Canad. Math. Bull. Vol. **38** (3), 1995 pp. 257–261

COUNTEREXAMPLES IN NONSTANDARD MEASURE THEORY

J. M. ALDAZ AND P. A. LOEB

ABSTRACT. We show that several "good" properties of the standard part map on regular Hausdorff spaces do not hold for arbitrary Hausdorff spaces.

1. **Introduction.** Special measures, called *Loeb measures* in the literature, have often been used to define or represent standard measures on topological Hausdorff spaces. For this purpose the measurability properties of the standard part map are crucial. Anderson showed ([An]) that if X is a Hausdorff space and μ a Radon measure on X, then $L({}^\star\mu)\,\mathrm{st}^{-1} = \mu$. In applications, however, one often starts with an internal measure; the corresponding Loeb measure is then "pushed down" via the standard part map st. It is therefore important to have measurability results for st with respect to general internal measures which are not obtained by the transfer of standard measures. Such results were obtained by Loeb for compact [L1] and locally compact spaces [L2] (see also [H]). With Landers and Rogge's significant extension of these results in [LR1], there is now a good theory regarding the measurability of the standard part map when the underlying space X is regular and Hausdorff. The basic ingredients of this theory are: 1) If $I \subset \mathrm{ns}({}^\star X)$ is internal, then $\mathrm{st}(I)$ is compact ([Lux]). It follows that if st is measurable, then the pushed down Loeb measure is Radon. 2) The map st is measurable iff the set $\mathrm{ns}({}^\star X)$ is measurable ([LR1]). To check the measurability of st, therefore, one only needs to check the measurability of one set. 3) The universal Loeb measurability of st is equivalent to its measurability with respect to all transfers of standard, countably additive measures (see [A1] or [R1]). This is an immediate consequence of Landers and Rogge's results on outer Loeb measures, and it means that the map st is universally Loeb measurable iff X is pre-Radon.

Since the theory of Borel measures is usually developed in the context of Hausdorff spaces, it is interesting to know whether any of these results holds for arbitrary Hausdorff spaces. It will be shown, by example, that none of them do. It appears, therefore, that regular spaces form the right topological setting for the representation of measures via Loeb measures.

The authors are indebted to the referee for pointing out relevant references, as well as for some corrections and improvements in the exposition of this paper.

Received by the editors May 24, 1994; revised September 20, 1994.
AMS subject classification: 28E05, 03H05.
Key words and phrases: Loeb measures, standard part map.

https://doi.org/10.4153/CMB-1995-038-5 Published online by Cambridge University Press

2. **Examples.** Whenever we speak of a measure we mean a finite measure. A countably additive Borel measure μ is Radon if it is inner regular with respect to the compact sets, and τ-smooth if for every collection of open sets $\{O_\alpha\}$, closed under finite unions, $\mu(\bigcup_\alpha O_\alpha) = \sup_\alpha \mu(O_\alpha)$. Let X be a Hausdorff space with topology $\mathcal{T}$. The Borel sets of X are denoted by $\sigma(\mathcal{T})$. The space X is Radon if every countably additive Borel measure on X is Radon, and pre-Radon if every τ-smooth measure on X is Radon.

It is well known (see Counterexample 5.5.3 of [MH], or Aufgaben 28.4 and 23.3 of [LR2]) that for a Hausdorff space X, $\mathrm{st}(I)$ may fail to be compact even if the internal set I is contained in the set of near standard points. We present a related example (part i of the following theorem) which in addition will allow us to show that the measurability of $\mathrm{ns}({}^\star X)$ need not be equivalent to the measurability of st, and that a space can be pre-Radon (and even Radon) without being universally Loeb measurable (Theorem 1, parts ii) and iii)).

A natural generalization of Radon measures is the notion of a compact measure, *i.e.*, a measure that is inner regular with respect to a compact family of sets. (Recall that a family of sets is compact if every subfamily with the finite intersection property has nonempty intersection.) Since standard measures of the form $L(\nu)\,\mathrm{st}^{-1}$ are inner regular with respect to the family $\{\mathrm{st}(I) : I \text{ internal}, I \subset \mathrm{ns}({}^\star X)\}$, it is reasonable to ask whether this family has to be compact, even if the individual sets $\mathrm{st}(I)$ are not compact. The answer is no (part iv) of the following theorem). This issue is related to the question as to whether the images of Loeb measures must be compact (asked by D. Ross in [Ro] and answered negatively in [A2]).

The standard part map is said to be *measurable* with respect to an internal measure if for every Borel set B, $\mathrm{st}^{-1}(B)$ is measurable with respect to the corresponding completed Loeb σ-algebra $L\big({}^\star\sigma(\mathcal{T})\big)$. The standard part map is universally Loeb measurable if it is measurable with respect to all finite, finitely additive internal measures. We shall assume that the saturation of the nonstandard model is larger than the cardinality of the topology of any standard space under consideration. Recall that a Borel measure is continuous if every point has measure zero, and discrete if there is a countable set with full measure. Every Borel measure can be decomposed into a continuous and a discrete part.

THEOREM 1. *i) There exists a Hausdorff space X and an internal set $I \subset \mathrm{ns}({}^\star X)$ such that $\mathrm{st}(I)$ is not compact.*

ii) There is a pre-Radon Hausdorff space X such that $\mathrm{ns}({}^\star X)$ is universally Loeb measurable, and yet the map st, *although measurable with respect to all transfers of standard countably additive measures, is not universally Loeb measurable.*

iii) There exists a Radon space X such that $\mathrm{ns}({}^\star X)$ is not universally Loeb measurable.

iv) There exists a Hausdorff space X such that the family $\{\mathrm{st}(I) : I \text{ internal}, I \subset \mathrm{ns}({}^\star X)\}$ is not compact.

PROOF. i) Consider the "half-disk" or "mushroom space" topology (see [S], p. 96) on the closed unit square S with vertices (0,0), (1,1). This topology is defined as follows. The intersection of the open upper half plane with S (denoted here by U) has the usual

https://doi.org/10.4153/CMB-1995-038-5 Published online by Cambridge University Press

euclidean topology, while for $x \in H := S \setminus U$, a neighborhood basis is given by the sets $\{x\} \cup \big(B(x,r) \cap U\big)$. (As usual, $B(x,r)$ denotes the disc of radius r centered at x.) The internal set $I_\epsilon = {}^\star[0,1] \times \{\epsilon\}$, where ϵ is a positive infinitesimal, is contained in $\mathrm{ns}({}^\star S)$, but $\mathrm{st}(I_\epsilon)$ is an uncountable discrete set and hence not compact.

ii) It is clear that the "mushroom" square S is pre-Radon since U has the euclidean and H the discrete topology; the only τ-smooth measures the latter set admits are discrete. Let ν be any internal, finite, finitely additive measure on ${}^\star S$. We may assume, without loss of generality, that $L(\nu)$ is continuous. Let $E = \{({}^\star x, 0) : x \in [0,1]\}$. By the continuity of $L(\nu)$ and saturation (see Theorem 1 of [LR1]), $L(\nu)(E) = 0$. Since $\mathrm{ns}({}^\star S) = E \cup {}^\star U$ and ${}^\star U$ is ν-measurable, the set $\mathrm{ns}({}^\star S)$ is universally Loeb measurable.

Next it will be shown that st is measurable with respect to all transfers of standard countably additive measures. Let μ be a Borel measure on S. Again we may assume that μ is continuous. Note that $\mathrm{st}^{-1}(H) \cap {}^\star U$ is contained in the set ${}^\star A_n$, where $A_n := [0,1] \times (0, 1/n)$. Since $A_n \downarrow \emptyset$, $L({}^\star\mu)({}^\star A_n) = \mu(A_n) \longrightarrow 0$. Therefore $\mathrm{st}^{-1}(H) \cap {}^\star U$ is $L({}^\star\mu)$-measurable, with measure zero. Also $L({}^\star\mu)(E) = 0$, whence $L({}^\star\mu)\,\mathrm{st}^{-1}(H) = 0$. Now let V be a Borel subset of S. On U we have the euclidean topology, so $\mathrm{st}^{-1}(V \cap U)$ is universally Loeb measurable. Thus $\mathrm{st}^{-1}(V)$ is $L({}^\star\mu)$-measurable, since it is the union of $\mathrm{st}^{-1}(V \cap U)$ and the $L({}^\star\mu)$-null set $\mathrm{st}^{-1}(V \cap H)$.

Finally, let ν be the internal linear Lebesgue measure on the segment $I_\epsilon = {}^\star[0,1] \times \{\epsilon\}$, where ϵ is a positive infinitesimal. Let $\mathrm{st}_{\mathbb{R}}$ be the standard part map on ${}^\star[0,1]$ with respect to the euclidean topology, and let λ be the Lebesgue measure on $[0,1]$. It is well known that for $B \subset [0,1]$, $\mathrm{st}_{\mathbb{R}}^{-1}(B)$ is $L({}^\star\lambda)$-measurable iff B is Lebesgue measurable. Hence $\mathrm{st}_{\mathbb{R}}^{-1}(B) \times \{\epsilon\}$ is $L(\nu)$-measurable iff B is Lebesgue measurable. But $\mathrm{st}^{-1}(B \times \{0\})$ can be expressed as the union of $\mathrm{st}_{\mathbb{R}}^{-1}(B) \times \{\epsilon\}$ with an $L(\nu)$-null set, so $\mathrm{st}^{-1}(B \times \{0\})$ is $L(\nu)$-measurable iff B is a Lebesgue set. Since the topology on H is discrete, all its subsets are open and therefore st is not $L(\nu)$-measurable.

iii) Again let ν be the internal linear Lebesgue measure on $\{(x,\epsilon) : x \in {}^\star[0,1], \epsilon$ a positive infinitesimal$\}$. Let $D \subset [0,1]$ be a nonmeasurable subset (with respect to Lebesgue measure) of the least possible cardinality. By Theorem 14.7ii) of [K], D does not admit any finite, continuous measure defined on all its subsets. If we regard D as contained in H, then $X := D \cup U$, with the topology inherited from the mushroom square, is a Radon space. By the same argument as in part ii), $\mathrm{st}^{-1}(D)$ is not $L(\nu)$-measurable. Since $\mathrm{ns}({}^\star X)$ differs from $\mathrm{st}^{-1}(D)$ by the null set $\mathrm{st}^{-1}(U)$, the result follows.

iv) Take the plane with the usual topology and add to it two ideal points, a and $-a$. Basic neighborhoods of the ideal points are formed by setting for each $n \in \mathbb{N}$, $N_n(a) = \{a\} \cup \{(x,y) : x > 0, y > n\}$ and $N_n(-a) = \{-a\} \cup \{(x,y) : x < 0, y > n\}$. Select a positive infinitesimal ϵ and set $I_n = \Big\{\big((-1)^n\epsilon, y\big) : y > n\Big\}$. Then for each $n \in \mathbb{N}$, I_n is an internal set contained in the near standard points. But $\bigcap_n \{\mathrm{st}(I_n) : n \in \mathbb{N}\} = \emptyset$, even though the family $\{\mathrm{st}(I_n) : n \in \mathbb{N}\}$ has the finite intersection property. ■

REMARKS. 1) One might hope to obtain better properties if attention is restricted to particularly well behaved classes of Hausdorff spaces, such as analytic spaces (continuous images of polish spaces). However, the example given in the proof of part iv)

https://doi.org/10.4153/CMB-1995-038-5 Published online by Cambridge University Press

of the theorem is analytic since it is a continuous image of the plane with two isolated points adjoined. (This example is a slight simplification of counterexample 100 in [S]; the original example also works here.)

2) If a Hausdorff space X admits a coarser regular Hausdorff topology, then the family $\{\mathrm{st}(I) : I \text{ internal}, I \subset \mathrm{ns}({}^*X)\}$ is compact, because the internal sets contained in $\mathrm{ns}({}^*X)$ are still contained in the near standard points with respect to the new topology. In particular, this is the case for the mushroom square S, where the euclidean topology is coarser and yet regular.

Landers and Rogge noted ([LR1], Corollary 3iv) that if $m(C)$ is defined as $\bigcap\{{}^*O : C \subset O \text{ and } O \in \mathcal{T}\}$, then for every closed subset C of a regular Hausdorff space X, $\mathrm{st}^{-1}(C) = m(C) \cap \mathrm{ns}({}^*X)$. This property is useful because it yields the equivalence between measurability of the map st and the set $\mathrm{ns}({}^*X)$. Unfortunately, as we now show, this property does not hold for arbitrary Hausdorff spaces; indeed, it is equivalent to regularity for pre-Hausdorff spaces. A space X is pre-Hausdorff if given any two points x, $y \in X$, either $m(x) = m(y)$ or $m(x) \cap m(y) = \emptyset$. Both regular and Hausdorff spaces are pre-Hausdorff (see [R2]). For a general (not necessarily Hausdorff) topological space X, if $B \subset X$, then $\mathrm{st}^{-1}(B)$ is defined as the union of monads of points in B.

PROPOSITION 2. *A topological space is regular if and only if it is pre-Hausdorff and for every closed set $C \subset X$,* $\mathrm{st}^{-1}(C) = m(C) \cap \mathrm{ns}({}^*X)$.

PROOF. The proof that regularity implies the condition $\mathrm{st}^{-1}(C) = m(C) \cap \mathrm{ns}({}^*X)$ is the same for a Hausdorff or non-Hausdorff space. To show that for a pre-Hausdorff space the condition implies regularity, we fix an arbitrary closed set $C \subset X$ and a point $x \notin C$. For every $y \in C$, $m(x) \cap m(y) = \emptyset$ since the complement of C is an open set that contains x but not y, so their monads cannot be equal. It follows from the condition that $m(x) \cap m(C) = m(x) \cap \mathrm{st}^{-1}(C) = \emptyset$. By saturation and downward transfer, there exist disjoint open sets U and V that contain x and C respectively, whence X is regular. ∎

Finally, we consider the measurability of st with respect to the Baire sets rather than the Borel sets; we will show that this measurability is equivalent to the measurability of the set $\mathrm{ns}({}^*X)$. Let $\mathcal{Z}(X)$ be the collection of zero sets of continuous real valued functions defined on X. The smallest σ-algebra making all such functions measurable (the Baire sets) is precisely $\sigma\big(\mathcal{Z}(X)\big)$. Our last result employs an argument of Landers and Rogge ([LR1] Corollary 3iv) by replacing arbitrary open sets with cozero sets.

PROPOSITION 3. *Let X be a Hausdorff space. For an arbitrary set $B \in \sigma\big(\mathcal{Z}(X)\big)$,* $\mathrm{st}^{-1}(B)$ *is $L\Big({}^*\sigma\big(\mathcal{Z}(X)\big)\Big)$-measurable if and only if this is true for* $\mathrm{ns}({}^*X)$.

PROOF. Define $m(Z) = \bigcap\{{}^*O : Z \subset O \text{ and } O \text{ is cozero}\}$. Let Z be a zero set. By [LR1] Theorem 1, it suffices to show that $m(Z) \cap \mathrm{ns}({}^*X) = \mathrm{st}^{-1}(Z)$. Suppose $x \notin \mathrm{st}^{-1}(Z)$ and $x \in \mathrm{ns}({}^*X)$. Then there is a continuous function $f \geq 0$ which vanishes on Z but for which $f\big(\mathrm{st}(x)\big) > 0$. Therefore the set $O = \big\{y : f(y) < 1/2 \cdot f\big(\mathrm{st}(x)\big)\big\}$ is cozero, contains Z, and $x \notin {}^*O$, whence $m(Z) \cap \mathrm{ns}({}^*X) \subset \mathrm{st}^{-1}(Z)$. Since the other inclusion is obvious, we have equality. ∎

REFERENCES

[A1] J. M. Aldaz, *A characterization of universal Loeb measurability for completely regular Hausdorff spaces,* Canad. J. Math. (4) **44**(1992), 673–690.

[A2] ———, *On compactness and Loeb measures,* Proc. Amer. Math. Soc., to appear.

[An] R. M. Anderson, *Star-finite representations of measure spaces,* Trans. Amer. Math. Soc. **271**(1982), 667–687.

[H] C. W. Henson, *Analytic sets, Baire sets and the standard part map,* Canad. J. Math. **31**(1979), 663–672.

[K] K. Kunen, *Inaccessibility Properties of Cardinals*, Ph.D. Thesis, Stanford University, 1968.

[L1] P. A. Loeb, *Weak limits of measures and the standard part map,* Proc. Amer. Math. Soc. **77**(1979), 128–135.

[L2] ———, *A functional approach to nonstandard measure theory,* Contemp. Math. **26**(1984), 251–261.

[LR1] D. Landers and L. Rogge, *Universal Loeb-measurability of sets and of the standard part map with applications,* Trans. Amer. Math. Soc. (1) **304**(1987), 229–243.

[LR2] ———, *Nichtstandard Analysis*, Springer-Verlag, 1994.

[Lux] W. A. J. Luxemburg, *A general theory of monads*. In: Applications of Model Theory to Algebra, Analysis and Probability, (ed. W. A. J. Luxemburg), Holt, Rinehart and Winston, 1969, 18–69.

[MH] M. Machover and J. Hirschfield, *Lectures in Non-Standard Analysis*, Lecture Notes in Math. **94**, Springer-Verlag, 1968.

[R1] H. Render, *Pushing down Loeb measures,* Math. Scand. **72**(1993), 61–84.

[R2] ———, *Nonstandard topology on function spaces with applications to hyperspaces,* Trans. Amer. Math. Soc. (1) **336**(1993), 101–119.

[Ro] D. Ross, *Compact measures have Loeb preimages,* Proc. Amer. Math. Soc. **115**(1992), 365–370.

[S] L. A. Steen and J. A. Seebach, *Counterexamples in Topology*, Springer-Verlag, 1986.

Departamento de Matemáticas
Facultad de Ciencias
Universidad Autónoma de Madrid
28049 Madrid
Spain
e-mail: aldaz@ccuam3.sdi.uam.es

Department of Mathematics
University of Illinois
1409 West Green Street
Urbana, Illinois 61801
U.S.A.
e-mail: loeb@math.uiuc.edu