# THE COARSENESS OF THE COMPLETE BIPARTITE GRAPH

LOWELL W. BEINEKE AND RICHARD K. GUY

**1. Introduction.** The *coarseness*, $c(G)$, of a graph $G$ is the maximum number of edge-disjoint, non-planar graphs whose union is $G$. The coarseness of the complete graph has been investigated elsewhere (**1**; **2**). We consider the coarseness of the complete bipartite, or 2-coloured, graph, $K_{m,n}$, consisting of sets of $m$ and $n$ vertices, each member of one set being joined by an edge to each member of the other. No members of one set are joined to each other.

Our results are summarized in the following theorem, where square brackets denote "integer part".

THEOREM. *If* $m = 3p + d$, $0 \leqq d \leqq 2$, *and* $n = 3q + e$, $0 \leqq e \leqq 2$, *then for* $d = 0$ *or* $1$ *and* $e = 0$ *or* $1$,

$$c(K_{m,n}) = pq + \min\left(\left[\frac{ep}{3}\right], \left[\frac{dq}{3}\right]\right). \tag{1}$$

*For* $d = 2$ *and* $e = 0$,

$$c(K_{m,n}) = pq + \left[\frac{q}{3}\right], \qquad p \geqq 1. \tag{2}$$

*For* $d = 2$ *and* $e = 1$,

$$c(K_{m,n}) \leqq pq + \min\left(\left[\frac{p+q}{3}\right], \left[\frac{2q}{3}\right], \left[\frac{8p+16q+2}{39}\right]\right), \tag{3}$$

$$c(K_{m,n}) \geqq pq + \max\left(\min\left(\left[\frac{p}{3}\right], \left[\frac{2q}{3}\right]\right), \left[\frac{q+2}{3}\right]\right), \tag{4}$$

*the last alternative being available only for* $p \geqq 2$, $q \geqq 7$. *Note that there is equality in* (3) *and* (4) *for* $p \geqq 2q$. *For* $d = e = 2$,

$$c(K_{m,n}) \leqq pq + \min\left(\left[\frac{p+2q}{3}\right], \left[\frac{2p+q}{3}\right], \left[\frac{16p+16q+4}{39}\right]\right), \tag{5}$$

$$c(K_{m,n}) \geqq pq + \left[\frac{p}{3}\right] + \left[\frac{q}{3}\right] + \left[\frac{p}{9}\right], \qquad p \geqq 1, \tag{6}$$

*where, in* (6), *we have assumed by symmetry that* $p \leqq q$.

---

Received March 19, 1968 and in revised form, August 12, 1968. The research of the first author was supported in part by the Office of Naval Research; the research of the second author was supported by the National Research Council of Canada, grant A-4011.

**2. Upper bounds.** Kuratowski's theorem (**3**) states that a non-planar graph must contain a subgraph homeomorphic to $K_5$, the complete graph on five vertices, or to $K_{3,3}$. Suppose that it is possible to find $y + z$ edge-disjoint subgraphs of $K_{m,n}$, where $y$ are homeomorphs of $K_5$ and $z$ are homeomorphs of $K_{3,3}$. Note that a homeomorph of $K_5$ contains five tetravalent vertices, which we call *essential*. Note also that since $K_{m,n}$ does not contain $K_5$ as a subgraph, a homeomorph of $K_5$ must contain some bivalent vertices. In fact, let $y_k$, $0 \leqq k \leqq 5$, be the number of $K_5$ homeomorphs which have $k$ of their essential vertices among the $m$, and hence $5 - k$ among the $n$. Consequently, we see that

$$\sum_{k=0}^{5} y_k = y. \tag{7}$$

In a $K_5$ homeomorph, there are at least $\binom{5-k}{2}$ bivalent vertices among the $m$, and at least $\binom{k}{2}$ among the $n$. These necessary minima of bivalent vertices are called *supporting*. Additional bivalent vertices may also occur, in pairs in the present context owing to the bipartite nature of the parent graph; these will be called *trivial*. There is some ambiguity in a particular case as to which vertices are supporting and which trivial, but since we are concerned only with counting arguments, *any* labelling consistent with the correct totals will suffice. Note that the essential and supporting vertices of any one subgraph are distinct, otherwise it would cease to be non-planar. The minimum number of edges in a $K_5$ homeomorph is

$$k(5-k) + 2\binom{k}{2} + 2\binom{5-k}{2} = k^2 - 5k + 20 \geqq 14. \tag{8}$$

Similarly, let $z_l$, $0 \leqq l \leqq 6$, be the number of $K_{3,3}$ homeomorphs with $l$ essential (trivalent) vertices among the $m$ and $6 - l$ among the $n$. In such a $K_{3,3}$ homeomorph there are $3|l - 3|$ supporting vertices* and at least

$$9 + 3|l - 3| \tag{9}$$

edges. Thus, we have

$$\sum_{l=0}^{6} z_l = z. \tag{10}$$

Let $m_{ij}$ be the number of vertices among the $m$ which are *essential* vertices

*The referee observed that if the homeomorph has, for example, two vertices of each part among the $m$, and one of each among the $n$, then five supporting vertices are needed, rather than three. To delay inequalities, we again define *supporting vertices* by the *necessary minimum* number. In the case mentioned, two vertices would arbitrarily be labelled as "trivial", although the name is inappropriate.

of exactly $i$ $K_5$ homeomorphs *and supporting* vertices of exactly $j$ other non-planar subgraphs. Define $n_{ij}$ similarly so that

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} m_{ij} = m = 3p + d, \tag{11}$$

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} n_{ij} = n = 3q + e. \tag{12}$$

Furthermore,

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} im_{ij} = \sum_{k=0}^{5} ky_k, \tag{13}$$

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} in_{ij} = \sum_{k=0}^{5} (5-k)y_k, \tag{14}$$

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} jm_{ij} = \sum_{k=0}^{5} \binom{5-k}{2} y_k + \sum_{l=0}^{2} 3(3-l)z_l, \tag{15}$$

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} jn_{ij} = \sum_{k=0}^{5} \binom{k}{2} y_k + \sum_{l=4}^{6} 3(l-3)z_l. \tag{16}$$

Each of the $m$ vertices of $K_{m,n}$ is $(3q+e)$-valent, so that at one of the $m_{ij}$ vertices, there are at most $(3q+e) - 4i - 2j$ edges which are candidates for incidence with an essential vertex in a $K_{3,3}$ homeomorph. Thus if $u_{ij}$ $(= 0, 1 \text{ or } 2)$ is the least non-negative residue of $e + 2i + j$, modulo 3, then there are $u_{ij}$ edges at such a vertex which either do not belong to any non-planar subgraph, or are in excess of the minimum numbers, (8) and (9), of edges of such a subgraph, by being incident with trivial vertices. Similarly, at each of the $n$ vertices, we define $v_{ij}$ analogously to $u_{ij}$, as the least non-negative residue of $d + 2i + j$, modulo 3. On counting the edges incident with the $m$ vertices, and with the $n$ vertices, and using formulas (8) and (9) we have:

$$\sum_{k=0}^{5} (k^2 - 5k + 20)y_k + \sum_{l=0}^{6} (9 + 3|l-3|)z_l \leqq mn - \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} m_{ij}u_{ij}, \tag{17}$$

$$\sum_{k=0}^{5} (k^2 - 5k + 20)y_k + \sum_{l=0}^{6} (9 + 3|l-3|)z_l \leqq mn - \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} n_{ij}v_{ij}. \tag{18}$$

On subtracting (13) and (15) from (17), and (14) and (16) from (18), we have:

$$\sum_{k=0}^{5} \tfrac{1}{2}(k^2 - 3k + 20)y_k + 9z + \sum_{l=4}^{6} 3(l-3)z_l \leqq 3mq + \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} m_{ij}(e - u_{ij} - i - j), \tag{19}$$

$$\sum_{k=0}^{5} \tfrac{1}{2}(k^2 - 7k + 30)y_k + 9z + \sum_{l=0}^{2} 3(3-l)z_l \leqq 3np + \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} n_{ij}(d - v_{ij} - i - j). \tag{20}$$

https://doi.org/10.4153/CJM-1969-121-3 Published online by Cambridge University Press

These two formulas imply:

$$(21)\quad 9(y+z) \leqq 9pq + 3dq + (e-1)m_{01} \leqq 9pq + 3dq + (e-1)(3p+d)$$

and

$$(22)\quad 9(y+z) \leqq 9pq + 3ep + (d-1)n_{01} \leqq 9pq + 3ep + (d-1)(3q+e),$$

respectively. A further subtraction of (15) from (19) and (16) from (20) yields:

$$(23)\quad \sum_{k=0}^{5} 3ky_k + 9z + \sum_{l=0}^{6} 3(l-3)z_l \leqq 9pq + 3dq + \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} m_{ij}(e - u_{ij} - i - 2j),$$

$$(24)\quad \sum_{k=0}^{5} 3(5-k)y_k + 9z + \sum_{l=0}^{6} 3(3-l)z_l \leqq 9pq + 3ep + \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} n_{ij}(d - v_{ij} - i - 2j).$$

If we add (23) and (24), note that all terms in the double sums are non-positive since the parentheses are congruent to zero (modulo 3) and are at most two, and omit a factor 3, we have:

$$(25)\qquad 5y + 6z \leqq 6pq + ep + dq.$$

From (17) or (18),

$$(26)\qquad 14y + 9z \leqq mn = 9pq + 3ep + 3dq + de.$$

Formula (26) with five times (25) yields:

$$(27)\qquad 39(y+z) \leqq 39pq + 8ep + 8dq + de,$$

so that

$$(28)\qquad c(K_{m,n}) \leqq pq + \left[\frac{8ep + 8dq + de}{39}\right].$$

If $d = e = 0$, this implies that

$$(29)\qquad c(K_{m,n}) \leqq pq,$$

while the same inequality is implied by (22) and (21) in the cases $d = 1, e = 0$ and $d = 0, e = 1$. If $d = e = 1$, the same formulas yield:

$$(30)\qquad c(K_{m,n}) \leqq pq + \min\left(\left[\frac{p}{3}\right], \left[\frac{q}{3}\right]\right).$$

If $d = 2$, $e = 0$, then formula (22) yields:

$$(31)\qquad c(K_{m,n}) \leqq pq + \left[\frac{q}{3}\right].$$

https://doi.org/10.4153/CJM-1969-121-3 Published online by Cambridge University Press

If $d = 2$, $e = 1$, then formulas (21), (22), and (28) yield:

$$c(K_{m,n}) \leqq pq + \left[\frac{2q}{3}\right], \tag{32}$$

$$c(K_{m,n}) \leqq pq + \left[\frac{p+q}{3}\right], \tag{33}$$

$$c(K_{m,n}) \leqq pq + \left[\frac{8p+16q+2}{39}\right], \tag{34}$$

which combine to prove formula (3) of the Theorem. If $d = e = 2$, the same three formulas yield:

$$c(K_{m,n}) \leqq pq + \left[\frac{p+2q}{3}\right], \tag{35}$$

$$c(K_{m,n}) \leqq pq + \left[\frac{2p+q}{3}\right], \tag{36}$$

$$c(K_{m,n}) \leqq pq + \left[\frac{16p+16q+4}{39}\right], \tag{37}$$

which combine to prove formula (5) of the Theorem.

## 3. Lower bounds.

*Construction* A. If $d$ or $e$ is zero and the other is not equal to two, it is clear that the right member of formula (1), which reduces to $pq$ in this case, can be attained by joining $p$ triples of the $m$ vertices to $q$ triples of the $n$, forming $pq$ $K_{3,3}$ graphs.

If $d = e = 1$, we may assume by symmetry that $p \leqq q$, and the right member of (1) is then

$$pq + \left[\frac{p}{3}\right] = pq + r, \tag{38}$$

where we write

$$p = 3r + f, \quad 0 \leqq f \leqq 2, \qquad q = 3s + g, \quad 0 \leqq g \leqq 2, \tag{39}$$

and (38) can be attained by Construction B, given below. Thus Construction A, with formula (29), and Construction B, with formula (30), serve to prove formula (1) of the Theorem.

*Construction* B. This will be given in greater generality than required for the proof of (1), since it contributes also to the proof of (4). In fact, we show that

$$c(K_{m,n}) \geqq pq + \min\left(\left[\frac{ep}{3}\right], \left[\frac{dq}{3}\right]\right), \tag{40}$$

by constructing that number of edge-disjoint non-planar graphs. By interchanging $m$ and $n$ if necessary, we may assume that

$$ep \leqq dq \tag{41}$$

so that the right member of (40) is

$$pq + \left[\frac{ep}{3}\right] = pq + er + \left[\frac{ef}{3}\right]. \tag{42}$$

We first select this number of triples $A_iA_jA_k$ from the $m$ vertices $A_1, A_2, \ldots, A_m$, writing them in $p$ rows of $q$ triples, with a partial, $(p + 1)$th row of $er + [ef/3]$ triples. Note that this last number is

$$\left[\frac{ep}{3}\right] \leqq \left[\frac{dq}{3}\right] = ds + \left[\frac{dg}{3}\right] \leqq 2s + \left[\frac{g+1}{3}\right],$$

so that the $(p + 1)$th row, when it occurs, is indeed a partial row, and in fact does not extend beyond the second of the three sets of triples now to be described.

The first row consists of

$$\left[\frac{q}{3}\right] = s + \left[\frac{g}{3}\right] = s$$

triples, each of which has $i, j, k = 1, 2, 3$; then

$$\left[\frac{q+1}{3}\right] = s + \left[\frac{g+1}{3}\right]$$

triples with $i, j, k = 2, 3, 4$; and finally,

$$\left[\frac{q+2}{3}\right] = s + \left[\frac{g+2}{3}\right]$$

triples with $i, j, k = 3, 4, 5$. The second row, and each succeeding one is obtained from the preceding one by adding three to every suffix $i, j, k$. In the $p$th and $(p + 1)$th rows, the addition is to be performed modulo $m$.

These triples are joined to triples chosen from the $n$ vertices $B_1, B_2, \ldots, B_n$, the suffixes being taken in rotation, 123, 456, . . . repeating cyclically, modulo $n$. In each row there are

$$\left[\frac{q}{3}\right] + \left[\frac{q+1}{3}\right] + \left[\frac{q+2}{3}\right] = q$$

triples, which contain $3q \leqq n$ vertices, hence no $B_w$ occurs twice in the same row. Moreover, $p$ repetitions of the suffixes 1 to $n$ suffice to provide all the required triples, because, comparing (42),

$$3\left(pq + er + \left[\frac{ef}{3}\right]\right) \leqq 3pq + e(3r + f) = np,$$

so that exactly $ef - 3[ef/3]$ $(= 0, 1 \text{ or } 2)$ vertices $B_w$ do not appear in the $p$th repetition. It is clear that no edge $A_iB_w$ is used more than once, since each row contains $3q \leqq n$ of the $B_w$ (and of the $A_i$), except perhaps in the cases

https://doi.org/10.4153/CJM-1969-121-3 Published online by Cambridge University Press

$i = 1$ and 2. These can also be verified, since the $p$th repetition of the suffixes 1 to $n$ extends into the $(p+1)$th row by an amount

$$np - 3pq = ep = 3er + ef \geqq 3\left(er + \left[\frac{ef}{3}\right]\right),$$

where the last parenthesis is the number of triples in the $(p+1)$th row. As already noted, $ef - 3[ef/3]$ of the vertices $B_w$ occur only $p - 1$ times. Construction B is illustrated in Figure 1, in the case $m = 26$, $n = 16$, the $A_i$ being represented by italic capitals, $A$ to $Z$, and the $B_w$ by lower case italic $a$ to $p$.

| | | | | |
|---|---|---|---|---|
| *ABC*<br>*abc* | *BCD*<br>*def* | *BCD*<br>*ghi* | *CDE*<br>*jkl* | *CDE*<br>*mno* |
| *DEF*<br>*pab* | *EFG*<br>*cde* | *EFG*<br>*fgh* | *FGH*<br>*ijk* | *FGH*<br>*lmn* |
| *GHI*<br>*opa* | *HIJ*<br>*bcd* | *HIJ*<br>*efg* | *IJK*<br>*hij* | *IJK*<br>*klm* |
| *JKL*<br>*nop* | *KLM*<br>*abc* | *KLM*<br>*def* | *LMN*<br>*ghi* | *LMN*<br>*jkl* |
| *MNO*<br>*mno* | *NOP*<br>*pab* | *NOP*<br>*cde* | *OPQ*<br>*fgh* | *OPQ*<br>*ijk* |
| *PQR*<br>*lmn* | *QRS*<br>*opa* | *QRS*<br>*bcd* | *RST*<br>*efg* | *RST*<br>*hij* |
| *STU*<br>*klm* | *TUV*<br>*nop* | *TUV*<br>*abc* | *UVW*<br>*def* | *UVW*<br>*ghi* |
| *VWX*<br>*jkl* | *WXY*<br>*mno* | *WXY*<br>*pab* | *XYZ*<br>*cde* | *XYZ*<br>*fgh* |
| *YZA*<br>*ijk* | *ZAB*<br>*lmn* | | | |

FIGURE 1

*Construction* C. Figure 2 is an incidence matrix for $K_{5,9}$, except that, in place of the usual unit entries denoting existence of an edge, various digits are used to indicate membership of particular non-planar subgraphs. For example, the digits 4 represent the edges joining the first three of the $m = 5$ vertices, represented by the first three rows of the matrix, to the fourth, fifth, and sixth

$$\begin{pmatrix} 2 & 3 & 3 & 4 & 4 & 4 & 3 & 2 & 2 \\ 1 & 3 & 1 & 4 & 4 & 4 & 3 & 1 & 3 \\ 2 & 2 & 1 & 4 & 4 & 4 & 1 & 1 & 2 \\ 2 & 3 & 1 & 1 & 2 & 3 & 3 & 1 & 2 \\ 1 & 2 & 3 & 1 & 2 & 3 & 1 & 2 & 3 \end{pmatrix}$$

FIGURE 2

of the $n = 9$ vertices, represented by the corresponding columns. Each of the digits 1, 2, and 3 corresponds to a $K_{3,3}$ homeomorph. For example, the twelve edges represented by the digit 3 join the first, second, and fourth of the $m = 5$ vertices to the second and seventh of the $n = 9$ and to the fifth of the $m = 5$, these last three joins being effected via the third, ninth, and sixth, respectively, of the $n = 9$. Thus

$$c(K_{5,9}) = 4, \tag{43}$$

since, by (31) it cannot exceed this value. The proof of (2) now follows from (31) and the present Construction C which consists in taking $s$ replicas of Figure 2, together with $3g$ additional columns, to form a $5 \times n$ matrix (where $n = 3q = 9s + 3g$), which contains $4s$ non-planar graphs, and can accommodate $g$ further $K_{3,3}$ graphs in the last $3g$ columns, with $6g$ edges remaining unused. This is a total of

$$4s + g = q + s = pq + \left[\frac{q}{3}\right]$$

graphs, since $p = 1$ for $m = 5$. For larger values of $m$ (and $p$), add $3(p - 1)$ rows to the $5 \times n$ matrix already constructed, which can accommodate $(p - 1)q$ additional $K_{3,3}$ graphs, giving a total of $q + s + (p - 1)q = pq + s$ non-planar graphs, showing that

$$c(K_{m,n}) \geqq pq + \left[\frac{q}{3}\right], \tag{44}$$

in this case, and proving (2).

If we take $d = 2$, $e = 1$ in Construction B, see formula (40), we obtain the first term in the maximum in formula (4). The proof of (4) is completed by Construction D, which shows that for $d = 2$, $e = 1$, $p \geqq 2$, $q \geqq 7$,

$$c(K_{m,n}) \geqq pq + \left[\frac{q+2}{3}\right], \tag{45}$$

and gives a sharper result than Construction B when $p < q + 2$.

*Construction* D. Figure 3 is a modified incidence matrix for $K_{8,22}$. The entries $A$ represent a $K_5$ homeomorph in which the first four of the $m = 8$ vertices are joined to the first of the $n = 22$ vertices, and the first four are joined among themselves via the second to the seventh of the $n$ vertices. Similarly, $B$ represents a $K_5$ homeomorph with the last four of the $m$ vertices joined to the first of the $n$, and joined among themselves via the eighth to the thirteenth of the $n$. Entries $X$, $Y$, and $Z$ correspond to $K_{3,3}$ homeomorphs of the kind represented by the digits 1, 2, and 3 in Figure 2. The numerical

| | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| *A* | *A* | *A* | *A* | 4 | 5 | 6 | 6 | 5 | 4 | 10 | 11 | 9 | 6 | 10 | 11 | 5 | 10 | 9 | 4 | 11 | 9 |
| *A* | *A* | 2 | 3 | *A* | *A* | 6 | 6 | 2 | 3 | 10 | 8 | 12 | 6 | 10 | 8 | 2 | 10 | 12 | 3 | 8 | 12 |
| *A* | 1 | *A* | 3 | *A* | 5 | *A* | 1 | 5 | 3 | 7 | 11 | 12 | 1 | 7 | 11 | 5 | 7 | 12 | 3 | 11 | 12 |
| *A* | 1 | 2 | *A* | 4 | *A* | *A* | 1 | 2 | 4 | 7 | 8 | 9 | 1 | 7 | 8 | 2 | 7 | 9 | 4 | 8 | 9 |
| *B* | *Y* | *X* | *Y* | *X* | *Z* | *Z* | *B* | *B* | *B* | 10 | 11 | 12 | *X* | 10 | 11 | *Y* | 10 | 12 | *Z* | 11 | 12 |
| *B* | *Y* | 2 | *Y* | 4 | *Z* | *Z* | *B* | 2 | 4 | *B* | *B* | 9 | *X* | *X* | *X* | 2 | *Y* | 9 | 4 | *Z* | 9 |
| *B* | 1 | *X* | 3 | *X* | *Z* | *Z* | 1 | *B* | 3 | *B* | 8 | *B* | 1 | *X* | 8 | *Y* | *Y* | *Y* | 3 | 8 | *Z* |
| *B* | *Y* | *X* | *Y* | *X* | 5 | 6 | 6 | 5 | *B* | 7 | *B* | *B* | 6 | 7 | *X* | 5 | 7 | *Y* | *Z* | *Z* | *Z* |

FIGURE 3

entries represent twelve $K_{3,3}$ graphs, thus after formula (33), this confirms that

$$c(K_{8,22}) = 17. \tag{46}$$

To complete Construction D, we append $3(p-2)$ rows to Figure 3, which can accommodate $7(p-2)$ $K_{3,3}$ graphs, and adjoin a $3p + 2 \times 3(q-7)$ matrix, which, by Construction C, formula (44), can contain

$$p(q-7) + \left[\frac{q-7}{3}\right]$$

non-planar graphs. We thus have a total of

$$17 + 7(p-2) + p(q-7) + \left[\frac{q-7}{3}\right] = pq + \left[\frac{q+2}{3}\right]$$

non-planar graphs, confirming (45) and proving (4).

It remains to prove (6), which we do with the aid of Construction E. We assume that $p \leqq q$ and write $p = 3r + f$, $0 \leqq f \leqq 2$, as before, and $r = 3t + h$, $0 \leqq h \leqq 2$. Figure 4 represents a $K_{29,29}$ graph containing six $K_5$ homeomorphs, denoted by $A$, $B$, $C$, $X$, $Y$, and $Z$, and 82 $K_{3,3}$ graphs, and with formula (37). shows that

$$c(K_{29,29}) = 88. \tag{47}$$

*Construction* E. Figure 5 shows $t$ copies of Figure 4 arranged so that they share the same two last rows and two last columns, which intersect in four unused edges. These $t$ copies occupy the first $27t + 2$ rows and first $27t + 2$ columns. The remainder of the figure consists of a $3(3h+f) \times (27t+2)$ matrix below, and a $(3p+2) \times 3(q-9t)$ matrix on the right, which, by (2), can accommodate

$$9t(3h+f) + \left[\frac{3h+f}{3}\right] \quad \text{and} \quad p(q-9t) + \left[\frac{q-9t}{3}\right]$$

non-planar graphs, respectively. The total number of non-planar graphs is thus $(9t)^2 + (88-81)t + 9t(3h+f) + h + p(q-9t) + s - 3t = pq + r + s + t$, which is the right member of (6).

https://doi.org/10.4153/CJM-1969-121-3 Published online by Cambridge University Press

| | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| — | 58 | 80 | 80 | 80 | 58 | 81 | 81 | 81 | 10 | *X* | *X* | 58 | 49 | 46 | 51 | 51 | 46 | *X* | *X* | 49 | 46 | 49 | 51 | 68 | 68 | 68 | 10 | 10 |
| 56 | 56 | 65 | 55 | 55 | — | 61 | 65 | 61 | *X* | 11 | *X* | 45 | 45 | 65 | 44 | 61 | 44 | *X* | *X* | 56 | 45 | 55 | 44 | 68 | 68 | 68 | 11 | 11 |
| 71 | 58 | 80 | 80 | 80 | 58 | 71 | — | 71 | *X* | *X* | 12 | 58 | 49 | 46 | 51 | 51 | 46 | *X* | *X* | 49 | 46 | 49 | 51 | 68 | 68 | 68 | 12 | 12 |
| 82 | 66 | 66 | 82 | 73 | 73 | — | 73 | 82 | 36 | 36 | 47 | 13 | *Y* | *Y* | 36 | 50 | 47 | 50 | 66 | *Y* | *Y* | 50 | 47 | 69 | 69 | 69 | 13 | 13 |
| 41 | 66 | 66 | — | 62 | 62 | 81 | 81 | 81 | 41 | 41 | 47 | *Y* | 14 | *Y* | 62 | 50 | 47 | 50 | 66 | *Y* | *Y* | 50 | 47 | 69 | 69 | 69 | 14 | 14 |
| 64 | 60 | 60 | 55 | 55 | 67 | 67 | 64 | — | 43 | 64 | 43 | *Y* | *Y* | 15 | 42 | 42 | 60 | 67 | 43 | *Y* | *Y* | 55 | 42 | 69 | 69 | 69 | 15 | 15 |
| 82 | — | 65 | 82 | 59 | 67 | 67 | 65 | 82 | 53 | 52 | 59 | 53 | 53 | 65 | 16 | *Z* | *Z* | 67 | 59 | 52 | 52 | *Z* | *Z* | 70 | 70 | 70 | 16 | 16 |
| 82 | 77 | 77 | 82 | — | 67 | 67 | 77 | 82 | 53 | 52 | 48 | 53 | 53 | 48 | *Z* | 17 | *Z* | 67 | 48 | 52 | 52 | *Z* | *Z* | 70 | 70 | 70 | 17 | 17 |
| 56 | 56 | — | 57 | 59 | 54 | 63 | 57 | 63 | 63 | 54 | 59 | 40 | 57 | 40 | *Z* | *Z* | 18 | 54 | 59 | 56 | 40 | *Z* | *Z* | 70 | 70 | 70 | 18 | 18 |
| 79 | *A* | *A* | 78 | 78 | 78 | 79 | 1 | 79 | 19 | 1 | 12 | 34 | 49 | 37 | 34 | 34 | 39 | 1 | 19 | 49 | 37 | 49 | 39 | 39 | 19 | 37 | 12 | 12 |
| *A* | 2 | *A* | 57 | 62 | 62 | 61 | 57 | 61 | 10 | 20 | 2 | 40 | 57 | 40 | 62 | 61 | 39 | 20 | 2 | 30 | 40 | 30 | 39 | 39 | 30 | 20 | 10 | 10 |
| *A* | *A* | 65 | 55 | 55 | 3 | 61 | 65 | 61 | 3 | 11 | 21 | 45 | 45 | 65 | 44 | 61 | 44 | 29 | 21 | 29 | 45 | 55 | 44 | 29 | 21 | 3 | 11 | 11 |
| 79 | 77 | 77 | 4 | *B* | *B* | 79 | 77 | 79 | 36 | 36 | 38 | 22 | 4 | 15 | 36 | 50 | 39 | 50 | 38 | 4 | 22 | 50 | 39 | 39 | 38 | 22 | 15 | 15 |
| 64 | 60 | 60 | *B* | 5 | *B* | 63 | 64 | 63 | 63 | 64 | 38 | 13 | 23 | 5 | 42 | 42 | 60 | 28 | 38 | 23 | 5 | 28 | 42 | 23 | 38 | 28 | 13 | 13 |
| 6 | 60 | 60 | *B* | *B* | 54 | 81 | 81 | 81 | 43 | 54 | 43 | 6 | 14 | 24 | 51 | 51 | 60 | 54 | 43 | 30 | 24 | 30 | 51 | 6 | 30 | 24 | 14 | 14 |
| 56 | 56 | 80 | 80 | 80 | 54 | 7 | *C* | *C* | 53 | 54 | 38 | 53 | 53 | 37 | 25 | 7 | 18 | 54 | 38 | 56 | 37 | 7 | 25 | 25 | 38 | 37 | 18 | 18 |
| 41 | 58 | 8 | 57 | 59 | 58 | *C* | 57 | *C* | 41 | 41 | 59 | 58 | 57 | 37 | 16 | 26 | 8 | 29 | 59 | 29 | 37 | 26 | 8 | 29 | 26 | 37 | 16 | 16 |
| 75 | 75 | 75 | 78 | 78 | 78 | *C* | *C* | 9 | 35 | 52 | 48 | 35 | 35 | 48 | 9 | 17 | 27 | 28 | 48 | 52 | 52 | 28 | 27 | 27 | 9 | 28 | 17 | 17 |
| *A* | *A* | *A* | 78 | 78 | 78 | 74 | 74 | 74 | 10 | 20 | 21 | 45 | 45 | 32 | 44 | 32 | 44 | 20 | 21 | — | 45 | 32 | 44 | *A* | 21 | 20 | 10 | 10 |
| *A* | *A* | *A* | 76 | 76 | 76 | 74 | 74 | 74 | 19 | 11 | 31 | 34 | 31 | 46 | 34 | 34 | 46 | 28 | 19 | 31 | 46 | 28 | — | *A* | 19 | 28 | 11 | 11 |
| 79 | 77 | 77 | *B* | *B* | *B* | 79 | 77 | 79 | 43 | 33 | 43 | 13 | 23 | 24 | 42 | 42 | 33 | 33 | 43 | 23 | 24 | — | 42 | 23 | *B* | 24 | 13 | 13 |
| 75 | 75 | 75 | *B* | *B* | *B* | 74 | 74 | 74 | 36 | 36 | 47 | 22 | 14 | 32 | 36 | 32 | 47 | 29 | — | 29 | 22 | 32 | 47 | 29 | *B* | 22 | 14 | 14 |
| 41 | 66 | 66 | 76 | 76 | 76 | *C* | *C* | *C* | 41 | 41 | 31 | 40 | 31 | 40 | 16 | 26 | 27 | — | 66 | 31 | 40 | 26 | 27 | 27 | 26 | *C* | 16 | 16 |
| 75 | 75 | 75 | 76 | 76 | 76 | *C* | *C* | *C* | 35 | 33 | 48 | 35 | 35 | 48 | 25 | 17 | 33 | 33 | 48 | 30 | — | 30 | 25 | 25 | 30 | *C* | 17 | 17 |
| 64 | 72 | 72 | 72 | 62 | 62 | 63 | 64 | 63 | 63 | 64 | 31 | 22 | 31 | 15 | 62 | 26 | 27 | *X* | *X* | 31 | 22 | 26 | 27 | 27 | 26 | 22 | 15 | 15 |
| 71 | 72 | 72 | 72 | 73 | 73 | 71 | 73 | 71 | 35 | 20 | 21 | 35 | 35 | 32 | 25 | 32 | 18 | 20 | 21 | *Y* | *Y* | 32 | 25 | 25 | 21 | 20 | 18 | 18 |
| 71 | 72 | 72 | 72 | 73 | 73 | 71 | 73 | 71 | 19 | 33 | 12 | 34 | 23 | 24 | 34 | 34 | 33 | 33 | 19 | 23 | 24 | *Z* | *Z* | 23 | 19 | 24 | 12 | 12 |
| 6 | 2 | 8 | 4 | 5 | 3 | 7 | 1 | 9 | 3 | 1 | 2 | 6 | 4 | 5 | 9 | 7 | 8 | 1 | 2 | 4 | 5 | 7 | 8 | 6 | 9 | 3 | — | — |
| 6 | 2 | 8 | 4 | 5 | 3 | 7 | 1 | 9 | 3 | 1 | 2 | 6 | 4 | 5 | 9 | 7 | 8 | 1 | 2 | 4 | 5 | 7 | 8 | 6 | 9 | 3 | — | — |

FIGURE 4

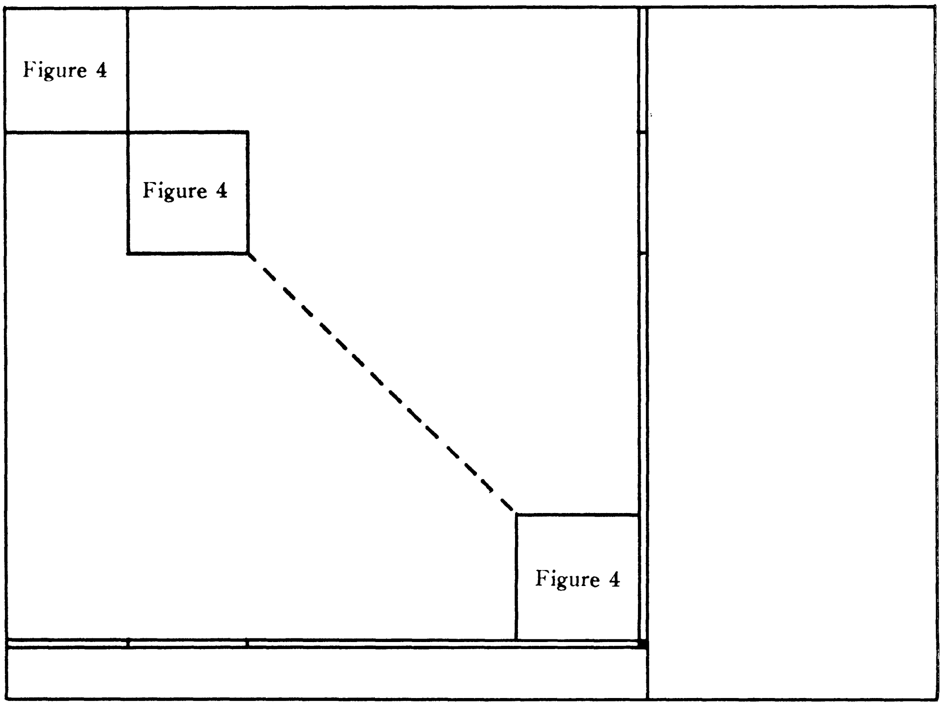


FIGURE 5

## REFERENCES

1. R. K. Guy, *On a coarseness conjecture of Erdős*, J. Combinatorial Theory *3* (1967), 38–42.
2. R. K. Guy and L. W. Beineke, *The coarseness of the complete graph*, Can. J. Math. *20* (1968), 888–894.
3. K. Kuratowski, *Sur le problème des courbes gauches en topologie*, Fund. Math. *15* (1930), 271–283.

*Purdue University,*
*Fort Wayne, Indiana;*
*University of Calgary,*
*Calgary, Alberta*

https://doi.org/10.4153/CJM-1969-121-3 Published online by Cambridge University Press