1 Examples and Basic Folds

SE&C constructions are much easier to analyze than origami constructions. The main reason for this is because it is easy to classify all the possible operations that can be performed with the tools of SE&C.

1: Given two points, we can use the straightedge to draw a line connecting them.
2: Given a point p and a length r, we can use the compass to draw a circle with radius r centered at the point p.
3: We can locate the points of intersection between combinations of circles and lines.

Studies of these basic SE&C operations can be found in a number of classic geometry texts; see (Courant and Robbins, 1941; Martin, 1998), for example. The idea is to show that, given a line segment of unit length, we can construct segments with the length of any rational number $a/b \in \mathbb{Q}$ as well as any expression involving rational numbers and the operation of taking square roots. Proving that these are the **only** kinds of lengths that can be constructed with SE&C requires carefully considering the kinds of equations we obtain when locating the intersection points of two circles, a circle and a line, or two lines, and then proving that repeated use of such intersections gives us the smallest field extension of the rationals that is closed under square roots. See (Cox, 2004, Section 10.1).

Performing such an analysis on straight-crease, single-fold origami is more perplexing because the tools we have are more flexible. There are many different ways in which we can determine a crease when folding, say, a square sheet of paper, especially if there are pre-creases already made in the paper. Classifying all the basic operations of origami and proving that there cannot be any more has been a controversial topic, let alone studying the algebra of such folds.

Before trying to formulate a list of basic origami operations, we will first familiarize ourselves with paper folding's variety by way of some construction examples. The majority of readers will not have seen these types of explicit geometric paper folding methods before, and exposure to such examples can be a big intuition-builder before undertaking more abstract analysis. Plus, they're fun. Readers are encouraged to try them.

1.1 Constructing an Equilateral Triangle

The following challenge appeared in *Mathematics Magazine* (Vol. 67, No. 2, April 1994, p. 123):

https://doi.org/10.1017/9781108778633.003 Published online by Cambridge University Press

Diversion 1.1 Starting with a square sheet of paper, fold it to produce a square having three-fourths its area. Only five folds are allowed.

The puzzle is referenced as coming from a book called *Mathematical Brain Benders* by Stephen Barr (1982). This puzzle is especially fun for origami practicioners who immediately conjecture that they can do it in fewer than five folds.

This challenge is similar to the following: Starting with a square piece of paper, fold it into a perfect equilateral triangle. To accomplish this, one would need to construct a 60° angle, which could be done by folding the sides of a 30°-60°-90° triangle in the square. In other words, we would need to construct line segments of length 1, 2, and $\sqrt{3}$ in our paper. It is standard, however, to always assume that our starting square has side length 1, so it would be more feasible to create a 30°-60°-90° triangle with side lengths 1/2, 1, and $\sqrt{3}/2$. Constructing $\sqrt{3}/2$ is exactly what we would need for the 3/4-area puzzle as well.

There are many ways to fold a 30°-60°-90° triangle in a square. In fact, it is not hard to find explicit methods for doing this in origami instruction books, especially books on modular origami (like (Fuse, 1990)), although such books usually do not mention that they are performing such a construction.

Figure 1.1 shows a standard method for producing such a 30°-60°-90° triangle. This can be shown synthetically, or we can merely note that if the square has side length 1, then PB must also have length 1, since it is the image of a side of the square under the fold. By symmetry, AP must also have length 1, and thus we have that the points APB form an equilateral triangle. (This immediately gives us that the fold made in Figure 1.1 produced the desired angles.)

A variation on this challenge is to fold an equilateral triangle of maximum area within our square piece of paper. Utilizing analytic techniques to discover what triangle orientation gives the maximal area can be a good exercise for calculus students (see (Hull, 2012)), but developing a folding method is another matter. Figure 1.2 shows the standard method of doing this as presented by Emily Gingras (Merrimack College class of 2003). The first picture is her "proof without words" that the angle θ shown is 15°, which proves that the other pictures give the proper equilateral triangle.

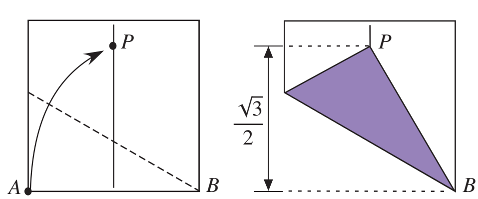

Figure 1.1 Producing a 30°-60°-90° triangle: First fold the square in half and unfold. Then fold the lower left corner up to the crease line, while making the crease go through the lower right corner.

https://doi.org/10.1017/9781108778633.003 Published online by Cambridge University Press

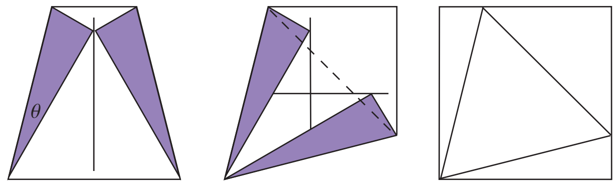

Figure 1.2 A "proof without words" for constructing the maximal equilateral triangle.

Both of these methods involve constructing a line segment whose length is an expression involving square roots, $\sqrt{3}/2$ in the first case and $2/\sqrt{2+\sqrt{3}}$ in the second. What kind of folding operation produced these lengths? In both cases we had a point being folded to a line (point A being folded to the half-way crease in Figure 1.1) where we also make sure that the crease passes through a second point (point B in Figure 1.1). This operation will be explored further in Chapter 2.

As an extra challenge, readers can try to use the method from Figure 1.2 to discover the classic method that paper folders use to fold a square into a regular hexagon with maximal area.

1.2 Dividing a Segment into 1/*n*ths

The problem of dividing the side of a piece of paper into n equal lengths is one which has been a favorite of origami geometry enthusiasts. References to it and various solutions for the cases where $n = 3$, 5, or 7 can be found in some origami books (see (Kasahara and Takahama, 1987; Kasahara, 1988)), on origami email lists, and on a variety of webpages. The challenge, of course, are the cases when n is odd, since folding lengths in half is simple and can generate all even numbers once the odds have been handled.

Such division methods have practical applications in origami as well. Many origami models start off by asking the folder to first divide the square into thirds or into a 5×5 grid. Interestingly, the most common method used by origamists to fold thirds is to use the method shown in Figure 1.3. The idea is to "eyeball" it by curving the paper into an S shape and easing the creases into their proper places. With practice this can

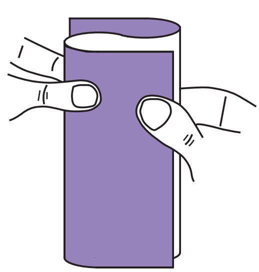

Figure 1.3 Folding thirds, the multifold method.

https://doi.org/10.1017/9781108778633.003 Published online by Cambridge University Press

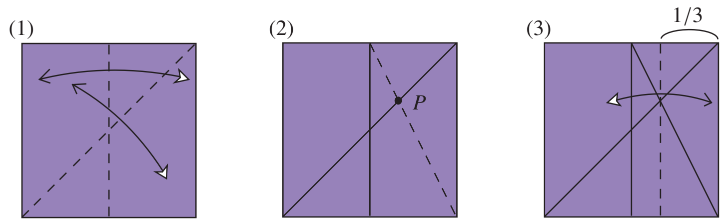

Figure 1.4 Folding thirds exactly. (1) Crease a diagonal and the 1/2 vertical crease. (2) Make a crease that connects the midpoint of the top edge and the lower right corner. Let this crease intersect the diagonal at P. (3) Make a crease at P perpendicular to the bottom and top sides. Then this last crease will be 1/3 from the right side.

be done very quickly and accurately, but it violates our rule of one fold at a time since it requires making two creases simultaneously (which is called a 2-fold or a multifold; these will be discussed in Chapter 4).

A mathematically precise, one-fold-at-a-time way to fold a square of paper into thirds is shown in Figure 1.4. While the origins of this method are unclear, Lang (1988) refers to it as the **crossing diagonals method**. The correctness of this method can be proven by similar triangles or by noticing that the point P is at the intersection of the lines $y = x$ and $y = -2x + 2$, where we assume that the square has side length 1 and lower left corner is at the origin. Thus $P = (2/3, 2/3)$.

This method can be generalized for arbitrary odd values of $n = 2k + 1$. Instead of making a vertical crease at the line $x = 1/2$, make it at $x = (2k - 1)/(2k)$. This is feasible because any odd factor of $2k$ is a smaller odd number than n. Thus, by induction, we can assume that dividing the side of our square into $1/(2k)$ths can be done. Then, using the same method as the 1/3 case, our point P would be at the intersection of the lines $y = x$ and $y = -2kx + 2k$, so $P = (2k/(2k + 1), 2k/(2k + 1))$, giving us a landmark for dividing the side into $1/n$ths.

The crossing diagonals method is sometimes used by origami designers; see John Montroll's Chess Board (Montroll, 1993), for example. However, since the method requires several crease lines to be made across the paper, it isn't viewed as ideal. A better method in this regard is based on **Haga's Theorem** (Kasahara and Takahama, 1987), which states that if we fold a corner of a square (or rectangular) sheet of paper to a point on a nonadjacent side, then several similar triangles can be found and the resulting crease can mark the sides of the paper at interesting lengths.

In particular, if we mark a point at $(1/(2k), 1)$ on the square and fold the lower left corner (the origin) to this point, as seen in Figure 1.5, then triangles A and B are similar. Also, triangle A is a right triangle and one leg, x, and the hypothenuse make up a side of the square, so the hypothenuse is $1 - x$. The Pythagorean Theorem then gives us that $x = (2k + 1)(2k - 1)/(8k^2)$. Letting the short leg of triangle B be y, the similarity relation gives us

$$\frac{y}{1/(2k)} = \frac{1 - 1/(2k)}{(2k + 1)(2k - 1)/(8k^2)},$$

https://doi.org/10.1017/9781108778633.003 Published online by Cambridge University Press

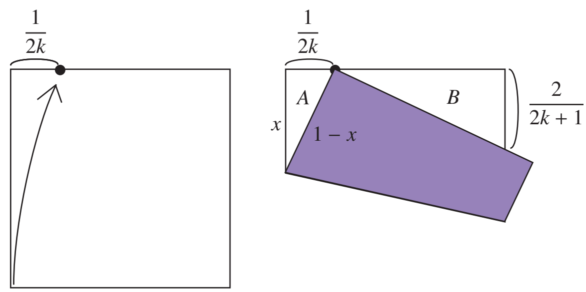

Figure 1.5 Haga's Theorem applied to the odd division problem.

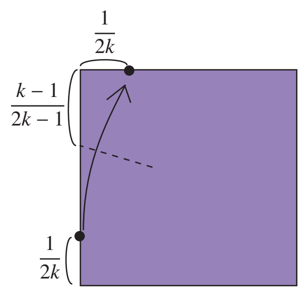

Figure 1.6 Noma's method.

which simplifies, amazingly enough, to $y = 2/(2k+1)$. Thus if divisions of $1/n = 1/(2k+1)$ are desired, constructing $1/(2k)$ and the one fold of Haga's Theorem will do the trick.

Diversion 1.2 (Geretschläger, 2002, 2008) Prove that the perimeter of triangle B in Figure 1.5 is always half the perimeter of the original square.

Haga's Theorem contains many other geometric morsels. See (Husimi and Husimi, 1979; Haga, 2002; Geretschläger, 2002) for more information.

However, it is possible to make any $1/n$ divisions along the side of a square without folding any creases all the way across the paper. The idea is to perform folds that only require making pinch marks on the perimeter of the paper. This would clearly be attractive for origami designers, making it possible to create any a/b mark on the perimeter without marring the paper's interior with extraneous creases.

Masamichi Noma (1992) developed such a method, and it is summarized in Figure 1.6. The idea is, if divisions of $1/n = 1/(2k-1)$ are the goal, to make pinch marks at length $1/(2k)$ on the left side of the top edge of the square and at the bottom side of the left edge. This gives us two marked points on the paper's perimeter. If we fold these two points together, we can pinch the paper only on the left side, so as to avoid

https://doi.org/10.1017/9781108778633.003 Published online by Cambridge University Press

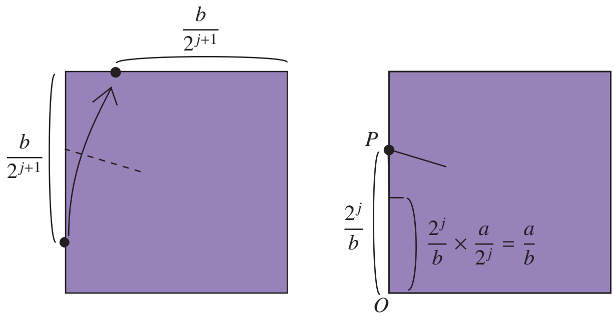

Figure 1.7 Noma's method used to construct a/b.

making a crease all the way across the square. This crease will intersect the left edge $(k-1)/(2k-1)$ from the top corner.

Diversion 1.3 Prove that Noma's method works.

Robert J. Lang has synthesized Noma's method, among others, to generate algorithms for producing folding sequences of pinch marks to create any rational length divisions. In (Lang, 2003) he suggests the following to apply Noma's method to create an arbitrary rational length a/b for integers $a < b$:

(1) Let 2^j be the largest power of 2 smaller than b.
(2) Construct lengths $b/2^{j+1}$ along the top and left sides of the square, as shown in Figure 1.7. (This is easy since the denomenators are just powers of 2.)
(3) Bring these two points together to make a crease pinch along the left side at point P.
(4) Then length OP (O being the lower left corner) will be $2^j/b$. (The same work needed in Figure 1.6 shows this.)
(5) Divide segment OP into $1/2^j$ths (which is easy). Taking a of these from O gives a length $(2^j/b)(a/2^j) = a/b$.

In all three of these division methods, none of the basic folding operations used are very complex. Each case involved only the "moves" of folding a crease between two existing points or folding one point onto another point. This is hardly surprising because only rational lengths were being constructed. But the variety and ingenuity of these methods are nonetheless a marvel.

1.3 Trisecting an Angle

The hallmark of origami geometric constructions has been the fact that paper folding can, fairly easily, trisect angles. The first known method for doing this was created by

https://doi.org/10.1017/9781108778633.003 Published online by Cambridge University Press

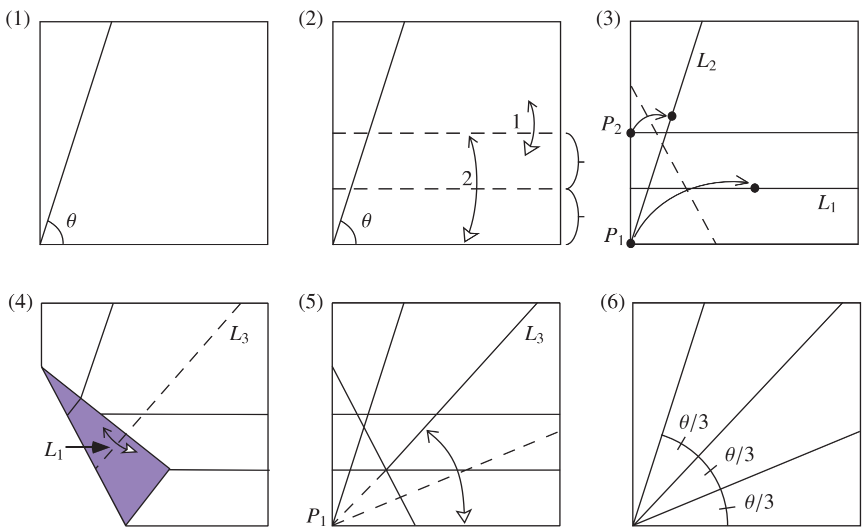

Figure 1.8 Abe's angle trisection method.

Hisashi Abe (Husimi, 1980) sometime in the late 1970s. His method for trisecting an arbitrary acute angle θ is shown in Figure 1.8 and proceeds as follows:

(1) Position your angle θ in the lower left corner of the square, as shown.
(2) Make a horizontal crease, labeled 1 in the figure, parallel to the bottom edge, and then fold and unfold the bottom edge to this crease line (labeled 2). Line 1 can be made at any height, although if θ is less than 45° then crease 1 might need to be closer to the bottom edge for the next step to be possible.
(3) Then, using the labeling in the figure, fold the corner P_1 onto L_1 **while at the same time** making point P_2 land on line L_2. This will require curling the paper over, lining up these two points onto their lines, and then pressing the crease flat.
(4) Leaving this last crease folded, you'll see part of L_1 reflected on this flap of paper. Refold this crease, extending it through the rest of the paper to crease line L_3.
(5) Unfold step (3) and extend the left side of L_3 – it will hit the corner P_1. Then fold the bottom side of the square to L_3 to bisect the angle L_3 makes with the bottom.
(6) Voilà! The angle θ has been trisected.

One way to prove that Abe's method works is shown in Figure 1.9. First, we need to establish that when crease L_3 is extended in step (5) of Figure 1.8, it will intersect point P_1. If we let F be the point where L_3 intersects the crease line from step (3), and draw the segment P_1F on the unfolded paper, then the acute angles between (the unextended) L_3 and L_1 and between P_1F and L_1 are equal, since the fold in step (3) superimposes them. (This is angle α in Figure 1.9(a).) Thus this angle acts as a vertical angle, and P_1F and L_3 must form a straight line.

https://doi.org/10.1017/9781108778633.003 Published online by Cambridge University Press

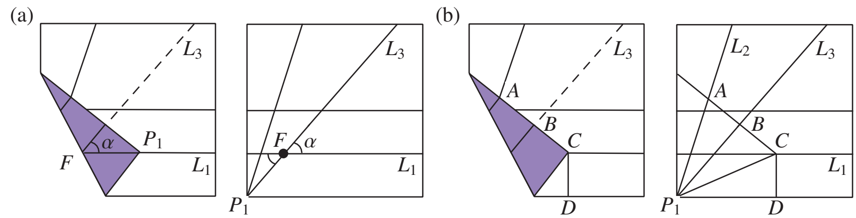

Figure 1.9 Proof of Abe's trisection.

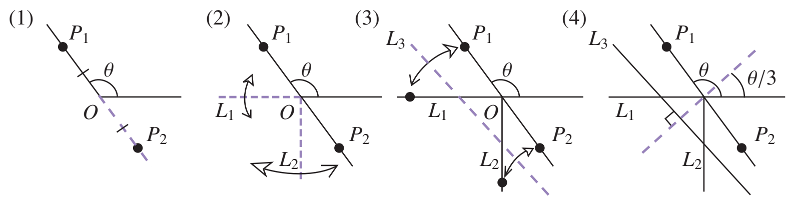

Figure 1.10 Justin's angle trisection method.

Then we can label points A, B, C, and D as in Figure 1.9(b), where C, A, and B are the images of P_1, P_2, and the point in between, respectively, under the step (3) trisection fold. (For D we drop a perpendicular to the bottom of the square.) This gives us that the lengths AB, BC, and CD are all congruent, and thus $\triangle ABP_1$, $\triangle BCP_1$, and $\triangle CDP_1$ are congruent right triangles, giving us the trisection.

While Abe's method as shown in Figure 1.8 only works for acute θ, readers are encouraged to explore how it can be extended for obtuse angles.

By now readers will have probably noticed the unusual folding step in this method that seems to give us the trisection, namely step (3) in Figure 1.8. In this step we have P_1 being folded to a line, which by itself is similar to what we had to do when constructing equilateral triangles in Section 1.1. But one point going to one line is not enough to uniquely determine a crease, and for step (3) we choose to nail down where P_1 will go on L_1 by also requiring P_2 to fold onto line L_2. This "two points folding onto two lines" origami operation is a move that is rarely seen in origami instructions, but it turns out to be the key that gives paper folding more muscle than SE&C constructions. (We'll see exactly why in Chapter 2.) In fact, any straight-crease, single-fold origami construction that goes beyond the constructible range of SE&C will require a move such as this.

Independently of Abe, the French mathematician Jacques Justin (1984) also developed an angle trisection method at roughly the same time. (See also (Justin, 1986b).) Justin's method, shown in Figure 1.10, allows the starting angle θ to be obtuse and positioned in the interior of the square. The method is as follows (see the corresponding pictures in Figure 1.10):

(1) Let θ be an angle at a point O. Let P_1 be a point on one side of θ, and extend the line P_1O so that we can find a point P_2 on this line but on the other side of O so that $P_1O \cong P_2O$.
(2) Extend the other side of the angle θ to become the line L_1. Fold line L_2 to be perpendicular to L_1 at the point O. (This is a move we haven't seen before; it involves folding L_1 onto itself making the crease go through O.)
(3) Now fold P_1 onto L_1 and P_2 onto L_2 simultaneously to create line L_3.
(4) Finally, fold a crease perpendicular to L_3 that goes through point O. This line will make an angle of $\theta/3$ with one side of angle θ.

Diversion 1.4 Prove that Justin's trisection method works.

There is a lot of interesting mathematics to be explored in this "two points to two lines" origami move. Questions to ponder might include: Given any two points and two lines, can this operation always be performed? Does it always result in a unique crease? What kinds of numbers (segment lengths) is it constructing for us? We will address these questions in Chapters 2 and 3.

1.4 Folding a Regular Heptagon

The easiest regular polygons to fold from a square are, well, the square, regular octagon, 16-gon, and other 2^n-gons. We saw earlier that equilateral triangles are not too hard to fold from a square, and this admits regular hexagons and dodecagons relatively easily.

Diversion 1.5 Devise a way to fold a regular pentagon from a square piece of paper. (See (Morassi, 1989) for an analysis of approximate and exact methods for this.)

However, the smallest regular n-gon that origami can produce that SE&C cannot is the heptagon.

The first published instructions for folding a regular heptagon appears to be those of Scimemi (1989) and, independently, Geretschläger (1997b) (although Justin (1986b) also provides the basic ingredients for such a construction). Both of these methods are nearly identical, however, which is not surprising because both follow a classic algebraic approach to the heptagon problem, as can also be seen in the non-folding heptagon construction given by Gleason (1988). Gleason's construction assumes the tools of straightedge, compass, and an angle trisector. Since we know origami can trisect angles, Gleason's method could be used, as is, to fold a regular heptagon. Alperin (2002) did just this, but such a strategy produces a lengthy and inelegant folding

https://doi.org/10.1017/9781108778633.003 Published online by Cambridge University Press

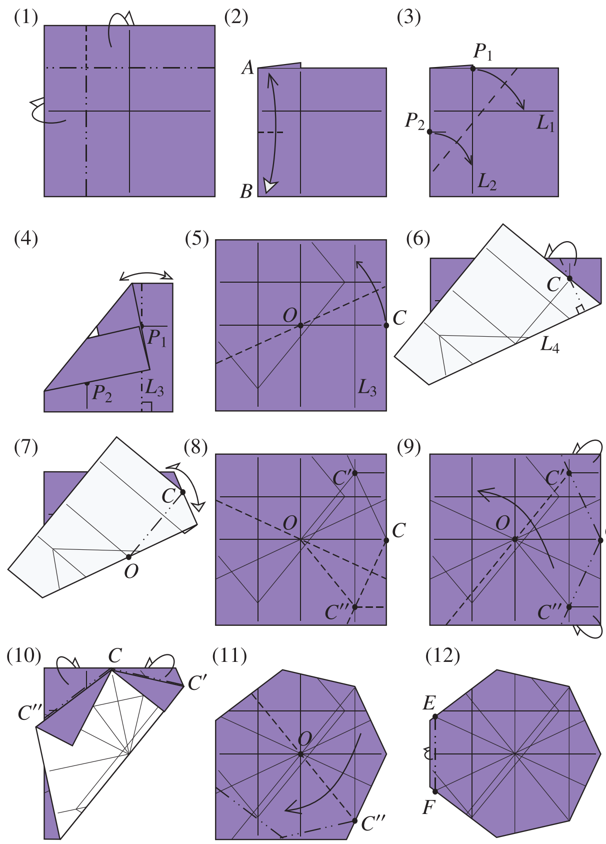

Figure 1.11 Folding a regular heptagon.

procedure. (We should note that mathematical purity demands that we find a folding procedure that is mathematically exact. However, when one physically makes a fold there will always be error present. So for practicality's sake it is always better to find folding sequences that help minimize error, either by being short or by encompassing folds that are easy to perform.)

Figure 1.11 shows a more cleaned-up way to fold a regular heptagon than those previously given. (Scimemi (1989) doesn't give an explicit folding sequence, and Geretschläger (1997b) has more folds than are necessary.) Our procedure is as follows:

(1) First, crease the paper in half from top to bottom and left to right. Then fold the top 1/4 behind and then the left 1/4 behind.
(2) Make a pinch crease on the left side by bringing points A and B together.
(3) Now we're ready for the fold that does the "magic." Fold point P_1 onto line L_1 and point P_2 onto line L_2 at the same time.

https://doi.org/10.1017/9781108778633.003 Published online by Cambridge University Press

(4) Notice where P_1 went after step (3). Mountain fold a vertical crease, perpendicular to the bottom edge, on the underneath layer of paper, creating line L_3. Crease sharply and then unfold everything.
(5) Notice where L_3 is on the unfolded sheet. Fold C, the midpoint on the right side, to line L_3 so that the crease goes through the center O of the paper.
(6) Step (5) created the folded edge L_4. Fold the right flap of paper behind, making the crease go through C while being perpendicular to L_4.
(7) Now fold and unfold line OC. (This crease already exists, but you want it to be made through **all** layers of paper.) Then unfold everything.
(8) Line CC' is one side of our heptagon. (C' is the image of C under the fold in step (5).) Repeat steps (5)–(7) on the bottom half of the paper, creating point C''.
(9) Fold CC' and CC'' behind. Then valley fold OC', extending it across the paper.
(10) Use the images of CC' and CC'' to fold two more sides of our heptagon. Then unfold OC'.
(11) Repeat steps (9)–(10) on the bottom half.
(12) Fold the left side behind with crease EF to complete the heptagon.

To see why this works, let us set up a coordinate system for the paper as follows: let O, the center of the paper, be the origin, and let the side of the square be of length 4. (We choose these coordinates to more easily illustrate the connection with Gleason's analysis in (Gleason, 1988).) Our goal is to show that the point C' in Figure 1.11 has coordinates $(2\cos(2\pi/7), 2\sin(2\pi/7))$, and thus points C, C', and C'' form three vertices of a heptagon of radius 2. Since these points are used to generate the other vertices in a logical way, this would prove the folded heptagon's validity.

The points in step (3) of Figure 1.11 are $P_1 = (0, 1)$ and $P_2 = (-1, -1/2)$, where L_1 is the x-axis and L_2 is the y-axis. Suppose that P_1 gets folded to the point $P_1' = (t, 0)$ on L_1 and P_2 gets folded to the point $P_2' = (0, s)$ on L_2.

The segment P_1P_1' has slope $-1/t$, and the crease line in step (3) must be the perpendicular bisector to this segment. So the slope of the crease line must be t and pass through the midpoint of P_1P_1', which is $(t/2, 1/2)$. Thus one formula for the crease line in step (3) is

$$y = tx - \frac{t^2}{2} + \frac{1}{2}.$$

On the other hand, segment P_2P_2' has slope $(2s+1)/2$ and midpoint $(-1/2, (2s-1)/4)$. Thus, another formula for our crease line is

$$y = \frac{-2}{2s+1}x - \frac{1}{2s+1} + \frac{2s-1}{4}.$$

Our aim is to find the value of t (the x-coordinate of P_1'), since this determines the location of line L_3 in step (5) and thus the location of C'. Equating the slopes of our two line equations gives $s = -(t+2)/(2t)$. This can then be substituted into the equation we get by equating the constant terms of our two line equations, resulting in a single equation in t. After simplifying, this becomes

$$t^3 + t^2 - 2t - 1 = 0. \tag{1.1}$$

Sure enough, $t = 2\cos(2\pi/7)$ satisfies this equation, proving that L_3 is in the proper place. (The other roots of Equation (1.1) are real and negative, and thus are not values of t that would make the fold in step (3) of Figure 1.11 work.) For readers who do not immediately believe our claims as to the solutions of Equation (1.1), we present an argument from (Gleason, 1988).

Consider the vertices of a regular heptagon as the seventh roots of unity in the complex plane, that is, the complex solutions of $z^7 - 1 = 0$. Factoring out the obvious $z - 1$ term for the $z = 1$ corner, we get the equation for the remaining six corners: $z^6 + z^5 + z^4 + z^3 + z^2 + z + 1 = 0$. Let $A = \cos(2\pi/7) + i\sin(2\pi/7)$ (the principle seventh root of 1). Since the reciprocals of complex numbers on the unit circle are the same as the complex conjugates, we have that $1/A = A^6$, $1/A^2 = A^5$, and $1/A^3 = A^4$. Plugging these into our factored heptagon equation, we see that A satisfies

$$A^3 + A^2 + A + 1 + \frac{1}{A} + \frac{1}{A^2} + \frac{1}{A^3} = 0. \tag{1.2}$$

But we also have that $A + 1/A = A + \overline{A} = 2\cos(2\pi/7)$. Furthermore, notice that

$$A^2 + \frac{1}{A^2} = \left(A + \frac{1}{A}\right)^2 - 2 \quad \text{and} \quad A^3 + \frac{1}{A^3} = \left(A + \frac{1}{A}\right)^3 - 3\left(A + \frac{1}{A}\right).$$

Substituting these into Equation (1.2), we get

$$\left(A + \frac{1}{A}\right)^3 + \left(A + \frac{1}{A}\right)^2 - 2\left(A + \frac{1}{A}\right) - 1 = 0.$$

Therefore, $A + 1/A = 2\cos(2\pi/7)$ is a solution to Equation (1.1). Similar machinations show that other two roots of Equation (1.1) are $2\cos(4\pi/7)$ and $2\cos(6\pi/7)$.

Figure 1.12 provides a geometric interpretation of what's going on. If A is the principle root of $(z^7 - 1)/(z - 1)$, then $1/A$ is just A reflected about the real axis, and $(A + 1/A)/2$ is the midpoint of the segment connecting these two points. Since this midpoint is on the real axis, it's just the real part of A, which is $\cos(2\pi/7)$. The same thing holds for the other roots A^2 and A^3. Therefore the roots of $(z^7 - 1)/(z - 1)$ after the substitution $z = A + 1/A$ will just be twice the real parts of the seventh roots of unity, excluding $z = 1$. There are only three such numbers, and the equation after substitution simply becomes Equation (1.1).

In fact, Equation (1.1) is the standard equation one tries to solve when confronting the regular heptagon construction problem. (See (Martin, 1998), for example.) In a very real sense, this equation is being solved in step (3) of the folding sequence. It is interesting to note that both the folding methods of Scimemi (1989) and of Geretschläger (1997b) incorporate basically the same fold seen in step (3) of Figure 1.11 to create a crease with the proper slope for the heptagon construction. That two independent researchers came upon the same fold to solve Equation (1.1) is not a coincidence – they were both trying to solve the same equation via folding. We will see in Chapter 2 how to go about solving general cubic equations with such folding operations.

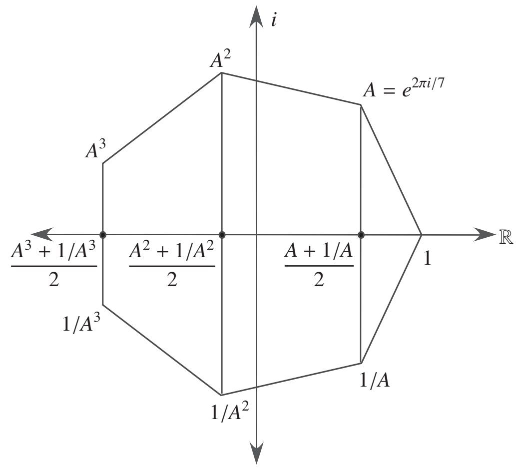

Figure 1.12 Geometric interpretation of $A + 1/A$ and the like.

1.5 The Basic Origami Operations

Now that we have seen several examples of origami constructions, we are in a better position to consider classifying the **basic origami operations**, or **BOOs** for short. This turns out to be more problematic than one might expect.

As seen at the beginning of this chapter, it is easy to classify what operations are possible under SE&C because we know exactly what our tools can do. The examples we've seen show that origami admits many different types of operations. When trying to make a list of them, it is not clear if one is really a special case of another, or whether we have found them all. For example, in the 1980s Huzita and Scimemi (1989) developed the following list of operations for origami:

O1: Given two points P_1 and P_2, we can fold a crease line connecting them.
O2: Given two lines, we can locate their point of intersection, if it exists.
O3: Given two points P_1 and P_2, we can fold the point P_1 onto P_2 (perpendicular bisector).
O4: Given two lines L_1 and L_2, we can fold the line L_1 onto the line L_2 (angle bisector).
O5: Given a point P and a line L, we can make a fold line perpendicular to L passing through the point P (perpendicular through a point).
O6: Given two points P_1 and P_2 and a line L, we can, whenever possible, make a fold that places P_1 onto the line L and passes through the point P_2.
O7: Given two points P_1 and P_2 and two lines L_1 and L_2, we can, whenever possible, make a fold that places P_1 onto L_1 and also places P_2 onto L_2.

Diversion 1.6 Under what conditions will operations O6 and O7 be possible? (This will be addressed in the next chapter.)

Readers who enjoy paying attention to details can examine this list to check it for completeness and optimality. That is, are any BOOs missing? Can any of BOOs O1–O7 be performed via a combination of the other BOOs?

We will address the latter question later. For the former, we can try to distinguish origami operations by what combination of points and lines they use. For example, should the following be considered an additional BOO?

O8: Given a point P and two nonparallel lines L_1 and L_2, we can make a fold perpendicular to L_2 that places P onto L_1.

Jacques Justin included this operation in his list of BOOs, which may predate the Huzita–Scimemi list (see (Justin, 1986b)). Also, Hatori (2003) independently proposed this BOO as an addition much later.

Since BOO O8 takes two lines and a point as input, and none of the Huzita–Scimemi BOOs do this, we could consider this to be a separate BOO. Sticklers for optimality may disagree.

Diversion 1.7 Show that O8 can be performed via a sequence of BOOs O1–O6.

However, with the addition of BOO O8, we can prove that no more basic operations are possible.

Theorem 1.1 *If we only allow one fold at a time, and assuming all our creases are straight lines, then the only folding operations possible are O1–O8.*

Robert J. Lang first proved this in 2003 (Lang, 2003) using vector geometry methods. We present a more elementary proof from Hull (2005) that follows Lang's basic argument.

Proof When we do origami, we only have two types of things to fold to each other: points and lines. Sometimes when we fold one of these to another, it uniquely determines the fold, like when folding one point to another point. But other times, like when folding a point to a line, there is still a **degree of freedom** left that must be removed before a specific fold is determined. The possible combinations of points and lines that can be folded to each other are as follows (note that we can ignore O2, since it involves no folding):

Case 1: Fold a point to another point – no degree of freedom.
Case 2: Fold a point to itself – one degree of freedom (the angle of the crease going through the point).

https://doi.org/10.1017/9781108778633.003 Published online by Cambridge University Press

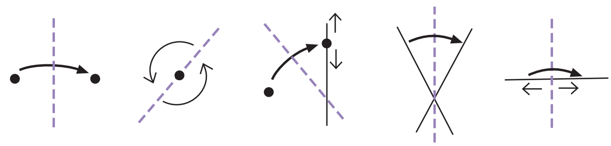

Figure 1.13 The five cases of points/lines being folded to each other.

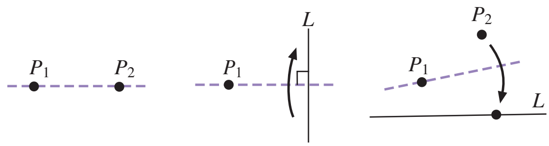

Figure 1.14 The subcases of folding a point P_1 to itself.

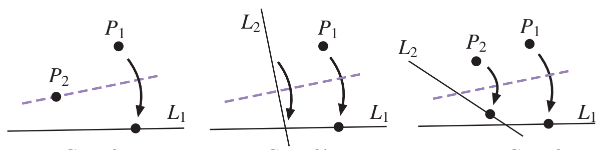

Figure 1.15 The subcases of folding a point P_1 to a line L_1.

Case 3: Fold a point to a line – one degree of freedom (the point can be anywhere on the line).
Case 4: Fold a line to another line – no degree of freedom.
Case 5: Fold a line to itself – one degree of freedom (see Figure 1.13).

Cases 1 and 4 are operations O3 and O4, respectively. The other cases all have a degree of freedom that needs to be removed to determine what folds they can become.

Case 2: *Fold a point P_1 to itself.* The only creases that can fold a point P_1 to itself are creases going through the point P_1, and there are an infinite number that do this. To specify a unique crease we need to combine this with another operation that also possesses a degree of freedom. This gives us three subcases, as illustrated in Figure 1.14:
Case 2a: Also fold another point P_2 to itself. This is operation O1.
Case 2b: Also fold a line L to itself. This is the perpendicular line operation O5.
Case 2c: Also fold another point P_2 to a line L. This is operation O6.
Case 3: *Fold a point P_1 to a line L_1.* Again, we have a degree of freedom because the point could be folded anywhere on the line. Thus we need to combine this with another degree of freedom (see Figure 1.15):
Case 3a: Also fold another point P_2 to itself. This again gives us operation O6.
Case 3b: Also fold another line L_2 to itself. This gives us operation O8.

https://doi.org/10.1017/9781108778633.003 Published online by Cambridge University Press

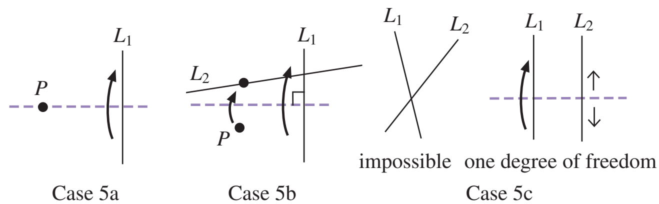

Figure 1.16 The subcases of folding a line L_1 to itself.

Case 3c: Also fold another point P_2 to another line L_2. This is operation O7.
We can now see that we've covered all seven of our BOOs. To make sure there are no more, however, we need to look at the last case.

Case 5: *Fold a line L_1 to itself.* Folding a line L_1 to itself is really just folding a crease perpendicular to L_1. But since this crease could intersect L_1 at any point, this gives us a degree of freedom. Combining with other operations gives the following (as shown in Figure 1.16):

Case 5a: Also fold a point P to itself. This gives us operation O5 again.

Case 5b: Also fold a point P to another line L_2. This gives us O8 again.

Case 5c: Also fold another line L_2 to itself. If these two lines intersect, then this is impossible, since the resulting crease would have to be perpendicular to both. Thus the two lines would have to be parallel, but this still results in one degree of freedom. Thus this combination is redundant—we would still need another operation to specify a unique crease, bringing us back to Case 5a or 5b.

This exhausts all the possibilities of folding points and lines to points and lines, completing the proof. □

Operations O1–O8 encompass everything that straight-crease, single-fold origami can do. This list does contain redundancies, however, and to eliminate them we need to be more specific about what is given to us at the start of our constructions.

For example, Alperin (2000) assumes the paper to be the entire complex plane with the given constructed points 0 and 1. He then chooses operations O1–O4, O6, and O7 to be his list of construction operations. (Actually, Alperin and several other writers use the word "axiom" to refer to allowed folds. But since O6 and O7 are not always possible, "operations" seems a more appropriate term.) One could equivalently assume that the four points $(\pm 1, \pm 1)$ are given, which, when the lines $y = \pm 1$ and $x = \pm 1$ are folded using O1, simulate the boundary of a square piece of paper.

Certainly the given points we start with and the operations O1–O8 guarantee that every crease line that is constructed will have a constructed point on it somewhere, as well as every point having a constructed line passing through it. This observation has led several people, including Martin (1998) and Hatori (2003) to prove that all we really need to characterize origami constructions are operations O2 and O7.

https://doi.org/10.1017/9781108778633.003 Published online by Cambridge University Press

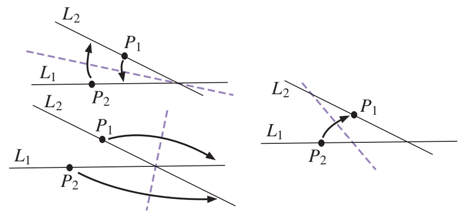

Figure 1.17 Generating O3 and O4 from O7.

Theorem 1.2 *Assuming that we are given at least two constructed points contained in nonparallel constructed lines (which may be identical), then any straight-crease, single-fold origami construction from this starting set can be completely described by combinations of operations O2 and O7.*

Proof By Theorem 1.1, we know that operations O1–O8 are all we need to consider. We obviously need to keep O2, but the others can be shown to be special cases of O7 where either we have $L_1 = L_2$ or some of the points P_1, P_2 lie on the lines L_1, L_2. This needs to be done carefully.

For the operations O3 and O4, let P_1 be on L_2 and P_2 be on L_1. (We know that all constructed lines contain constructed points and vice versa, so we can assume this from the premises of O3 and O4.) Then there will be at most three ways in which P_1 and P_2 can be folded onto L_1 and L_2, respectively, as shown in Figure 1.17. The first two cases will amount to folding L_1 onto L_2, or bisecting one of the angles made at their intersection, producing O4. (If L_1 and L_2 are parallel, then there will be only one way to do this.) The other case folds P_1 onto P_2, giving us O3.

In O6, we are given points P_1, P_2 and line L_1. Let L_2 be any line through P_2. So long as O6 is possible, we can have O7 fold P_2 to itself on line L_2 and fold P_1 onto L_1. (Note that we are not concerned here with when O6 is possible – we will take that up in the next chapter.)

For O1 and O5, let L_1 and L_2 be lines containing P_1 and P_2, respectively. Then, assuming L_1 and L_2 are not parallel, there are at most three different folds we could make that will leave P_1 on L_1 and P_2 on L_2: (a) making the crease pass through P_1 and P_2, (b) making the crease pass through P_1 and be perpendicular to L_2, and (c) making the crease through P_2 and perpendicular to L_1. While some of these may be identical, case (a) is operation O1. Cases (b) and (c) would give us O5. If L_1 and L_2 do happen to be parallel, then P_1 and P_2 could not have been the original two points in the construction, so there exist other points and lines that we can use to construct a different line through one of P_1 or P_2.

For O8 we can take an arbitrary point P_2 on L_2 and use O7 to fold P_1 onto L_1 while folding P_2 to a (probably) different place on L_2 to ensure that the crease line will be perpendicular to L_2. This covers all the operations O1–O8. □

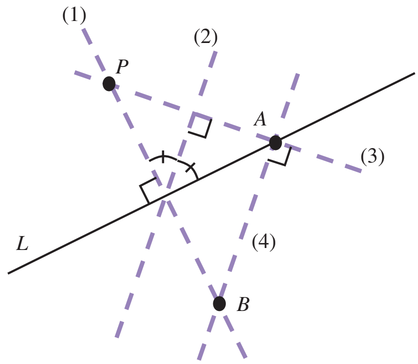

Figure 1.18 Constructing the reflection of P about L.

Remark 1.3 A common concern regarding the list of BOOs O1–O8 is whether or not we unfold the paper after each operation. Geometrically, all we care about is that each BOO creates a new line in the plane and that the intersection of lines creates new points. From this view, it does seem that every time we perform a BOO, we unfold the paper immediately to consider the new line formed in the paper (plane).

However, in practice origamists often leave the paper folded in order to perform an operation using the folded layers of paper as a guide. This was done several times in the heptagon construction in Section 1.4, for example. It seems conceivable that repeated applications of a BOO, especially O7, on a piece of paper without unfolding it might lead to a construction that could not be achieved by performing one BOO at a time, unfolding after each.

Yet this is not the case. The locations of any points or lines that are moved in the process of folding the paper flat with a BOO can be reconstructed if we immediately unfold the paper. (And therefore there is no need to keep the paper folded.) To prove this, suppose that L is a crease line constructed by some BOO. All we need to do is show that for any point P or other line L', we can construct the reflections of P and L' about L.

One method for constructing the reflection of P about L is shown in Figure 1.18. First use O5 to crease a line (1) perpendicular to L passing through P. Then use O4 to fold this crease line onto L, bisecting the angle between them to make the crease (2). Then use O5 again with P and the crease (2), labeling A as the point where this crease (3) intersects L. Finally, perform O5 again with the crease line (3) and the point A to make the crease (4). Where this crease intersects the crease line (1), called B in Figure 1.18, is the reflection of P about L.

Constructing the reflection of a line L' about L can be handled similarly. All we need is a point or two constructed on L'; by reflecting these about L, we can then create the reflection of L' by using O1.

https://doi.org/10.1017/9781108778633.003 Published online by Cambridge University Press

This does, of course, use the convention that we can think of our sheet of paper as being as large as we wish and that the boundary lines of the paper are just constructed lines like any other. In any case, we conclude that while in practice it is often much more efficient to leave the paper folded after performing an origami operation, we can keep our list of BOOs simply to O1–O8 (or O2 and O7, if we want to be really efficient) by unfolding the paper after each step without losing any origami construction power.

Some of the subtleties in the work of this section, especially Theorem 1.1, seem to be ignored by much of the literature in this area. Several researchers (see (Alperin, 2000), for example) create definitions of what it means to be origami constructible (as we will in Chapter 3) referring to the basic origami operations mentioned here, but they do so with no argument as to whether more operations might be possible. Other papers, for example (Auckly and Cleveland, 1995) and parts of (Alperin, 2000), ask what can be constructed by a deliberately reduced set of origami operations. Investigators in the origami community were very concerned with the question of whether more operations existed (see (Hull, 1996)), whereby Lang's proof of Theorem 1.1 is viewed as a breakthrough.

In some sense it doesn't matter which of the moves O1–O8 we choose for an official list of basic origami operations. As we will see in the next chapter, it is O7 that separates origami from SE&C constructions, and being able to reference the other operations makes for very convenient notation.

1.6 Historical Remarks

The first person to seriously analyze origami geometric constructions seems to have been T. Sundara Row (1901). The first person to introduce operation O7 seems to have been Margherita Beloch (1936). (See the remarks in Section 2.5 for more information.) However, none of these early researchers made a formal list of possible origami construction operations. The first such list to see print seems to have been in Jacques Justin's 1986 paper (Justin, 1986b). Justin states that his list was inspired by an unpublished list created by Peter Messer (1984). Messer's list contains the operations O1–O7, but not O8. Justin's list contains all O1–O8. It appears that Scimemi independently developed a list of origami construction operations, which became those listed in (Huzita and Scimemi, 1989) and only included O1–O7.

Complicating things further, George Martin published a paper in 1985 (Martin, 1985) that defines origami constructions using only operations O2 and O7, and he seems to have developed this without any knowledge of Beloch's work. Martin cites Row and a publication of Yates, but he also cites (Dayoub and Lott, 1977), which describes how one can use a Mira, a geometric construction tool that allows one to reflect points about a line in the same way origami does, to trisect an arbitrary angle. In doing so Dayoub and Lott use the Mira to perform an operation very similar to O7 in origami. Martin then refines this method for the Mira, publishing his own paper on Mira constructions that contains exactly the form of operation O7 where the two given lines are perpendicular, whereupon Martin proves that the Mira can be used to construct cube roots (Martin, 1979). So it could be that Martin was inspired by the Mira to devise his version of operation O7 for origami.

Both Justin's paper (Justin, 1986b) and the Huzita–Scimemi paper (Huzita and Scimemi, 1989) were published in hard-to-find publications, or at least in periodicals that weren't indexed in the standard mathematical abstracts. This is likely why subsequent work, like (Auckly and Cleveland, 1995; Geretschläger, 1997a; Alperin, 2000; Hatori, 2003) make no mention of Messer, Justin, or Scimemi (or Beloch, for that matter).