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Some fixed-point theorems on locally convex linear topological spaces

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Let (E, τ) be a locally convex linear Hausdorff topological space. We have proved mainly the following results.

(i) Let f be nonexpansive on a nonempty τ-sequentially complete, τ-bounded, and starshaped subset M of E and let $(I-f)$ map τ-bounded and τ-sequentially closed subsets of M into τ-sequentially closed subsets of M. Then f has a fixed-point in M.

(ii) Let f be nonexpansive on a nonempty, τ-sequentially compact, and starshaped subset M of E. Then f has a fixed-point in M.

(iii) Let (E, τ) be τ-quasi-complete. Let X be a nonempty, τ-bounded, τ-closed, and convex subset of E and M be a τ-compact subset of X. Let F be a commutative family of nonexpansive mappings on X having the property that for some $f_1 \in F$ and for each $x \in X$, τ-closure of the set

$$\left\{f_1^n(x) : n = 1, 2, \ldots\right\}$$

contains a point of M. Then the family F has a common fixed-point in M.

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Introduction

Some results concerning fixed-point theorems for nonexpansive mappings on linear topological spaces have recently been obtained by Taylor [10] and also Tarafdar [9]. These results hold for nonexpansive mappings on a complete bounded set of a linear topological space. In the first section of this paper we have shown that similar results can be obtained by weakening the completeness condition to sequential completeness. The main tool in this paper will be the Minkowski functional of a balanced, convex (that is, absolutely convex) bounded subset obtained from a given bounded set of locally convex linear topological spaces.

Kakutani [6], Markov [8], and Day [2] have investigated the fixed-point theorems for a commutative family of linear continuous self mappings on a compact convex subset of a linear topological space. DeMarr [3], Belluce and Kirk [1], and others, have considered the fixed-point theorems for a commutative family of nonexpansive mappings (not necessarily linear) on a Banach space. In Section 2 of our paper we have established that the result of Belluce and Kirk [1], which includes that of DeMarr, can be extended to the case of a locally convex linear topological space. As in [1] our proof depends on a lemma of DeMarr and a theorem of Göhde [4]. To suit our requirements we have also extended the above lemma and theorem to the locally convex linear topological space situation. These, particularly the extension of Göhde's Theorem, have their own interest.

1.

Throughout this paper each locally convex linear topological space will be assumed Hausdorff. Let (E, τ) be a locally convex linear topological space. Then a family $[p_\alpha : \alpha \in I]$ of seminorms defined on E is said to be an associated family of seminorms for τ if the family $[\rho U : \rho > 0]$, where $U = \bigcap_{i=1}^{n} U_{\alpha_i}$ and $U_{\alpha_i} = \left\{x : p_{\alpha_i}(x) < 1\right\}$, forms a base of neighbourhoods of 0 for τ. The set U is also given by $U = \{x : p(x) < 1\}$ where $p(x)$ is the seminorm $\max\left[p_{\alpha_1}, p_{\alpha_2}, \ldots, p_{\alpha_n}\right]$.

A family $[p_\alpha : \alpha \in I]$ of seminorms defined on E is said to be an

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augmented associated family for τ if $[p_\alpha : \alpha \in I]$ is an associated family for τ and has the further property that, given $\alpha, \beta \in I$, the seminorm $\max[p_\alpha, p_\beta] \in [p_\alpha : \alpha \in I]$. We shall denote an associated family and an augmented associated family for τ by $A(\tau)$ and $A^*(\tau)$ respectively. It is well known (see [7], p. 203) that given a locally convex linear topological space (E, τ) there always exists a family $[p_\alpha : \alpha \in I]$ of seminorms defined on E such that $[p_\alpha : \alpha \in I] = A^*(\tau)$. Conversely each family $[p_\alpha : \alpha \in J]$ of seminorms defined on E with the property that for each $x \in E$ with $x \neq 0$ there is at least one $\alpha \in I$ such that $p_\alpha(x) \neq 0$ always determines a unique locally convex topology τ on E such that $A(\tau) = [p_\alpha : \alpha \in J]$ and $A(\tau)$ can be extended to $A^*(\tau)$ by adjoining to $A(\tau)$ all seminorms of the form $\max\left[p_{\alpha_1}, p_{\alpha_2}, \ldots, p_{\alpha_n}\right]$ for each finite subset $[\alpha_1, \alpha_2, \ldots, \alpha_n]$ of the index set J.

DEFINITION. Let (E, τ) be a locally convex linear topological space. Then a mapping f of a subset $M \subset E$ into itself is said to be $A(\tau)$- $(A^*(\tau))$-nonexpansive on M if, for all $x, y \in M$,

$$p_\alpha\big(f(x)-f(y)\big) \leq p_\alpha(x-y) \quad \text{for each } p_\alpha \in A(\tau) \quad \big(A^*(\tau)\big) .$$

(For equivalent definitions see [10] and [9].)

It is trivial to see that if f is $A^*(\tau)$-nonexpansive then f is also $A(\tau)$-nonexpansive. It is also true (see [9]) that if f is $A(\tau)$-nonexpansive then f is also $A^*(\tau)$-nonexpansive. Hence, instead of saying that f is $A(\tau)$- or $A^*(\tau)$-nonexpansive, we will simply say that f is nonexpansive in either case.

In what follows the following construction will be crucial. Let M be a τ-bounded set of a locally convex linear topological space (E, τ) and let $A^*(\tau) = [p_\alpha : \alpha \in I]$. Let us consider the family $\{U_\alpha : \alpha \in I\}$ where $U_\alpha = \{x : p_\alpha(x) \leq 1\}$. Then the family $\{U_\alpha : \alpha \in I\}$ is a base of closed absolutely convex neighbourhoods of 0.

Since M is τ-bounded, we can select a number $\lambda_\alpha > 0$ for each $\alpha \in I$ such that $M \subset \lambda_\alpha U_\alpha$. Then clearly $B = \bigcap_\alpha \lambda_\alpha U_\alpha$ is τ-bounded,

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τ-closed and absolutely convex and $M \subset B$. The linear span of B in E is equal to $E_B = \bigcup_{n=1}^{\infty} nB$ and B is an absolutely convex α-body (that is, has an algebraic interior point). The Minkowski functional of B is a norm $\|\cdot\|_B$ on E_B. Thus E_B is a normed space with the norm $\|\cdot\|_B$ and the closed unit ball B. The norm topology on E_B is finer than the topology on E_B induced by τ (for details see [7], p. 252 or [5], pp. 207-208). Now since p_α is the Minkowski functional of U_α and $\|\cdot\|_B$ is the Minkowski functional of B and $B \subset \lambda_\alpha U_\alpha$ we can easily see that, for each $x \in E_B$, $p_\alpha(x) \le \lambda_\alpha \|x\|_B$.

Thus, for each $\alpha \in I$, we have

(1) $$p_\alpha\left(\frac{x}{\lambda_\alpha}\right) \le \|x\|_B .$$

We now prove that

(2) $$\sup_\alpha p_\alpha\left(\frac{x}{\lambda_\alpha}\right) = \|x\|_B \quad \text{for each} \quad x \in E_B .$$

Let $x \in E_B$. We assume that $\sup_\alpha p_\alpha\left(\frac{x}{\lambda_\alpha}\right) < \|x\|_B$ and deduce a contradiction. Let $\sup_\alpha p_\alpha\left(\frac{x}{\lambda_\alpha}\right) = \lambda$. Then we have $p_\alpha\left(\frac{x}{\lambda_\alpha}\right) \le \lambda < \|x\|_B$ for each $\alpha \in I$.

Now $p_\alpha\left(\frac{x}{\lambda_\alpha}\right) \le \lambda$ implies that $\frac{x}{\lambda} \in \lambda_\alpha U_\alpha$ for each $\alpha \in I$; that is, $\frac{x}{\lambda} \in B$. But $\|x\|_B > \lambda$ implies that $x \notin \lambda B$. Thus we have a contradiction.

We are now in a position to prove the following theorem.

THEOREM 1.1. *Let* (E, τ) *be a locally convex linear topological space and* $A^*(\tau) = [p_\alpha : \alpha \in I]$. *If* f *is a nonexpansive mapping on a* τ*-bounded set* $M \subset E$, *then* f *is also nonexpansive on* M *with respect to the norm* $\|\cdot\|_B$ *where* $\|\cdot\|_B$ *has the meaning as explained above.*

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Proof. Let $x, y \in M$. Then since f is nonexpansive on M, $p_\alpha(f(x)-f(y)) \le p_\alpha(x-y)$ for each $\alpha \in I$. Hence

$$\sup_\alpha p_\alpha\left(\frac{f(x)-f(y)}{\lambda_\alpha}\right) \le \sup_\alpha p_\alpha\left(\frac{x-y}{\lambda_\alpha}\right) .$$

Thus $\|f(x)-f(y)\|_B \le \|x-y\|_B$ from (2).

DEFINITION. A subset X of E is called starshaped if there exists a point $p \in X$ such that for each $x \in X$ and real t with $0 < t < 1$, $tx + (1-t)p \in X$. p is called a star centre of X. Each convex subset of E is thus starshaped.

The following result, with M assumed to be complete, is known (see [10] and [9]). Here we have relaxed the completeness condition by sequential completeness.

THEOREM 1.2. *Let* (E, τ) *be a locally convex linear topological space and* $A^*(\tau) = [p_\alpha : \alpha \in I]$. *Let* M *be a nonempty, starshaped,* τ*-bounded, and* τ*-sequentially complete subset of* E, *and* f *be a nonexpansive mapping on* M. *Then* 0 *lies in* $\|\cdot\|_B - \mathrm{cl}(I-f)M$ *and hence in* $\tau - \mathrm{cl}(I-f)M$ *where* I *is the identity map on* M, $\mathrm{cl}\, A$ *stands for the closure of a subset* A *of* E *and* $\|\cdot\|_B$ *has the meaning as explained earlier.*

Proof. We have already mentioned that E_B is a normed space with the norm $\|\cdot\|_B$ and with B as the unit ball. Since the norm topology on E_B has a base of neighbourhoods of 0 consisting of τ-closed sets, namely the scalar multiples of B and M is τ-sequentially complete, we know that M is a $\|\cdot\|_B$-sequentially complete subset of E_B (apply 18, 4.4 (b) of [7] to the topology on E_B induced by τ and the $\|\cdot\|_B$-topology on E_B). Let p be the star centre of M. For each t, $0 < t < 1$, we define

$$f_t(x) = tf(x) + (1-t)p , \quad x \in M .$$

Then clearly f_t maps M into itself. Moreover,

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$$\|f_t(x)-f_t(y)\|_B = \|t(f(x)-f(y))\|_B \le t\|x-y\|_B$$

for all $x, y \in M$ as f is nonexpansive on M with respect to the norm $\|\cdot\|_B$ by Theorem 1.1. Thus f_t is a contraction on M with respect to the norm $\|\cdot\|_B$. Now since M is $\|\cdot\|_B$-complete, by Banach's contraction mapping principle, f_t has a unique fixed-point x_t, say, in M. By the definition of f_t, we have

$$(I-f)(x_t) = x_t - \tfrac{1}{t}(f_t(x_t)-(1-t)p) = \left(1 - \tfrac{1}{t}\right)(x_t-p) .$$

Hence

$$\|(I-f)(x_t)\|_B \le \left|1 - \tfrac{1}{t}\right|\left(\|x_t\|_B+\|p\|_B\right) \le 2\left|1 - \tfrac{1}{t}\right| \to 0 \text{ as } t \to 1 ,$$

because x_t and p are in the unit ball of E_B. Thus $0 \in \|\cdot\|_B - \mathrm{cl}(I-f)M \subset \tau - \mathrm{cl}(I-f)M$. The last inclusion follows from the fact that the $\|\cdot\|_B$-topology on E_B is finer than the topology induced on E_B by τ.

REMARK. This theorem includes Theorem 2.2 in [10] (also Lemma 3.1 in [9]) when E is a locally convex linear topological space. Also we note that here we have obtained a stronger result under a weaker hypothesis.

COROLLARY 1.1. *Let* (E, τ) *be a locally convex linear topological space and* $A^*(\tau) = [p_\alpha : \alpha \in I]$. *Let* f *be nonexpansive on a nonempty,* τ*-sequentially complete,* τ*-bounded, and starshaped subset of* E, *and let* $(I-f)$ *map* τ*-bounded and* τ*-sequentially closed subsets of* M *into* τ*-sequentially closed subsets of* M. *Then* f *has a fixed point in* M.

A point $p \in \tau - \mathrm{cl}\, M$ is a τ-sequential limit point of M if there exists a sequence $\{p_n\}$, $p_n \in M$, such that $p_n \to p$ in the τ-topology. M is called τ-sequentially closed if each τ-sequential limit point of M belongs to M.

Proof. Since M is τ-sequentially complete and E is Hausdorff, it follows that M is τ-sequentially closed. (Let $p_n \to p$ in the τ-topology and $p_n \in M$. Then $\{p_n\}$ is a τ-Cauchy sequence and, therefore, $p \in M$.) Hence, by hypothesis $(I-f)M$ is τ-sequentially closed. By Theorem 1.2, $0 \in \|\cdot\|_B - \mathrm{cl}(I-f)M$. But $\|\cdot\|_B - \mathrm{cl}(I-f)M \subset \tau\text{-sequential-cl}(I-f)M$ because it follows that each point in $\|\cdot\|_B - \mathrm{cl}(I-f)M$ is a τ-sequential limit point of $(I-f)M$ as a $\|\cdot\|_B$-topology is finer than the τ-topology. Hence $0 \in (I-f)M$. This completes the proof.

COROLLARY 1.2. *Let* (E, τ) *be a locally convex linear topological space and* $A^*(\tau) = [p_\alpha : \alpha \in I]$. *Let* f *be nonexpansive on a nonempty,* τ-*sequentially compact, and starshaped subset* M *of* E . *Then* f *has a fixed-point in* M .

Proof. M being τ-sequentially compact is τ-bounded and τ-sequentially complete. Hence, by Theorem 1.2 and by the reason given in Corollary 1.1,

$$0 \in \|\cdot\|_B - \mathrm{cl}(I-f)M \subset \tau\text{-sequential-cl}(I-f)M .$$

Thus there exists a sequence $\{y_n\}$, $y_n \in (I-f)M$, such that $y_n \to 0$ in the τ-topology. Now since f is nonexpansive on M , it follows that f is p_α-continuous for each $\alpha \in I$. Hence f is τ-continuous and, therefore, $(I-f)$ is τ-continuous. Then it follows that $(I-f)M$ is sequentially compact as M is. Now it is easy to see that $0 \in (I-f)M$. This completes the proof.

2.

Before we prove the main result (Theorem 2.1) of this section we need to prove two lemmas. The following result, which we write as a lemma, was proved by Göhde ([4], Theorem 5) in a normed space. We extend this to a locally convex linear topological space and also weaken the convexity hypothesis to the starshaped convexity.

LEMMA 2.1. *Let* (E, τ) *be a locally convex linear topological space and* $A^*(\tau) = [p_\alpha : \alpha \in I]$. *Let* f *be nonexpansive on a nonempty,* τ-*closed,* τ-*bounded, and starshaped subset* M *of* E . *Further assume that there exists a* τ-*compact subset* L *of* M *such that, for each* $x \in M$,

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$$\tau - \mathrm{cl}\{f^n(x) : n = 1, 2, \ldots\} \cap L \neq \emptyset .$$

Then there exists at least one fixed-point of f *in* L .

Proof. Let p be the star centre of M . For each t , $0 < t < 1$, we define

$$f_t(x) = tf(x) + (1-t)p , \quad x \in M .$$

Then, in exactly the same way as in the proof of Theorem 1.2, we can show that f_t is a contraction on M with respect to the norm $\|\cdot\|_B$ where $\|\cdot\|_B$ has the meaning as explained in the beginning of Section 1.

For any $x \in M$, $\left\{f_t^n(x)\right\}$ is a $\|\cdot\|_B$-Cauchy sequence and there are points in M which are displaced by f_t with respect to $\|\cdot\|_B$ by an arbitrary small amount. Let

$$\|f_t(x_t) - x_t\|_B \le (1-t) .$$

Then we have

$$\begin{aligned}\|f(x_t) - x_t\|_B &= \|f(x_t) - \{tf(x_t) + (1-t)p\} + f_t(x_t) - x_t\|_B \\ &\le \|f(x_t) - tf(x_t)\|_B + (1-t)\|p\|_B + \|f_t(x_t) - x_t\|_B \\ &\le (1-t)\left(\|f(x_t)\|_B + \|p\|_B + 1\right) \\ &\le 3(1-t) ,\end{aligned}$$

as $f(x_t)$ and p are in the unit ball of E_B . Thus there are points in M which will be displaced by f (with respect to $\|\cdot\|_B$) by an arbitrary small amount.

By the above inequality and the $\|\cdot\|_B$-nonexpansion of f on M (due to Theorem 1.1) we have that, for each positive integer n ,

$$\left\|f^{n+1}(x_t) - f^n(x_t)\right\| \le 3(1-t) .$$

Thus, from (1) of Section 1, we have

$$p_\alpha\left(\frac{f^{n+1}(x_t) - f^n(x_t)}{\lambda_\alpha}\right) \le 3(1-t) \quad \text{for each } \alpha \in I . \tag{3}$$

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Now, by our hypothesis, there is a $y_t \in L$ such that y_t is a τ-limit point of $\left\{f^n(x_t)\right\}$. Clearly y_t is also a p_α-limit point of $\left\{f^n(x_t)\right\}$ for each $\alpha \in I$. Now let $\alpha \in I$ be arbitrary. Then, since y_t is a p_α-limit point of $\left\{f^n(x_t)\right\}$, for any $\varepsilon > 0$, there is a positive integer n such that

$$p_\alpha\left(f^n(x_t)-y_t\right) < \varepsilon . \tag{4}$$

Let ε be arbitrarily chosen. Then since $\lambda_\alpha > 0$, there is a positive integer m such that

$$p_\alpha\left(f^m(x_t)-y_t\right) < \lambda_\alpha\varepsilon . \tag{5}$$

From (3), (5), and the p_α-nonexpansion of f, we have

$$p_\alpha\left(\frac{f(y_t)-yt}{\lambda_\alpha}\right) \le p_\alpha\left(\frac{f(y_t)-f^{m+1}(x_t)}{\lambda_\alpha}\right) + p_\alpha\left(\frac{f^{m+1}(x_t)-f^m(x_t)}{\lambda_\alpha}\right) + p_\alpha\left(\frac{f^m(x_t)-y_t}{\lambda_\alpha}\right) < \varepsilon + 3(1-t) + \varepsilon .$$

Since ε is arbitrary, we must have

$$p_\alpha\left(\frac{f(y_t)-y_t}{\lambda_\alpha}\right) \le 3(1-t) . \tag{6}$$

Now we consider a sequence $\{t_i\}$ of real numbers such that $0 < t_i < 1$ for each i and $\lim\limits_{i\to\infty} t_i = 1$.

As L is τ-compact, the sequence $\left\{y_{t_i}\right\}$ has a τ-cluster point y in L. Clearly y is also a p_α-cluster points of $\left\{y_{t_i}\right\}$ and hence we can select a subsequence $\left\{y_{t_{n_i}}\right\}$ of $\left\{y_{t_i}\right\}$ such that $y_{t_{n_i}} \to y$ as the p_α-topology satisfies the first axiom of countability. In view of (6) we have

$$\lim_{i\to\infty} p_\alpha\left(f\left(y_{t_{n_i}}\right)-y_{t_{n_i}}\right) \le \lim_{i\to\infty} 3\lambda_\alpha\left(1-t_{n_i}\right) = 0 .$$

Again, since f is p_α-nonexpansive on M, it follows that f is p_α-continuous on M. Hence $f\left(y_{t_{n_i}}\right) \to f(y)$ in the p_α-topology; that is, $\lim_{i\to\infty} p_\alpha\left(f\left(y_{t_{n_i}}\right)-f(y)\right) = 0$.

We now have

$$p_\alpha(f(y)-y) \le p_\alpha\left(f(y)-f\left(y_{t_{n_i}}\right)\right) + p_\alpha\left(f\left(y_{t_{n_i}}\right)-y_{t_{n_i}}\right) + p_\alpha\left(y_{t_{n_i}}-y\right) ,$$

where $i = 1, 2, \ldots$. Taking the limit as $i \to \infty$, we have

$$p_\alpha(f(y)-y) = 0 .$$

Since α is arbitrary, $p_\alpha(f(y)-y) = 0$ for each $\alpha \in I$. Again, since E is Hausdorff, $f(y) = y$. This completes the proof.

REMARK. In proving the above theorem, if we start at the outset with an arbitrary $\alpha \in I$, then it is true that f_t and $\{f_t(x)\}$ are respectively a contraction and a Cauchy sequence with respect to the semi-norm p_α. But then x_t will depend on α and hence the technique of Göhde's applied α-wise does not work. Thus it seems that the use of $\|\cdot\|_B$, as made in the above proof, is appropriate.

The next lemma was proved by DeMarr [3] in a Banach space.

LEMMA 2.2. *Let (X, τ) be a locally convex linear topological space and $A^*(\tau) = [p_\alpha : \alpha \in I]$. Let M be a nonempty, τ-compact subset of X and K the convex hull of M. If, for any $\beta \in I$, the p_β-diameter $\partial(M, \beta)$ of M is greater than 0, then there exists an element $u \in K$ such that*

$$\sup\{p_\beta(x-u) : x \in M\} < \partial(M, \beta) .$$

Proof. The proof of [3], with slight adjustment, will do. Since M is τ-compact, M is p_α-compact for each $\alpha \in I$. Thus there exist points x_0, x_1 in M such that $p_\beta(x_0-x_1) = \partial(M, \beta)$. Let $M_\beta \subset M$ be the maximal so that $\{x_0, x_1\} \subset M_\beta$ and $p_\beta(x-y) = \partial(M, \beta)$ for all distinct $x, y \in M_\beta$. M_β is clearly nonempty. Also, since M is p_β-compact, it follows that M_β is finite. Let $M_\beta = \{x_0, x_1, \ldots, x_n\}$. We define

$$u = \sum_{k=0}^{n} \frac{1}{n+1} x_k \in K .$$

Since M is p_β-compact, there exists a point $y_0 \in M$ such that $p_\beta(y_0-u) = \sup\{p_\beta(x-u) : x \in M\}$. Again, since $p_\beta(y_0-x_k) \le \partial(M, \beta)$ for all $k = 0, 1, \ldots, n$, we have

$$p_\beta(y_0-u) \le \sum_{k=0}^{n} \frac{1}{n+1} p_\beta(y_0-x_k) \le \partial(M, \beta) .$$

Now $p_\beta(y_0-u) = \partial(M, \beta)$ would imply that $p_\beta(y_0-x_k) = \partial(M, \beta) > 0$ for all $k = 0, 1, \ldots, n$. But this would then imply, by definition of M_β, that $y_0 \in M_\beta$; that is, $y_0 = x_k$ for some $k = 0, 1, \ldots, n$, which would contradict that $p_\beta(y_0-x_k) = \partial(M, \beta) > 0$ for all $k = 0, 1, \ldots, n$. Hence $p_\beta(y_0-u) < \partial(M, \beta)$. This completes the proof.

REMARK. In [3] it is assumed that K is closed. This is extraneous. We now state and prove our main theorem of this section.

THEOREM 2.1. *Let (E, τ) be a quasi-complete locally convex linear topological space and $A^*(\tau) = [p_\alpha : \alpha \in I]$. Let X be a nonempty, τ-bounded, τ-closed and convex subset of E and M be a τ-compact subset of X. Let F be a nonempty commutative family of nonexpansive mappings on X having the property that for some $f_1 \in F$ and for each $x \in X$,*

$$\tau - \mathrm{cl}\left\{f_1^n(x) : n = 1, 2, \ldots\right\} \cap M \neq 0 .$$

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Then the family F *has a common fixed point in* M .

Proof. The proof proceeds in the general line of argument of the proof of Theorem 1 in [1]. Let K be a nonempty, τ-closed, and convex subset of X such that $f(K) \subseteq K$ for each $f \in F$. Let $x \in K$. Then $\left\{f_1^n(x)\right\} \subseteq K$. By hypothesis we have $K \cap M \supset \tau - \mathrm{cl}\left\{f_1^n(x)\right\} \cap M \neq \emptyset$.

Applying Zorn's Lemma we obtain a subset X^* of X which is minimal with respect to being nonempty, τ-closed, convex, and being mapped into itself by each $f \in F$. We set $M^* = X^* \cap M$. $M^* \neq \emptyset$ by the above inclusion relation. By our Lemma 2.1 it follows that f_1 has a nonempty τ-closed fixed-point set H in M^* . Now using commutativity of F and proceeding exactly as in [1] we can find a subset H^* of H which is minimal with respect to being nonempty, τ-closed, and mapped into itself by each $f \in F$. Let $g \in F$. Then g , being nonexpansive on X , is p_α-continuous for each $\alpha \in I$ and hence τ-continuous on X . Therefore $g(H^*)$ is τ-closed as H^* is τ-compact. Now for each $f \in F$, $f\big(g(H^*)\big) = g\big(f(H^*)\big) \subset g(H^*)$. Hence the minimality of H^* implies that $g(H^*) = H^*$. Hence H^* is mapped onto itself by each $f \in F$.

Let W be the convex τ-closure of H^* . Then W is τ-compact, as H^* is so, and E is quasi-complete. We now prove that $\partial(W, \alpha) = 0$ for each $\alpha \in I$ where $\partial(W, \alpha)$ is the p_α-diameter of W . We assume that $\partial(W, \beta) > 0$ for some $\beta \in I$ and deduce a contradiction. Then, by applying our Lemma 2.2 to the compact set W , there is a point $x \in W$ such that

$$\sup\{p_\beta(x \quad z) : z \in W\} = r < \partial(W, \beta) \quad .$$

As in [1] we set

$$C_1^\beta = \{w \in W : p_\beta(w-z) \leq r \quad \text{for all} \quad z \in H^*\}$$

and

$$C_2^\beta = \{w \in X^* : p_\beta(w-z) \leq r \quad \text{for all} \quad z \in H^*\} \ .$$

Then $C_1^\beta = C_2^\beta \cap W$. Since $f(H^*) = H^*$ for each $f \in F$, by using

p_β-nonexpansion of each $f \in F$, we can show that $f\left(C_2^\beta\right) \subset C_2^\beta$. Clearly C_2^β is nonempty and convex. Also C_2^β is τ-closed. (For let y be a τ-limit point of C_2^β. Then since X^* is τ-closed, $y \in X^*$. Also y being a τ-limit point of C_2^β is a p_β-limit point of C_2^β. Let ε be arbitrarily given. Then there exists a $w \in C_2^\beta$ such that $p_\beta(y-w) < \varepsilon$. Now for any $z \in H^*$, $p_\beta(y-z) \le p_\beta(y-w) + p_\beta(w-z) < \varepsilon + r$. Since ε is arbitrary, $p_\beta(y-z) \le r$. Hence $y \in C_2^\beta$.) Hence $C_2^\beta = X^*$ by the minimality of X^*. Thus we obtain that $C_1^\beta = W$. Let W' be the convex p_β-closure of H^*. Then we have $\partial(W, \beta) \le \partial(W', \beta) = \partial(H^*, \beta)$ as $W \subset W'$, each τ-limit point of a set being also a p_β-limit point of the set. Hence there must be points u and v in H^* such that $p_\beta(u-v) > r$. But since $H^* \subset W = C_1^\beta$, $p_\beta(u-v) \le r$. Thus we obtain a contradiction. Hence $\partial(W, \alpha) = 0$ for each $\alpha \in I$. Since E is Hausdorff, this implies that H^* consists of a single point which must be a fixed point of each $f \in F$. This completes the proof.

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