,\varepsilon}=\int_0^T\Delta f_\varepsilon^0(s)ds+\int_0^T g_\rho^{0,\varepsilon}(s)y_\rho^\varepsilon(s)ds$$
$$+\sum_{j=1}^{m}\left[\ell_\tau\left(\tau_j^\varepsilon,\xi_j^\varepsilon\right)\tau_j+\langle\ell_\xi\left(\tau_j^\varepsilon,\xi_j^\varepsilon\right),\xi_j\rangle\right]+o(1).$$

where

$$(3.16)\qquad \begin{cases}\Delta f_\varepsilon(s)=f(s,x^\varepsilon(s),u(s))-f(s,x^\varepsilon(s),u^\varepsilon(s))\\ \Delta f_\varepsilon^0(s)=f^0(s,x^\varepsilon(s),u(s))-f^0(s,x^\varepsilon(s),u^\varepsilon(s))\\ g_\rho^\varepsilon(s)=\displaystyle\int_0^1 f_x\left(s,x^\varepsilon(s)+\sigma\left(x_\rho^\varepsilon(s)-x^\varepsilon(s)\right),u_\rho^\varepsilon(s)\right)d\sigma,\\ g_\rho^{0,\varepsilon}(s)=\displaystyle\int_0^1 f_x^0\left(s,x^\varepsilon(s)+\sigma\left(x_\rho^\varepsilon(s)-x^\varepsilon(s)\right),u_\rho^\varepsilon(s)\right)d\sigma,\end{cases}$$

and

$$\operatorname{sgn}\tau=\begin{cases}1, & \tau>0,\\ 0, & \tau=0,\\ -1, & \tau<0.\end{cases}$$

PROOF: For any $\varepsilon>0$ and $(x_0,u(\cdot),\xi(\cdot))\in X\times\mathcal{U}\times\mathcal{K}_m$, we define

$$(3.17)\qquad F_\varepsilon(x_0,u(\cdot),\xi(\cdot))=\{d_{\Omega^0(\varepsilon)}\left(x^0(x_0,u(\cdot),\xi(\cdot))\right)^2$$
$$+d_\Omega(x_0,x(T;x_0,u(\cdot),\xi(\cdot)))^2\}^{1/2},$$

with

$$\Omega^0(\varepsilon)=\left(-\infty,\bar{x}^0-\varepsilon\right],$$

and

$$(3.18)\qquad \begin{cases}d_{\Omega^0(\varepsilon)}\left(x^0\right)=\inf\limits_{r\in\Omega^0(\varepsilon)}\left|r-x^0\right|, & \forall x^0\in\mathbb{R},\\ d_\Omega(x_0,x_1)=\inf\limits_{(y_0,y_1)\in\Omega}\{|x_0-y_0|^2+|x_1-y_1|^2\}^{1/2}, & \forall(x_0,x_1)\in X\times X.\end{cases}$$

From Lemma 3.1, we see that $F_\varepsilon(\cdot,\cdot,\cdot)$ is continuous on $(X\times\mathcal{U}\times\mathcal{K}_m, \overline{d})$ and

$$(3.19)\qquad \begin{cases} F_\varepsilon(x_0, u(\cdot), \xi(\cdot)) > 0, & \forall(x_0, u(\cdot), \xi(\cdot)) \in X\times\mathcal{U}\times\mathcal{K}_m, \\ F_\varepsilon(\overline{x}_0, \overline{u}(\cdot), \overline{\xi}(\cdot)) = \varepsilon. \end{cases}$$

Thus, by Ekeland's variational principle [9], we can find a triplet $(x_0^\varepsilon, u^\varepsilon(\cdot), \xi^\varepsilon(\cdot)) \in X\times\mathcal{U}\times\mathcal{K}_m$, such that

$$(3.20)\qquad F_\varepsilon(x_0^\varepsilon, u^\varepsilon(\cdot), \xi^\varepsilon(\cdot)) \leqslant F_\varepsilon(\overline{x}_0, \overline{u}(\cdot), \overline{\xi}(\cdot)).$$

$$(3.21)\qquad \overline{d}((x_0^\varepsilon, u^\varepsilon(\cdot), \xi^\varepsilon(\cdot)), (\overline{x}_0, \overline{u}(\cdot), \overline{\xi}(\cdot))) \leqslant \sqrt{\varepsilon},$$

$$(3.22)\qquad -\sqrt{\varepsilon}\overline{d}\Big(\big(\widehat{x}_0, \widehat{u}(\cdot), \widehat{\xi}(\cdot)\big), (x_0^\varepsilon, u^\varepsilon(\cdot), \xi^\varepsilon(\cdot))\Big) \leqslant F_\varepsilon\big(\widehat{x}_0, \widehat{u}(\cdot), \widehat{\xi}(\cdot)\big) - F_\varepsilon(x_0^\varepsilon, u^\varepsilon(\cdot), \xi^\varepsilon(\cdot)), \quad \forall\big(\widehat{x}_0, \widehat{u}(\cdot), \widehat{\xi}(\cdot)\big) \in X\times\mathcal{U}\times\mathcal{K}_m.$$

Now, for $x_0 \in X$, $u(\cdot)\in\mathcal{U}$, $\xi_j \in K$, $r_j \in \mathbb{R}$ $(1\leqslant j\leqslant m)$ and $\rho_0 > 0$ fixed, such that

$$(3.23)\qquad 0 \leqslant \tau_{j-1}^\varepsilon + \rho r_{j-1} < \tau_j^\varepsilon + \rho r_j \leqslant T, \qquad 1 < j \leqslant m, \qquad 0 < \rho \leqslant \rho_0.$$

This is possible due to (2.13) and (3.21). From [14], we know that there exists a measurable set $E_\rho^\varepsilon \subset [0, T]$, with the following properties

$$(3.24)\qquad \operatorname{meas} E_\rho^\varepsilon = \rho T,$$

$$(3.25)\qquad \rho\int_0^t e^{A(t-s)}[f(s, x^\varepsilon(s), u(s)) - f(s, x^\varepsilon(s), u^\varepsilon(s))]ds = \int_{[0,t]\cap E_\rho^\varepsilon} e^{A(t-s)}[f(s, x^\varepsilon(s), u(s)) - f(s, x^\varepsilon(s), u^\varepsilon(s))]ds + o(\rho),$$

uniformly in $t\in[0, T]$ and

$$(3.26)\qquad \rho\int_0^T [f^0(s, x^\varepsilon(s), u(s)) - f^0(s, x^\varepsilon(s), u^\varepsilon(s))]ds = \int_{E_\rho^\varepsilon} [f^0(s, x^\varepsilon(s), u(s)) - f^0(s, x^\varepsilon(s), u^\varepsilon(s))]ds + o(\rho),$$

where the $o(\rho)$ in (3.25) and (3.26) are uniform in ε. Then, we define

$$(3.27)\qquad \begin{cases} x_{0,\rho}^\varepsilon = x_0^\varepsilon + \rho x_0, \\ u_\rho^\varepsilon(\cdot) = u(\cdot)\chi_{E_\rho^\varepsilon}(\cdot) + u^\varepsilon(\cdot)\chi_{[0,T]\setminus E_\rho^\varepsilon}(\cdot), \\ \xi_\rho^\varepsilon(\cdot) = \sum_{j=1}^m (\xi_j^\varepsilon + \rho\xi_j)\chi_{[\tau_j^\varepsilon+\rho r_j, T]}(\cdot), \end{cases}$$

and denote

$$(3.28)\qquad x_\rho^\varepsilon(\cdot) = x\big(\cdot; x_{0,\rho}^\varepsilon, u_\rho^\varepsilon(\cdot), \xi_\rho^\varepsilon(\cdot)\big), \qquad x_\rho^{0,\varepsilon} = x^0\big(x_{0,\rho}^\varepsilon, u_\rho^\varepsilon(\cdot), \xi_\rho^\varepsilon(\cdot)\big).$$

By (3.22), we have

(3.29)

$$-\sqrt{\varepsilon}[|x_0| + T + \sum_{j=1}^m (|r_j| + |\xi_j|)] \leqslant \frac{1}{\rho}[F_\varepsilon\big(x_{0,\rho}^\varepsilon, u_\rho^\varepsilon(\cdot), \xi_\rho^\varepsilon(\cdot)\big) - F_\varepsilon(x_0^\varepsilon, u^\varepsilon(\cdot), \xi^\varepsilon(\cdot))]$$

$$= \frac{1}{\rho}\frac{d_{\Omega^0(\varepsilon)}\big(x_\rho^{0,\varepsilon}\big)^2 - d_{\Omega^0(\varepsilon)}\big(x^{0,\varepsilon}\big)^2 + d_\Omega\big(x_{0,\rho}^\varepsilon, x_\rho^\varepsilon(T)\big)^2 - d_\Omega(x_0^\varepsilon, x^\varepsilon(T))^2}{F_\varepsilon\big(x_{0,\rho}^\varepsilon, u_\rho^\varepsilon(\cdot), \xi_\rho^\varepsilon(\cdot)\big) + F_\varepsilon(x_0^\varepsilon, u^\varepsilon(\cdot), \xi^\varepsilon(\cdot))}.$$

By the continuity of $F_\varepsilon(\cdot, \cdot, \cdot)$, we know that

$$(3.30)\qquad F_\varepsilon\big(x_{0,\rho}^\varepsilon, u_\rho^\varepsilon(\cdot), \xi_\rho^\varepsilon(\cdot)\big) = F_\varepsilon(x_0^\varepsilon, u^\varepsilon(\cdot), \xi^\varepsilon(\cdot)) + o(1),$$
$$\text{as } \rho \to 0 \text{ uniformly in } \varepsilon \in (0, 1].$$

On the other hand, by setting

$$y_\rho^\varepsilon(t) = \frac{x_\rho^\varepsilon(t) - x^\varepsilon(t)}{\rho}$$

and

$$y_\rho^{0,\varepsilon} = \frac{x_\rho^{0,\varepsilon} - x^{0,\varepsilon}}{\rho}$$

we see that (3.14) and (3.15) are satisfied. Furthermore, by the strict convexity of X^*, we can define [15]

$$(3.31)\qquad \begin{cases} (\varphi^\varepsilon, \psi^\varepsilon) = \dfrac{d_\Omega(x_0^\varepsilon, x^\varepsilon(T))\nabla d_\Omega(x_0^\varepsilon, x^\varepsilon(T))}{F_\varepsilon(x_0^\varepsilon, u^\varepsilon(\cdot), \xi^\varepsilon(\cdot))} \in X^* \times X^*, \\ \psi^{0,\varepsilon} = \dfrac{d_{\Omega^0(\varepsilon)}(x_0^\varepsilon)\nabla d_{\Omega^0(\varepsilon)}(x_0^\varepsilon)}{F_\varepsilon(x_0^\varepsilon, u^\varepsilon(\cdot), \xi^\varepsilon(\cdot))} \in [0, 1]. \end{cases}$$

Then (3.13) follows from (3.29). Also, we see that (3.11) and (3.12) hold. □

Next, we define $G_\rho^\varepsilon(\cdot, \cdot)$ to be the evolution operator generated by $A + g_\rho^\varepsilon(t)$, that is, $G_\rho^\varepsilon(\cdot, \cdot)$ satisfies the following:

$$(3.32)\quad G_\rho^\varepsilon(t, s)x = e^{A(t-s)}x + \int_s^t e^{A(t-\tau)}g_\rho^\varepsilon(\tau)G_\rho^\varepsilon(\tau, s)x\,d\tau, \qquad x \in X, 0 \leqslant s \leqslant t \leqslant T.$$

It is important that (from [8]), we have

$$(3.33)\quad G_\rho^\varepsilon(t, s)x = e^{A(t-s)}x + \int_s^t G_\rho^\varepsilon(t, \tau)g_\rho^\varepsilon(\tau)e^{A(\tau-s)}x\,d\tau, \qquad x \in X, 0 \leqslant s \leqslant t \leqslant T.$$

The following result gives a representation for the solution $y_\rho^\varepsilon(\cdot)$ of (3.14).

Lemma 3.3. *The solution $y_\rho^\varepsilon(\cdot)$ of (3.14) can be written as follows:*

(3.34)

$$\begin{aligned} y_\rho^\varepsilon(t) &= G_\rho^\varepsilon(t,0)x_0 + \int_0^t G_\rho^\varepsilon(t,s)\Delta f_\varepsilon(s)ds + \sum_{j=1}^m \Big\{ G_\rho^\varepsilon(t,\tau_j^\varepsilon+\rho r_j)\xi_j\chi_{[\tau_j^\varepsilon+\rho r_j,T]}(t) \\ &- r_j G_\rho^\varepsilon(t,\rho_j^\varepsilon \vee (\tau_j^\varepsilon+\rho r_j))\frac{e^{A\rho|r_j|}-I}{\rho|r_j|}\xi_j^\varepsilon \chi_{[\tau_j^\varepsilon\vee(\tau_j^\varepsilon+\rho r_j),T]}(t), \\ &- \frac{1}{\rho}(\operatorname{sgn} r_j)e^{A(t-\tau_j^\varepsilon\wedge(\tau_j^\varepsilon+\rho r_j))}\xi_j^\varepsilon\chi_{[\tau_j^\varepsilon\wedge(\tau_j^\varepsilon+\rho r_j),\tau_j^\varepsilon\vee(\tau_j^\varepsilon+\rho r_j))}(t) \\ &- \frac{1}{\rho}(\operatorname{sgn} r_j)\int_0^t G_\rho^\varepsilon(t,s)g_\rho^\varepsilon(s)e^{A(s-\tau_j^\varepsilon\wedge(\tau_j^\varepsilon+\rho r_j))}\xi_j^\varepsilon\chi_{[\tau_j^\varepsilon\wedge(\tau_j^\varepsilon+\rho r_j),\tau_j^\varepsilon\vee(\tau_j^\varepsilon+\rho r_j))}(s)ds\Big\} \\ &+ o(1), \qquad \forall t\in[0,T], \end{aligned}$$

where $o(1)\to 0$ as $\rho\to 0$, uniformly in $t\in[0,T]$.

Proof: First of all, for the equation

$$z(t) = h(t) + \int_0^t e^{A(t-s)}g_\rho^\varepsilon(s)z(s)ds, \qquad t\in[0,T], \tag{3.35}$$

by some direct computation, we can show that the solution is given by

$$z(t) = h(t) + \int_0^t G_\rho^\varepsilon(t,s)g_\rho^\varepsilon(s)h(s)ds, \qquad t\in[0,T]. \tag{3.36}$$

On the other hand, by (3.33), we have

$$\begin{aligned} &\sum_{j=1}^m \int_0^t G_\rho^\varepsilon(t,s)g_\rho^\varepsilon(s)e^{A(s-\tau_j^\varepsilon-\rho r_j)}\xi_j\chi_{[\tau_j^\varepsilon+\rho r_j,T]}(s)ds \\ &= \sum_{j=1}^m \int_{\tau_j^\varepsilon+\rho r_j}^t G_\rho^\varepsilon(t,s)g_\rho^\varepsilon(s)e^{A(s-\tau_j^\varepsilon-\rho r_j)}\xi_j ds\chi_{[\tau_j^\varepsilon+\rho r_j,T]}(t) \\ &= \sum_{j=1}^m \left[G_\rho^\varepsilon(t,\tau_j^\varepsilon+\rho r_j) - e^{A(t-\tau_j^\varepsilon-\rho r_j)}\right]\xi_j\chi_{[\tau_j^\varepsilon+\rho r_j,T]}(t). \end{aligned} \tag{3.37}$$

Similarly, we have

(3.38)

$$\begin{aligned} &\sum_{j=1}^m \int_0^t G_\rho^\varepsilon(t,s)g_\rho^\varepsilon(s)e^{A(s-\tau_j^\varepsilon\vee(\tau_j^\varepsilon+\rho r_j))}\frac{e^{A\rho|r_j|}-I}{\rho|r_j|}\xi_j^\varepsilon\chi_{[\tau_j^\varepsilon\vee(\tau_j^\varepsilon+\rho r_j),T]}(s)ds \\ &= \sum_{j=1}^m \left[G_\rho^\varepsilon(t,\tau_j^\varepsilon\vee(\tau_j^\varepsilon+\rho r_j)) - e^{A(t-\tau_j^\varepsilon\vee(\tau_j^\varepsilon+\rho r_j))}\right]\frac{e^{A\rho|r_j|}-I}{\rho|r_j|}\xi_j^\varepsilon\chi_{[\tau_j^\varepsilon\vee(\tau_j^\varepsilon+\rho r_j),T]}(t), \end{aligned}$$

$$(3.39)\qquad \int_0^t G_\rho^\varepsilon(t,s)g_\rho^\varepsilon(s)e^{As}x_0\,ds = G_\rho^\varepsilon(t,0)x_0 - e^{At}x_0$$

and

(3.40)
$$\int_0^t G_\rho^\varepsilon(t,s)g_\rho^\varepsilon(s)\int_0^s e^{A(s-\tau)}\Delta f(\tau)d\tau\,ds = \int_0^t\int_0^t G_\rho^\varepsilon(t,s)g_\rho^\varepsilon(s)e^{A(s-\tau)}ds\Delta f_\varepsilon(\tau)d\tau$$
$$= \int_0^t \left[G_\rho^\varepsilon(t,s) - e^{A(t-s)}\right]\Delta f_\varepsilon(s)ds.$$

Thus, (3.34) follows. □

Our next goal is to take the limits. The main point of the following result is the existence of a nontrivial ω^*-limit point of $\{(\varphi^\varepsilon,\psi^\varepsilon,\psi^{0,\varepsilon})\}$.

Lemma 3.4. *Along some subsequence of* $\{(\varphi^\varepsilon,\psi^\varepsilon,\psi^{0,\varepsilon})\}$, *we have*

$$(3.41)\qquad (\varphi^\varepsilon,\psi^\varepsilon,\psi^{0,\varepsilon}) \xrightarrow{*} \left(\overline{\varphi},\overline{\psi},\overline{\psi}^0\right) \neq 0.$$

Proof: let us note that (3.13) holds for all $(x_0, u(\cdot), \xi(\cdot)) \in X \times \mathcal{U} \times \mathcal{K}_m$. In particular, it holds for the case that $r_j = 0$, $1 \leqslant j \leqslant m$. In this case, we have (see (3.34))

(3.42)
$$y_\rho^\varepsilon(t) = G_\rho^\varepsilon(t,0)x_0 + \int_0^t G_\rho^\varepsilon(t,s)\Delta f_\varepsilon(s)ds + \sum_{j=1}^m G_\rho^\varepsilon(t,\tau_j^\varepsilon+\rho r_j)\xi_j\chi_{[\tau_j^\varepsilon+\rho r_j, T]}(t)$$
$$\forall \in [0,T].$$

In what follows, we take

$$(3.43)\qquad \rho = \varepsilon^{1/4}.$$

Then, from (3.14), we see that for any given $(x_0, u(\cdot), \xi(\cdot)) \in X \times \mathcal{U} \times \mathcal{K}_m$,

$$(3.44)\qquad \lim_{\varepsilon\to 0}\left|x_\rho^\varepsilon(t) - x^\varepsilon(t)\right| = 0, \qquad \forall t \neq \overline{\tau}_j, 1 \leqslant j \leqslant m,$$

and by (3.6) and (3.21),

$$(3.45)\qquad \lim_{\varepsilon\to 0}|x^\varepsilon(t) - \overline{x}(t)| = 0, \qquad \forall t \neq \overline{\tau}_j, 1 \leqslant j \leqslant m.$$

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Thus, it is not hard to show that

$$\lim_{\varepsilon\to 0} G^\varepsilon_\rho(t, s)x = G(t, s)x, \quad \forall x \in X, \text{ uniformly in } s, t \text{ with } 0 \leqslant s \leqslant t \leqslant T, \tag{3.46}$$

where $G(\cdot, \cdot)$ is the evolution operator generated by $A + f_x(t, \bar{x}(t), \bar{u}(t))$, that is, (2.14) holds. Hence, in the case $r_j = 0$, $1 \leqslant j \leqslant m$, by (3.42), we have that as $\varepsilon \to 0$,

$$\begin{aligned} y^\varepsilon_\rho(t) \to \bar{y}(t) = G(t, 0)x_0 &+ \int_0^t G(t, s)\left[f(s, \bar{x}(s), u(s)) - f(s, \bar{x}(s), \bar{u}(s))\right] ds \\ &+ \sum_{j=1}^m G(t, \bar{\tau}_j)\xi_j \chi_{[\bar{\tau}_j, T]}(t), \qquad t \in [0, T]. \end{aligned} \tag{3.47}$$

As a consequence, we also have (as $\varepsilon \to 0$)

$$\begin{aligned} y^{0,\varepsilon}_\rho \to \bar{y}^0 = &\int_0^T [f^0(s, \bar{x}(s), u(s)) - f^0(s, \bar{x}(s), \bar{u}(s))] ds \\ &+ \int_0^T \langle f^0_x(s, \bar{x}(s), \bar{u}(s)), \bar{y}(s)\rangle ds \\ &+ \sum_{j=1}^m \left[\langle \ell_\xi(\bar{\tau}_j, \bar{\xi}_j), \xi_j\rangle + r_j \ell(\bar{\tau}_j, \bar{\xi}_j)\right]. \end{aligned} \tag{3.48}$$

Next, by definition of $(\varphi^\varepsilon, \psi^\varepsilon, \psi^{0,\varepsilon})$ and the convexity of the set Ω, we have

$$\begin{cases} \langle \varphi^\varepsilon, \hat{x}_0 - x^\varepsilon_0\rangle + \langle \psi^\varepsilon, \hat{x}_1 - x^\varepsilon(T)\rangle \leqslant 0, \qquad \forall (\hat{x}_0, \hat{x}_1) \in \Omega, \\ \psi^{0,\varepsilon} \geqslant 0, \\ |\varphi^\varepsilon|^2_{X^*} + |\psi^\varepsilon|^2_{X^*} + |\psi^{0,\varepsilon}|^2 = 1. \end{cases} \tag{3.49}$$

Thus, combining (3.13) with the above, we obtain

$$\begin{aligned} \langle \varphi^\varepsilon, x_0 - (\hat{x}_0 - \bar{x}_0)\rangle + \langle \psi^\varepsilon, \bar{y}(T) - (\hat{x}_1 - \bar{x}(T))\rangle + \psi^{0,\varepsilon}\bar{y}^0 \geqslant -\delta_\varepsilon, \\ \forall (\hat{x}_0, \hat{x}_1) \in \Omega, \end{aligned} \tag{3.50}$$

with $\delta_\varepsilon \to 0$ as $\varepsilon \to 0$. We should note that the above holds uniformly for all $x_0 \in X$, $u(\cdot) \in \mathcal{U}$ and $\xi(\cdot) = \sum_{j=1}^m \xi_j \chi_{[\bar{\tau}_j, T]}(\cdot) \in \mathcal{K}_m$, with $|x_0|$, $|\xi_j| (1 \leqslant j \leqslant m)$ being bounded. Now, we let

$$\widehat{\mathcal{R}} = \left\{ \begin{pmatrix} \eta \\ \zeta \end{pmatrix} \in X \times X \mid \zeta = G(T, 0)\eta + z, \eta \in X, z \in \mathcal{R} + \sum_{j=1}^m G(T, \bar{\tau}_j)K \right\} \tag{3.51}$$

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and let

$$X_1 = \text{span}\left(\mathcal{R} + \sum_{j=1}^{m} G(T, \overline{\tau}_j)K - Q + \overline{x}(T) - G(T, 0)\overline{x}_0\right). \tag{3.52}$$

Then, X_1 is finite codimensional in X. It is not hard to see that

$$\text{span}\left(\widehat{\mathcal{R}} - \Omega + \begin{pmatrix} \overline{x}_0 \\ \overline{x}(T) \end{pmatrix}\right) = \left\{ \begin{pmatrix} x \\ G(T, 0)x + x_1 \end{pmatrix} \,\middle|\, x \in X,\, x_1 \in X_1 \right\}. \tag{3.53}$$

By some direct argument, we can show that [15] $\widehat{\mathcal{R}} - \Omega$ is finite codimensional in $X \times X$. Thus, similar to [10, 15], we can find a subsequence (still denote it by itself) $\{(\varphi^\varepsilon, \psi^\varepsilon, \psi^{0,\varepsilon})\}$ satisfying (3.41). ∎

It is clear that

$$\langle \overline{\varphi}, x_0 \rangle + \langle \overline{\psi}, \overline{y}(T) \rangle + \overline{\psi}^0 \overline{y}^0 \geqslant 0, \tag{3.54}$$

and

$$\overline{\psi}^0 \in [0, 1]. \tag{3.55}$$

Hereafter, we are always along the subsequence given in (3.41). It is important that this subsequence is independent of $(x_0, u(\cdot), \xi(\cdot)) \in X \times \mathcal{U} \times \mathcal{K}_m$. Next, we would like to introduce the approximating costate.

Lemma 3.5. *Let $\psi_\rho^\varepsilon(\cdot)$ be the solution of the following:*

$$\begin{aligned} \psi_\rho^\varepsilon(s) = e^{A^*(T-s)}\psi^\varepsilon + \int_s^T e^{A^*(\tau-s)} g_\rho^\varepsilon(\tau)^* \psi_\rho^\varepsilon(\tau)\,d\tau \\ + \int_s^T \psi^{0,\varepsilon} e^{A^*(\tau-s)} g_\rho^{0,\varepsilon}(\tau)^*\,d\tau, \qquad s \in [0, T]. \end{aligned} \tag{3.56}$$

Then, (3.13) implies

$$\begin{aligned} -\sqrt{\varepsilon}\left[|x_0| + T + \sum_{j=1}^{m}(|r_j| + |\xi_j|)\right] \leqslant \langle \varphi^\varepsilon + \psi_\rho^\varepsilon(0), x_0\rangle \\ + \int_0^T [\langle \psi_\rho^\varepsilon(s), \Delta f_\varepsilon(s)\rangle + \psi^{0,\varepsilon}\Delta f_\varepsilon^0(s)]\,ds \\ + \sum_{j=1}^{m} \Big\{ \langle \psi_\rho^\varepsilon(\tau_j^\varepsilon + \rho r_j) + \psi^{0,\varepsilon}\ell_\xi(\tau_j^\varepsilon, \xi_j^\varepsilon), \xi_j \rangle \end{aligned}$$

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$$
\begin{aligned}
&- r_j[\langle\psi_\rho^\varepsilon(\tau_j^\varepsilon \vee (\tau_j^\varepsilon + \rho r_j)), \frac{e^{A\rho|r_j|} - I}{\rho|r_j|}\xi_j^\varepsilon\rangle - \psi^{0,\varepsilon}\ell_\tau(\tau_j^\varepsilon, \xi_j^\varepsilon)] \\
&- \frac{1}{\rho}(\operatorname{sgn} r_j)\int_{\tau_j^\varepsilon\wedge(\tau_j^\varepsilon+\rho r_j)}^{\tau_j^\varepsilon\vee(\tau_j^\varepsilon+\rho r_j)} \langle\psi_\rho^\varepsilon(s), g_\rho^\varepsilon(s)e^{A(s-\tau_j^\varepsilon\wedge(\tau_j^\varepsilon+\rho r_j))}\xi_j^\varepsilon\rangle ds \\
&-\frac{1}{\rho}(\operatorname{sgn} r_j)\psi^{0,\varepsilon}\int_{\tau_j^\varepsilon\wedge(\tau_j^\varepsilon+\rho r_j)}^{\tau_j^\varepsilon\vee(\tau_j^\varepsilon+\rho r_j)} \langle g_\rho^{0,\varepsilon}(s)^*, e^{A(s-\tau_j^\varepsilon\wedge(\tau_j^\varepsilon+\rho r_j))}\xi_j^\varepsilon\rangle ds\Big\} + o(1).
\end{aligned}
\tag{3.57}
$$

PROOF: By (3.13), we have

$$
\begin{aligned}
&- \sqrt{\varepsilon}\left[|x_0| + T + \sum_{j=1}^m(|r_j| + |\xi_j|)\right] \leqslant \langle\varphi^\varepsilon, x_0\rangle + \langle\psi^\varepsilon, y_\rho^\varepsilon(T)\rangle + \psi^{0,\varepsilon}y_\rho^{0,\varepsilon} + o(1) \\
&= \langle\varphi^\varepsilon, x_0\rangle + \langle G_\rho^\varepsilon(T,0)^*\psi^\varepsilon + \int_0^T \psi^{0,\varepsilon}G_\rho^\varepsilon(s,0)^*g_\rho^{0,\varepsilon}(s)^*ds, x_0\rangle \\
&+ \int_0^T\left[\langle G_\rho^\varepsilon(T,s)^*\psi^\varepsilon + \int_s^T \psi^{0,\varepsilon}G_\rho^\varepsilon(\tau,s)^*g_\rho^{0,\varepsilon}(\tau)^*d\tau, \Delta f_\varepsilon(s)\rangle + \psi^{0,\varepsilon}\Delta f_\varepsilon^0(s)\right]ds \\
&+ \sum_{j=1}^m\Big\{\langle G_\rho^\varepsilon(T, \tau_j^\varepsilon + \rho r_j)^*\psi^\varepsilon + \int_{\tau_j^\varepsilon+\rho r_j}^T \psi^{0,\varepsilon}G_\rho^\varepsilon(s, \tau_j^\varepsilon + \rho r_j)^*g_\rho^{0,\varepsilon}(s)^*ds, \xi_j\rangle \\
&- r_j\Big\langle G_\rho^\varepsilon(T, \tau_j^\varepsilon \vee (\tau_j^\varepsilon + \rho r_j))^*\psi^\varepsilon \\
&+ \int_{\tau_j^\varepsilon\vee(\tau_j^\varepsilon+\rho r_j)}^T \psi^{0,\varepsilon}G_\rho^\varepsilon(s, \tau_j^\varepsilon \vee (\tau_j^\varepsilon+\rho r_j))^*g_\rho^{0,\varepsilon}(s)^*ds, \frac{e^{A\rho|r_j|}-I}{\rho|r_j|}\xi_j^\varepsilon\Big\rangle \\
&- \frac{1}{\rho}(\operatorname{sgn} r_j)\int_0^T\Big\langle G_\rho^\varepsilon(T,s)^*\psi^\varepsilon \\
&+ \int_s^T \psi^{0,\varepsilon}G_\rho^\varepsilon(\tau,s)^*g_\rho^{0,\varepsilon}(\tau)^*d\tau, g_\rho^\varepsilon(s)e^{A(s-\tau_j^\varepsilon\wedge(\tau_j^\varepsilon+\rho r_j))}\xi_j^\varepsilon\Big\rangle \\
&\qquad \chi_{[\tau_j^\varepsilon\wedge(\tau_j^\varepsilon+\rho r_j), \tau_j^\varepsilon\vee(\tau_j^\varepsilon+\rho r_j))}(s)ds \\
&- \frac{1}{\rho}(\operatorname{sgn} r_j)\int_0^T \psi_\rho^{0,\varepsilon}\langle e^{A^*(s-\tau_j^\varepsilon\wedge(\tau_j^\varepsilon+\rho r_j))}g_\rho^{0,\varepsilon}(s)^*, \xi_j^\varepsilon\rangle \\
&\qquad \chi_{[\tau_j^\varepsilon\wedge(\tau_j^\varepsilon+\rho r_j), \tau_j^\varepsilon\vee(\tau_j^\varepsilon+\rho r_j))}(s)ds \\
&+ \psi^{0,\varepsilon}[\ell_\tau(\tau_j^\varepsilon, \xi_j^\varepsilon)r_j + \langle\ell_\xi(\tau_j^\varepsilon, \xi_j^\varepsilon), \xi_j\rangle]\qquad\Big\} + o(1).
\end{aligned}
$$

Set

$$(3.58)\qquad \psi_\rho^\varepsilon(s)=G_\rho^\varepsilon(T,s)^*\psi^\varepsilon+\int_s^T\psi^{0,\varepsilon}G_\rho^\varepsilon(\tau,s)^*g_\rho^{0,\varepsilon}(\tau)^*d\tau,\qquad s\in[0,T].$$

We see that $\psi_\rho^\varepsilon(\cdot)$ is the solution of (3.56). Then (3.57) follows easily. □

Now, we are ready to complete the proof of our main result.

COMPLETION OF THE PROOF OF THEOREM 2.1: We are going to take the limits in (3.57) term by term. To this end, we first note that by (3.56) and Gronwall's inequality, we see that there exists a constant C, such that

$$(3.59)\qquad |\psi_\rho^\varepsilon(s)|_{X^*}\leqslant C,\qquad s\in[0,T],\ \varepsilon,\ \rho>0.$$

Hence, by (3.46) and (3.58) (or (3.56)), we may assume that (note $\rho=\varepsilon^{1/4}$) as $\varepsilon\to0$,

$$(3.60)\qquad \langle\psi_\rho^\varepsilon(s),x\rangle\to\langle\overline{\psi}(s),x\rangle,\qquad \forall x\in X,\ \text{uniformly in } s\in[0,T],$$

and

$$(3.61)\qquad \psi_\rho^\varepsilon(\cdot)\overset{*}{\rightharpoonup}\overline{\psi}(\cdot),\qquad \text{in } L^\infty(0,T;X^*),$$

where

$$(3.62)\qquad \overline{\psi}(s)=G(T,s)^*\overline{\psi}+\int_s^T\overline{\psi}^0G(\tau,s)^*f_x^0(\tau,\overline{x}(\tau),\overline{u}(\tau))^*d\tau,\qquad s\in[0,T].$$

Then, we see that

$$(3.63)\qquad \lim_{\varepsilon\to0}\langle\varphi^\varepsilon+\psi_\rho^\varepsilon(0),x_0\rangle\to\langle\overline{\varphi}+\overline{\psi}(0),x_0\rangle,$$

$$(3.64)\qquad \begin{aligned}&\lim_{\varepsilon\to0}\int_0^T\left[\langle\psi_\rho^\varepsilon(s),\Delta f_\varepsilon(s)\rangle+\psi^{0,\varepsilon}\Delta f_\varepsilon{}^0(s)\right]ds\\&=\int_0^T\Big\{\langle\overline{\psi}(s),f(s,\overline{x}(s),u(s))-f(s,\overline{x}(s),\overline{u}(s))\rangle\\&\quad+\overline{\psi}^0[f^0(s,\overline{x}(s),u(s))-f(s,\overline{x}(s),\overline{u}(s))]\Big\}ds,\end{aligned}$$

$$(3.65)\qquad \lim_{\varepsilon\to0}\langle\psi_\rho^\varepsilon(\tau_j^\varepsilon+\rho r_j)+\psi^{0,\varepsilon}\ell_\xi(\tau_j^\varepsilon,\xi_j^\varepsilon),\xi_j\rangle=\langle\overline{\psi}(\overline{\tau}_j)+\overline{\psi}^0\ell_\xi(\overline{\tau}_j,\overline{\xi}_j),\xi_j\rangle.$$

By the definition of ξ_j^ε and (3.43), we have

$$(3.66)\qquad \left|\frac{e^{A\rho|r_j|}-I}{\rho|r_j|}\xi_j^\varepsilon-A\overline{\xi}_j\right|\leqslant\left|\frac{e^{A\rho|r_j|}-I}{\rho|r_j|}\overline{\xi}_j-A\overline{\xi}_j\right|+\left|\frac{e^{A\rho|r_j|}-I}{|r_j|}\right|\varepsilon^{1/4}\to0.$$

Hence,

$$\lim_{\varepsilon\to 0}\left[\left\langle \psi^\varepsilon_\rho(\tau^\varepsilon_j \vee (\tau^\varepsilon_j+\rho r_j)),\, \frac{e^{A\rho|r_j|}-I}{\rho|r_j|}\xi^\varepsilon_j\right\rangle - \psi^{0,\varepsilon}\ell_\tau(\tau^\varepsilon_j,\xi^\varepsilon_j)\right] = [\langle\overline{\psi}(\overline{\tau}_j),\, A\overline{\xi}_j\rangle - \overline{\psi}^0\ell_\tau(\overline{\tau}_j,\overline{\xi}_j)]. \tag{3.67}$$

Finally, we would like to consider the last two terms in the right hand side of (3.57). To this end, let us first assume that $r_j > 0$. Then, we claim that (note (3.43))

$$\lim_{\varepsilon\to 0}\frac{1}{\rho}\int_{\tau^\varepsilon_j}^{\tau^\varepsilon_j+\rho r_j}\langle\psi^\varepsilon_\rho(s),\, g^\varepsilon_\rho(s)e^{A(s-\tau^\varepsilon_j)}\xi^\varepsilon_j\rangle ds = r_j\langle\overline{\psi}(\overline{\tau}_j),\, \int_0^1 f_x(\overline{\tau}_j+0,\overline{x}(\tau_j)-\sigma\overline{\xi}_j,\overline{u}(\overline{\tau}_j+0))\,d\sigma\overline{\xi}_j\rangle. \tag{3.68}$$

In fact, for any $s\in[\tau^\varepsilon_j,\tau^\varepsilon_j+\rho r_j)$,

$$\begin{aligned}|x^\varepsilon(s)-x^\varepsilon(\tau^\varepsilon_j)| &\leqslant \left|\left(e^{As}-e^{A\tau^\varepsilon_j}\right)\overline{x}_0\right| + C|x^\varepsilon_0-\overline{x}_0|\\
&+\int_0^{\tau^\varepsilon_j}\left|\left(e^{A(s-\tau)}-e^{A(\tau^\varepsilon_j-\tau)}\right)f(\tau,\overline{x}(\tau),\overline{u}(\tau))\right|d\tau\\
&+C\int_0^{\tau^\varepsilon_j}|f(\tau,x^\varepsilon(\tau),u^\varepsilon(\tau))-f(\tau,\overline{x}(\tau),\overline{u}(\tau))|\,d\tau\\
&+\int_{\tau^\varepsilon_j}^{s}\left|e^{A(s-\tau)}f(\tau,x^\varepsilon(\tau),u^\varepsilon(\tau))\right|d\tau\\
&+\sum_{i=1}^{j}\left|e^{A(s-\tau^\varepsilon_i)}-e^{A(\tau^\varepsilon_j-\tau^\varepsilon_i)}\overline{\xi}_i\right|\chi_{[\tau^\varepsilon_i,T]}(s)+C\sum_{i=1}^m|\xi^\varepsilon_i-\overline{\xi}_i|.\end{aligned} \tag{3.69}$$

On the other hand,

$$\begin{aligned}|x^\varepsilon(\tau^\varepsilon_j)-\overline{x}(\overline{\tau}_j)| &\leqslant \left|\left(e^{A\tau^\varepsilon_j}-e^{A\overline{\tau}_j}\right)\overline{x}_0\right| + C|x^\varepsilon_0-\overline{x}_0|\\
&+\int_0^{\tau^\varepsilon_j\wedge\overline{\tau}_j}\left|\left(e^{A(\tau^\varepsilon_j-\tau)}-e^{A(\overline{\tau}_j-\tau)}\right)f(\tau,\overline{x}(\tau),\overline{u}(\tau))\right|d\tau\\
&+C\int_0^{\tau^\varepsilon_j\wedge\overline{\tau}_j}|f(\tau,x^\varepsilon(\tau),u^\varepsilon(\tau))-f(\tau,\overline{x}(\tau),\overline{u}(\tau))|\,d\tau\\
&+C\int_{\tau^\varepsilon_j\wedge\overline{\tau}_j}^{\tau^\varepsilon_j\vee\overline{\tau}_j}(|f(\tau,x^\varepsilon(\tau),u^\varepsilon(\tau))|+|f(\tau,\overline{x}(\tau),\overline{u}(\tau))|)d\tau\\
&+\sum_{i=1}^{j}\left|\left(e^{A(\tau^\varepsilon_j-\tau^\varepsilon_i)}-e^{A(\overline{\tau}_j-\overline{\tau}_i)}\right)\overline{\xi}_i\right|+C\sum_{i=1}^m|\xi^\varepsilon_i-\overline{\xi}_i|.\end{aligned} \tag{3.70}$$

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Hence, we see that

$$\lim_{\varepsilon\to 0}\sup_{s\in\left[\tau_j^\varepsilon,\tau_j^\varepsilon+\rho r_j\right)}|x^\varepsilon(s)-\overline{x}(\overline{\tau}_j)|=0, \tag{3.71}$$

and

$$\lim_{\varepsilon\to 0}\sup_{s\in\left[\tau_j^\varepsilon,\tau_j^\varepsilon+\rho r_j\right)}\left|x_\rho^\varepsilon(s)-[\overline{x}(\overline{\tau}_j)-\overline{\xi}_j]\right|=0. \tag{3.72}$$

Then, it follows that

$$\begin{aligned}
&\Big|\frac{1}{\rho}\int_{\tau_j^\varepsilon}^{\tau_j^\varepsilon+\rho r_j}\langle\psi_\rho^\varepsilon(s),\,g_\rho^\varepsilon(s)e^{A\left(s-\tau_j^\varepsilon\right)}\xi_j^\varepsilon\rangle ds\\
&\qquad -r_j\langle\overline{\psi}(\overline{\tau}_j),\,\int_0^1 f_x\big(\overline{\tau}_j+0,\,\overline{x}(\tau_j)-\sigma\overline{\xi}_j,\,\overline{u}(\overline{\tau}_j+0)\big)d\sigma,\,\overline{\xi}_j\rangle\Big|\\
&\leqslant C\sup_{s\in[0,\,\rho r_j]}\left|e^{As}\xi_j^\varepsilon-\overline{\xi}_j\right|+C\frac{1}{\rho}\,\mathrm{meas}\{u^\varepsilon\neq\overline{u}\}\\
&+C\sup_{s\in[\tau_j^\varepsilon,\,\tau_j^\varepsilon+\rho r_j]}\Big|\int_0^1\big[f_x\big(s,\,x^\varepsilon(s)+\sigma\big(x_\rho^\varepsilon(s)-x^\varepsilon(s)\big),\,\overline{u}(s)\big)\\
&\qquad -f_x\big(\overline{\tau}_j+0,\,\overline{x}(\overline{\tau}_j)-\sigma\overline{\xi}_j,\,\overline{u}(\overline{\tau}_j+0)\big)\big]\,d\sigma\Big|\\
&+\frac{1}{\rho}\left|\left(\int_{\tau_j^\varepsilon}^{\tau_j^\varepsilon+\rho r_j}-\int_{\overline{\tau}_j}^{\tau_j+\rho r_j}\right)\langle\psi_\rho^\varepsilon(s),\,\int_0^1 f_x\big(\overline{\tau}_j+0,\,\overline{x}(\overline{\tau}_j)-\sigma\overline{\xi}_j,\,\overline{u}(\overline{\tau}_j+0)\big)d\sigma\overline{\xi}_j\rangle\right|\\
&+\frac{1}{\rho}\left|\int_{\overline{\tau}_j}^{\overline{\tau}_j+\rho r_j}\langle\psi_\rho^\varepsilon(s)-\overline{\psi}(\overline{\tau}_j),\,\int_0^1 f_x\big(\overline{\tau}_j+0,\,\overline{x}(\tau_j)-\sigma\overline{\xi}_j,\,\overline{u}(\overline{\tau}_j+0)\big)d\sigma\overline{\xi}_j\rangle ds\right|\\
&\leqslant C\sup_{s\in[0,\,\rho r_j]}\left|e^{As}\xi_j^\varepsilon-\overline{\xi}_j\right|+C\frac{1}{\rho}\sqrt{\varepsilon}+C\frac{1}{\rho}\left|\tau_j^\varepsilon-\overline{\tau}_j\right|\\
&+C\sup_{s\in\left[\tau_j^\varepsilon,\,\tau_j^\varepsilon+\rho r_j\right)}\Big|\int_0^1\big[f_x\big(s,\,x^\varepsilon(s)+\sigma\big(x_\rho^\varepsilon(s)-x^\varepsilon(s)\big),\,\overline{u}(s)\big)\\
&\qquad -\,f_x\big(\overline{\tau}_j+0,\,\overline{x}(\overline{\tau}_j)-\sigma\overline{\xi}_j,\,\overline{u}(\overline{\tau}_j+0)\big)\big]\,d\sigma\Big|\\
&+\frac{1}{\rho}\left|\int_{\overline{\tau}_j}^{\overline{\tau}_j+\rho r_j}\langle\psi_\rho^\varepsilon(s)-\overline{\psi}(\overline{\tau}_j),\,\int_0^1 f_x\big(\overline{\tau}_j+0,\,\overline{x}(\tau_j)-\sigma\overline{\xi}_j,\,\overline{u}(\overline{\tau}_j+0)\big)d\sigma\overline{\xi}_j\rangle ds\right|.
\end{aligned}$$

Hence (3.68) holds. Similarly, we have (still let $r_j>0$),

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\psi^{0,\varepsilon}\frac{1}{\rho}\int_{\tau_j^\varepsilon}^{\tau_j^\varepsilon+\rho r_j}\langle g_\rho^{0,\varepsilon}(s),\,e^{A\left(s-\tau_j^\varepsilon\right)}\xi_j^\varepsilon\rangle ds\\
&=r_j\overline{\psi}^0\int_0^1\langle f_x^0\big(\overline{\tau}_j+0,\,\overline{x}(\tau_j)-\sigma\overline{\xi}_j,\,\overline{u}(\overline{\tau}_j+0)\big)d\sigma,\,\overline{\xi}_j\rangle.
\end{aligned} \tag{3.73}$$

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For the case $r_j < 0$, we can similarly show that

$$\begin{aligned}(3.74)\qquad &\lim_{\varepsilon\to 0}\frac{1}{\rho}\int_{\tau_j^\varepsilon+\rho r_j}^{\tau_j^\varepsilon}\langle\psi_\rho^\varepsilon(s),\, g_\rho^\varepsilon(s)e^{A(s-\tau_j^\varepsilon)}\xi_j^\varepsilon\rangle ds\\ &= -r_j\langle\overline{\psi}(\overline{\tau}_j),\, \int_0^1 f_x\big(\overline{\tau}_j-0,\, \overline{x}(\tau_j)-\sigma\overline{\xi}_j,\, \overline{u}(\overline{\tau}_j-0)\big)d\sigma\overline{\xi}_j\rangle,\end{aligned}$$

and

$$\begin{aligned}(3.75)\qquad &\lim_{\varepsilon\to 0}\psi^{0,\varepsilon}\frac{1}{\rho}\int_{\tau_j^\varepsilon+\rho r_j}^{\tau_j^\varepsilon}\langle g_\rho^{0,\varepsilon}(s),\, e^{A(s-\tau_j^\varepsilon)}\xi_j^\varepsilon\rangle ds\\ &= -r_j\overline{\psi}^0\int_0^1\langle f_x^0\big(\overline{\tau}_j-0,\overline{x}(\tau_j)-\sigma\overline{\xi}_j,\overline{u}(\overline{\tau}_j-0)\big)d\sigma,\, \overline{\xi}_j\rangle.\end{aligned}$$

Then, we end up with

$$\begin{aligned}(3.76)\qquad 0 \leqslant\ &\langle\overline{\varphi}+\overline{\psi}(0),\, x_0\rangle\\ &+\int_0^T\Big\{\langle\overline{\psi}(s),\, f(s,\overline{x}(s),u(s))-f(s,\overline{x}(s),\overline{u}(s))\rangle\\ &\quad+\overline{\psi}^0[f^0(s,\overline{x}(s),u(s))-f(s,\overline{x}(s),\overline{u}(s))]\Big\}\,ds\\ &+\sum_{j=1}^m\{\langle\overline{\psi}(\overline{\tau}_j)+\overline{\psi}^0\ell_\xi(\overline{\tau}_j,\overline{\xi}_j),\,\xi_j\rangle - r_j[\langle\overline{\psi}(\overline{\tau}_j),\, A\overline{\xi}_j\rangle-\overline{\psi}^0\ell_\tau(\overline{\tau}_j\overline{\xi}_j)]\}\\ &-\sum_{r_j>0}r_j\Big\{\langle\overline{\psi}(\overline{\tau}_j),\,\int_0^1 f_x\big(\overline{\tau}_j+0,\overline{x}(\tau_j)-\sigma\overline{\xi}_j,\overline{u}(\overline{\tau}_j+0)\big)d\sigma\overline{\xi}_j\rangle\\ &\quad+\overline{\psi}^0\int_0^1\langle f_x^0\big(\overline{\tau}_j+0,\overline{x}(\tau_j)-\sigma\overline{\xi}_j,\overline{u}(\overline{\tau}_j+0)\big)d\sigma,\,\overline{\xi}_j\rangle\Big\}\\ &-\sum_{r_j<0}r_j\Big\{\langle\overline{\psi}(\overline{\tau}_j),\,\int_0^1 f_x\big(\overline{\tau}_j-0,\overline{x}(\tau_j)-\sigma\overline{\xi}_j,\overline{u}(\overline{\tau}_j-0)\big)d\sigma\overline{\xi}_j\rangle\\ &\quad+\overline{\psi}^0\int_0^1\langle f_x^0\big(\overline{\tau}_j-0,\overline{x}(\tau_j)-\sigma\overline{\xi}_j,\overline{u}(\overline{\tau}_j-0)\big)d\sigma,\,\overline{\xi}_j\rangle\Big\}.\end{aligned}$$

By setting

$$(3.77)\qquad \begin{cases}\psi^0 &= -\overline{\psi}^0,\\ \psi(t) &= -\overline{\psi}(t), \qquad t\in[0,T].\end{cases}$$

we see from (3.62) that $\psi(\cdot)$ is a solution of (2.19). Then, our conclusion follows from (3.76) easily. ∎

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