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TAME NEAR-RINGS AND N-GROUPS

by S. D. SCOTT

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Throughout this paper any near-ring N will be left distributive, zero symmetric and have an identity. Furthermore, all N-groups will be unitary. All groups considered will be written additively. This does not imply commutativity.

Suppose V is a group and S a set of endomorphisms of V containing the inner automorphisms. Let N be the near-ring of maps of V into V generated by S. It is easily seen that V is a unitary N-group. Suppose W is an N-subgroup of V. Since $WS \subseteqq W$, W is a normal subgroup and, by (**6**, 6.6, p. 174), W is a submodule of V i.e. the kernel of an N-homomorphism on V.

This leads to a definition. Let V be an N-group. We call V *tame* if every N-subgroup of V is a submodule. A near-ring N with a faithful tame N-group will be called *tame.*

Tame near-rings and N-groups were first investigated by the author in his Ph.D. dissertation (**7**, Chs 4, 5 & 6). Many of the results of this paper may be found in (**7**), although the proofs given here are often simpler.

The example given above has a much stronger property than tameness. In Section six 2-tame and compatible N-groups are defined and these are looked at in more detail in subsequent sections.

1. Nilpotency in tame near-rings

If S is a subset of a near-ring N, then $R(S)$ will denote the right ideal of N generated by S. With the aid of certain propositions and lemmas we investigate nilpotency in tame near-rings.

Proposition 1.1. *Let N be a near-ring and V a tame N-group. If M is a right N-subgroup of N, then $vR(M) = vM$ for all v in V.*

Proof. Since, for v in V, vM is a submodule of V, $(vM: v)$ is a right ideal of N. Clearly $M \leqq (vM: v)$, $R(M) \leqq (vM: v)$ and $vR(M) \leqq vM$. The result now follows.

Lemma 1.2. *Let N be a tame near-ring, M a right N-subgroup of N, and S_i, $i = 1, 2$, non-empty subsets of N. If one of the subsets S_1M, S_1MS_2 or MS_2 of N is $\{0\}$, then the same applies on replacing M by $R(M)$.*

Proof. If $S_1M=\{0\}$, then $M\leqq(0:S_1)$, $R(M)\leqq(0:S_1)$ and $S_1R(M)=\{0\}$. Suppose that $S_1MS_2=\{0\}$ and V is a faithful tame N-group. By 1.1 it follows that for all α in S_1, β in S_2 and v in V

$$v\alpha R(M)\beta=v\alpha M\beta=\{0\}.$$

Thus $VS_1R(M)S_2=\{0\}$ and $S_1R(M)S_2=\{0\}$. Taking $S_1=\{1\}$ the case where $MS_2=\{0\}$ follows.

Proposition 1.3. *Let N be a tame near-ring and $H_1,\dots,H_k$, right N-subgroups of N for $i=1,\dots,k$. If $H_1H_2\dots H_k=\{0\}$, then*

$$R(H_1)R(H_2)\dots R(H_k)=\{0\}.$$

Proof. This follows by repeated application of 1.2.

Corollary. *If M is a nilpotent right N-subgroup of a tame near-ring N, then $R(M)$ is nilpotent.*

Theorem 1.4. *If N is a tame near-ring and A the sum of all nilpotent right ideals of N, then A is an ideal of N.*

Proof. We have $A=\sum_{i\in I}R_i$, where $\{R_i,\ i\in I\}$ is the set of all nilpotent right ideals of N. Thus for any α in N

$$\alpha A=\sum_{i\in I}\alpha R_i$$

and each αR_i is a nilpotent right N-subgroup of N. By the above corollary each $R(\alpha R_i)$ is nilpotent. Thus for each α in N

$$\alpha A\leqq\sum_{i\in I}R(\alpha R_i)\leqq A$$

and it follows that A is an ideal.

We note that there exist near-rings where the sum of all nilpotent right ideals need no longer be an ideal (see (**1**, p. 204)) but, as 1.4 indicates, tame near-rings resemble rings more closely. Further evidence of this similarity is seen in many of the results that follow.

2. Direct sums

Let N be a near-ring with an N-group V. It may happen that $N=A\oplus B$ where A and B are ideals, but this split in N does not induce a corresponding split in V (e.g. if p and q are distinct primes, then $\mathbf{Z}_{pq}=\mathbf{Z}_p\oplus\mathbf{Z}_q$ and any group of exponent pq is a faithful unitary $\mathbf{Z}_{pq}$-group). However, for tame N-groups the situation is simpler.

In order to investigate this situation we need an alternative characterisation of tame. The following proposition provides this characterisation which will be of importance in defining 2-tame, and compatible *N-groups.*

Proposition 2.1. *An N-group V is tame if, and only if, for v and w in V and α in N there exists β in N such that*

$$(v+w)\alpha - v\alpha = w\beta.$$

Proof. If V is tame, then wN is a submodule of V containing w. Thus $(v+w)\alpha - v\alpha$ is in wN and β exists.

Suppose for given v, w and α, β exists. Let W be an N-subgroup of V. If w is in W clearly $(v+w)\alpha - v\alpha$ is in W. Thus W is a submodule of V (normality follows on taking $\alpha = 1$). The proposition is proved.

Theorem 2.2. *Let N be a near-ring with a faithful tame N-group V. If $N = A \oplus B$ where A and B are ideals of N, then $V = U \oplus W$ with U and W submodules of V such that $(0: W) = A$ and $(0: U) = B$.*

Proof. We have $1 = e_1 + e_2$ where e_1 is in A and e_2 is in B. The e_i, $i = 1, 2$, are central orthogonal idempotents.

Define U to be the subset Ve_1 of V and W the subset Ve_2 of V. Now $U = \{v \in V: ve_2 = 0\}$, since $Ue_2 = \{0\}$ and, if $ve_2 = 0$ for some v in V, then $v = ve_1 + ve_2 = ve_1$. To show U is a submodule of V it is sufficient to show that it is an N-subgroup. Now

$$UN = Ve_1N = VNe_1 \subseteqq U$$

since e_1 is central. In particular $u(-1) = -u$ is in U for all u in U. We must therefore show that U is closed under addition. Let v_1e_1 and v_2e_1 be elements of U. By 2.1

$$(v_1e_1 + v_2e_1)e_2 - v_1e_1e_2 = v_2e_1\beta$$

for some β in N. Now $v_1e_1e_2 = 0$ and

$$(v_1e_1 + v_2e_1)e_2 = v_2e_1\beta.$$

Hence

$$\begin{aligned}(v_1e_1 + v_2e_1)e_2^2 &= (v_1e_1 + v_2e_1)e_2 \\ &= v_2e_1\beta e_2.\end{aligned}$$

Since e_2 is central $v_2e_1\beta e_2 = v_2e_1e_2\beta = 0$. Thus $(v_1e_1 + v_2e_1)e_2 = 0$ and $v_1e_1 + v_2e_1$ is in U. It follows that U is a submodule of V. Similarly W is a submodule of V. If v is in V, then $v = ve_1 + ve_2$ and $V = U + W$. Also, if $v_1e_1 = v_2e_2$ is in $U \cap W$ where v_1 and v_2 are in V, then $v_1e_1 = v_2e_2e_1 = \{0\}$. Hence $V = U \oplus W$. Now $A = e_1N$ and $W = Ve_2$. Since $e_2e_1 = 0$, $A \leqq (0: W)$. Also $B \leqq (O: U)$ and $(0: U) + (0: W) = N$. If α is in $(0: U) \cap (0: W)$, then since the sum $U + W$ is direct, $V\alpha = \{0\}$. Since V is faithful $(0: U) \cap (0: W) = \{0\}$. Thus $(0: U) \oplus (0: W) = N$ and, since $A \leqq (0: W)$ and $B \leqq (0: U)$, it follows that $A = (0: W)$ and $B = (0: U)$. The theorem is now proved.

3. Finitely generated tame N-groups

Let V be a tame N-group. A submodule U of V is *finitely generated* if there exists v_i, $i = 1, \ldots, k$, in V such that

$$U = \sum_{i=1}^{k} v_iN.$$

A standard result of ring theory (see (**2**, p. 96)) states that if N is a ring with maximal condition on right ideals and V a finitely generated N-module (in the ring sense), then a submodule, of V is finitely generated. We shall show that this holds for tame N-groups.

A tame N-group V will be called *Noetherian* if the submodules of V satisfy the maximal condition.

The following proposition is the analogue of the corresponding ring result.

Proposition 3.1. *A tame N-group V is Noetherian if, and only if, every submodule of V is finitely generated.*

If S is a set of subgroups of a group, then an element of S will be called an *S-subgroup.* The next lemma, a lattice theory result, will be used again in Section five.

Lemma 3.2. *Let V be a group and V_i, $i=0,\ldots,k$, a properly ascending finite sequence of normal subgroups of V, such that $V_0=\{0\}$ and $V_k=V$. Let S be a set of subgroups of V such that $H\cap W$ and $H+W$ are in S for all H in S and W in $\{V_i : i=0,\ldots,k-1\}$. If the S-subgroups of V between V_i and V_{i+1} satisfy minimal (maximal) condition for $i=0,\ldots,k-1$, then the elements of S satisfy minimal (maximal) condition.*

Proof. We shall prove the result for minimal condition as the proof for maximal condition is entirely similar.

For $k=1$ the result is trivial. Assume that the S-subgroups of V between $\{0\}$ and V_{k-1} satisfy minimal condition, and let

$$H_1 \geqq H_2 \geqq \ldots$$

be a descending chain of S-subgroups of V. By the assumptions there exists a positive integer m such that $H_m\cap V_{k-1}=H_{m+n}\cap V_{k-1}$ and $H_m+V_{k-1}=H_{m+n}+V_{k-1}$ for $n=0,1,2,\ldots$. Since $H_m\geqq H_{m+n}$, it follows by the "modular law" that $H_m=H_{m+n}$ for $n=0,1,2,\ldots$. The proof is now complete.

Proposition 3.3. *Let N be a near-ring with maximal condition on right ideals and V a faithful tame N-group. If there exists v in V such that $vN=V$, then V is Noetherian.*

Proof. The obvious N-homomorphism of N onto vN establishes a one to one correspondence between the submodules of vN and the right ideals of N containing $(0:v)$. The proposition follows.

Theorem 3.4. *Let N be a near-ring with maximal condition on right ideals and V a tame N-group. If V is finitely generated, then any submodule of V is finitely generated.*

Proof. We have V is a finite sum $\sum_{i=1}^{k} v_iN$ where v_i, $i=1,\ldots k$, are in V. The result will follow by 3.1 if we show that V is Noetherian. We proceed by induction. The case $k=1$ follows from 3.3 Let $U=v_2N+\ldots+v_kN$. The submodule U of V may be assumed to be Noetherian by 3.1. Now V/U *is* N-isomorphic to $v_1N/(v_1N)\cap U$ and V/U is Noetherian by 3.3. From the isomorphism theorems it follows that the submodules of V between U and V satisfy maximal condition. By 3.2 V is Noetherian and the theorem is proved.

4. Tame near-rings with minimal condition (the radical)

Let V be an N-group (assumed unitary). We shall call V *minimal* if it is of type 2 (see (**6**, 3.5, p. 77)). This additional terminology is valuable for the case of tame N-groups. Indeed the only radical that will concern us here is $J_2(N)$ (*see* (**6**, 5.1, *p.* 136)), which we denote by $J(N)$.

If N is a near-ring and V an N-group, then an N-group U/W will be called a *factor* of V if $U \geqq W$ are submodules of V. The factor U/W will be called *minimal* if $U > W$ and there exist no submodules of V lying properly between W and U. Thus R_1/R_2 is a minimal factor of a near-ring N, if $R_1 > R_2$ are right ideals of N and there exist no right ideals of N lying properly between R_2 and R_1.

We aim to show that for a tame near-ring N with minimal condition, $J(N)$ is nilpotent. First we show that a minimal factor of N is a minimal N-group.

Proposition 4.1. *If N is a near-ring, V a tame N-group, M a right N-subgroup of N and v an element of V, then $M+(0:v)$ is a right ideal of N.*

Proof. Let δ be the natural N-homomorphism of N onto vN. Since vM is a submodule of vN, the inverse image $M+(0:v)$, of vM under δ is a right ideal of N.

Lemma 4.2. *Let N be a tame near-ring with minimal condition on right ideals. If R_1/R_2 is a minimal factor of N, then R_1/R_2 is a minimal N-group.*

Proof. Out of all minimal factors of N, N-isomorphic to R_1/R_2, choose one R/H such that R is minimal. We show that R/H is a minimal N-group. If $K < R$ is a right ideal of N, then $K \leqq H$, otherwise $K+H=R$ and R/H is N-isomorphic to the minimal factor $K/K\cap H$. Let M be a right N-subgroup of N such that $H<M<R$ and let V be a faithful tame N-group. Since the intersection $\cap(0:u)$ over all u in V is zero, there exists v in V such that $(0:v)\cap R<R$ and therefore $(0:v)\cap R \leqq H$. *By* 4.1 $M+(0:v)$ is a right ideal of N. Thus

$$R\cap[M+(0:v)] = M+(0:v)\cap R$$

is a right ideal of N. But $(0:v)\cap R \leqq H < M$ and we see that M is a right ideal of N. This contradiction completes the proof.

Theorem 4.3. *If N is a tame near-ring with minimal condition on right ideals, then $J(N)$ is nilpotent.*

Proof. Suppose $J(N)$ is not nilpotent and $H \leqq J(N)$ is a right ideal of N minimal for being non-nilpotent. If $R(H^2) < H$, then H^2 is nilpotent and so is H. Thus $R(H^2) = H$. We have $\{0\}H = \{0\}$ and by Zorn's Lemma we may find a right ideal K_1 *of* N maximal for the property that $K_1H = \{0\}$. Since N has an identity we may assume that $K_1 \neq N$. By minimal condition there exists a right ideal $K_2 > K_1$ with K_2/K_1 a minimal factor of N. By 4.2, K_2/K_1 is a minimal N-group. Since $H \leqq J(N)$, $[K_2/K_1]H = \{0\}$ and $K_2H \subseteqq K_1$. Thus $K_2H^2 \subseteqq K_1H = \{0\}$. Hence $H^2 \subseteqq (0:K_2)$, $H = R(H^2) \leqq (0:K_2)$ and it follows that $K_2H = \{0\}$. This contradiction to the maximality of K_1 establishes the theorem.

5. Chain conditions on tame near-rings

In this section it is proved that for tame near-rings minimal condition on right ideals implies minimal and maximal condition on right N-subgroups. This proof will be accomplished in a sequence of propositions and lemmas. We need certain preliminaries.

If V is an N-group then a submodule W of V will be called *absolutely reducible* in V, if it is a direct sum of submodules of V which are minimal N-groups.

The following result may be proved in the same way as for rings (see (**2**, p. 87)).

Proposition 5.1. *If N is a near-ring, V an N-group and W a submodule of V, then W is absolutely reducible in V if, and only if, W is a sum of submodules of V which are minimal N-groups.*

If a submodule W of an N-group V is absolutely reducible in V, then $WJ(N)=\{0\}$ (see (**6**, 2.29, p. 49)). Furthermore, if V has minimal condition on submodules, then W is a finite direct sum of submodules which are minimal N-groups (see (**2**, p. 88)).

If N is a non-zero tame near-ring with minimal condition on right ideals, then soc N will denote the sum of all minimal right ideals of N. By 4.2 and 5.1 soc N is absolutely reducible in N. The above comment tells us that this direct sum is finite.

The following proposition can be deduced from (**6**, 5.32, p. 146).

Proposition 5.2. *If N is a near-ring with minimal condition on right ideals and T is an ideal of N, then $J(N/T)=(J(N)+T)/T$.*

Clearly for a tame near-ring N with minimal condition, $RJ(N)=\{0\}$ for all right ideals $R\leqq \text{soc } N$. We now establish the converse.

Lemma 5.3. *Let N be a tame near-ring with minimal condition on right ideals and R a right ideal of N such that $RJ(N)=\{0\}$. It follows that $R\leqq \text{soc } N$.*

Proof. Assume $R\nleqq \text{soc } N$ and let $H\leqq R$ be a right ideal of N minimal for the property that $H\nleqq \text{soc } N$. Set $K=(\text{soc } N)\cap H$. Clearly $K<H$ and K contains every right ideal of N properly contained in H. By (**6**, 5.32, p. 146) and 5.2

$$N/J(N)=R_1\oplus\ldots\oplus R_k$$

where the R_i, $i=1,\ldots,k$, are minimal right ideals and minimal $N/J(N)$-groups. Thus the R_i are minimal N-groups. Let ρ be in H but not in K. Since H is a unitary $N/J(N)$-group one of the ρR_i, $i=1,\ldots,k$, say ρR_j, must be such that $\rho R_j\nleqq K$. Set $\rho R_j=M$. Since R_j is a minimal N-group so is M. Hence $M\cap K=\{0\}$. Also, since $M\leqq H$, $M+K\leqq H$ and $M+K>K$. Now K is maximal in H and, by 4.2, $M+K=H$.

If $K=\{0\}$, then $H=M$ and H is a minimal right ideal. Thus we may assume that $K\neq\{0\}$. Let V be a faithful tame N-group. There exists v in V such that $(0:v)\cap K<K$. Now either $(0:v)\cap H\geqq K$ or $(0:v)\cap H<K$. If $(0:v)\cap H\geqq K$, then $(0:v)\cap K=K$. Hence $(0:v)\cap H<K$. By 4.1, $M+(0:v)$ is a right ideal of N and thus

$$H\cap[M+(0:v)]=M+(0:v)\cap H \quad (=K_1 \text{ say})$$

is a right ideal of N. Since $(0:v)\cap H<K$, $K_1<H$ and $K_1\leqq K$. Thus $M\leqq K$ and we have a contradiction that completes the proof.

Lemma 5.4. *Let N be a tame near-ring with minimal condition on right ideals. If S is a non-empty subset of N such that $SJ(N)=\{0\}$, then $S\subseteq$ soc N.*

Proof. Let α be in S. We have $\alpha J(N)=\{0\}$. Since $NJ(N)\subseteq J(N)$, $\alpha NJ(N)=\{0\}$. By 1.3, $R(\alpha N)J(N)=\{0\}$ and by 5.3, $R(\alpha N)\leqq$ soc N. Since N has an identity α is in $R(\alpha N)$ and therefore in soc N. The lemma now follows.

Lemma 5.5. *If N is a tame near-ring with minimal condition on right ideals, then* soc N *is an ideal of N and N/*soc *N is tame.*

Proof. Let V be a faithful tame N-group. If it is shown that there exists a submodule U of V such that $(U\colon V)=$ soc N, then it will follow that soc N is an ideal of N and N/soc N will have V/U as a faithful tame N/soc N-group. We have soc N is a finite direct sum $R_1\oplus R_2\oplus\ldots\oplus R_k$ of minimal right ideals which are minimal N-groups. Set

$$U=\sum\sum vR_i$$

where the sum is over all i in $\{1,\ldots,k\}$ and v in V. We have for each i and v that $vR_i=\{0\}$ or vR_i is N-isomorphic to R_i. Since V soc $N\neq\{0\}$, $U\neq\{0\}$ and it follows from 5.1 that U is absolutely reducible in V. Obviously soc $N\leqq(U\colon V)$. Also $V(U\colon V)\subseteq U$ and, by the absolute reducibility of U

$$V(U\colon V)J(N)\subseteq UJ(N)=\{0\}.$$

Since V is faithful $(U\colon V)J(N)=\{0\}$. By 5.3, $(U\colon V)\leqq$ soc N and the lemma follows.

Our next lemma is the main step in proving that minimal condition on right ideals implies, for tame near-rings, maximal and minimal condition on right N-subgroups.

Lemma 5.6. *If N is a tame near-ring with minimal condition on right ideals, then there exists a positive integer k and right ideals*

$$\{0\}=R_0<R_1<R_2<\ldots<R_k=N$$

such that R_{i+1}/R_i, $i=0,\ldots,k-1$, are minimal factors of N.

Proof. By 4.3, $J(N)^m=\{0\}$ for some positive integer m. We assume m is minimal and proceed by induction. If $m=1$, $J(M)=\{0\}$ and by (**6**, 5.32, p. 146), N is a finite direct sum

$$H_1\oplus\ldots\oplus H_n$$

of minimal right ideals of N. With $k=n$, $R_0=\{0\}$ and $R_i=H_1\oplus\ldots\oplus H_i$, $i=1,\ldots,k$, the lemma follows in this case.

Assume $m>1$. Now $J(N)^{m-1}J(N)=\{0\}$ and by 5.4, $J(N)^{m-1}\subseteq$ soc N. Also by 5.2

$$J(N/\text{soc } N)=(J(N)+\text{soc } N)/\text{soc } N$$

and, since

$$(J(N)+\text{soc } N)^{m-1}\subseteq J(N)^{m-1}+\text{soc } N\subseteq \text{soc } N,$$

we conclude that

$$[J(N/\text{soc } N)]^{m-1}=\{0\}.$$

Since, by 5.5, $N/\text{soc } N$ is tame it follows from the induction assumption that there exists a positive integer s and right ideals K_i, $i=0,\ldots,s$, of $N/\text{soc } N$ such that

$$\{0\}=K_0<K_1<\ldots<K_s=N/\text{soc}N$$

where K_{i+1}/K_i, $i=0,\ldots,s-1$, are minimal factors of $N/\text{soc } N$. Now there exists right ideals K_i', $i=0,\ldots,s$, of N containing soc N and such that $K_i'/\text{soc } N=K_i$. Also soc N is a finite direct sum $B_1\oplus\ldots\oplus B_r$ of minimal right ideals of N. Set $K=r+s$, $R_0=\{0\}$, $R_i=B_1\oplus\ldots\oplus B_i$ for $i\leqq r$ and $R_i \doteq K'_{i-r}$ for $i<r$ and $i\leqq k$. The fact that the factors R_{i+1}/R_i, $i=0,\ldots,k-1$. are minimal follows from the isomorphism theorems. The lemma is now proved.

Theorems 5.7. *If N is a tame near-ring with minimal condition on right ideals, then N has minimal and maximal condition on right N-subgroups.*

Proof. By 5.6, N has a finite sequence

$$\{0\}=R_0<R_1<R_2<\ldots<R_k=N$$

of right ideals of N such that R_{i+1}/R_i, $i=0,\ldots,k-1$, are minimal factors of N. Let S be the set of all right N-subgroups of N. By 4.2, there are no elements of S lying properly between R_{i+1} and R_i for $i=0,\ldots,k-1$. It follows from 3.2 that the elements of S satisfy both chain conditions. The theorem is therefore proved.

6. Compatible and 2-tame N-groups

We return to the example of a tame N-group given in the introduction (viz. a near-ring generated by a semi-group S of endomorphisms of V where S contains Inn (V), the inner automorphisms of V). As indicated there, such an N-group has a much stronger property than simply being tame.

Suppose μ or $-\mu$ is in S and v in V. We have $(v+w)\mu-v\mu$ is equal to either $v\mu+w\mu-v\mu$ or $w\mu$ for all w in V. Since Inn $(V)\subseteqq S$, it follows that there exists β in N such that

$$(v+w)\mu-v\mu=w\beta$$

for all w in V. Now suppose that for an α in N, there exists γ in N such that

$$(v+w)\alpha-v\alpha=w\gamma$$

for all w in V and take $\lambda=\alpha+\mu$. Computing we see that

$$\begin{aligned}(v+w)\lambda-v\lambda&=(v+w)(\alpha+\mu)-v(\alpha+\mu)\\&=(v+w)\alpha+(v+w)\mu-v\mu-v\alpha\\&=(v+w)\alpha-v\alpha+v\alpha+w\beta-v\alpha\\&=w[\gamma+\beta\kappa]\end{aligned}$$

where κ is the inner automorphism of V induced by $v\alpha$. It follows therefore by induction that given any α in N there exists γ in N such that

$$(v+w)\alpha-v\alpha=w\gamma$$

for all w in V. This is a stronger version of 2.1. We are ready for a definition.

Let N be a near-ring. An N-group V will be called *compatible* if, given v in V and α in N, there exists β in N such that

$$(v+w)\alpha - v\alpha = w\beta$$

for all w in V.

Example. Let V be an Ω-group belonging to some variety $\mathcal{V}$. Let $V(x, \mathcal{V})$ be the polynomials in x over V (see (**6**, p. 215) or (**4**, pp. 12–13)). Now $V(x, \mathcal{V})$ is a near-ring (not necessarily zero-symmetric) under substitution (for x) and pointwise addition. If $(x)p$ is in $V(x, \mathcal{V})$, then the maps of V into V of the form $u \to up$ $(u \in V)$ form a near-ring (this near-ring, the near-ring of polynomial maps (**4**), is not necessarily zero-symmetric). We denote the zero-symmetric part of this near-ring by $P_0(V)$. We claim that V is a compatible $P_0(V)$-group. Indeed, if α is in $P_0(V)$ and v in V, then there exists $(x)p$ in $V(x, \mathcal{V})$ where α is the map $u \to up$ $(u \in V)$. Clearly $(x)q = (v+x)p - vp$ is in $V(x, \mathcal{V})$ and thus with β the map $u \to uq$ $(u \in V)$, it follows that

$$(v+u)\alpha - v\alpha = u\beta$$

for all u in V. This example illustrates the importance of compatible N-groups.

The notion of a 2-tame N-group will now be introduced. Let V be an N-group. Suppose that for each v in V and α in N there exists β *in* N such that

$$(v+w_i)\alpha - v\alpha = w_i\beta$$

for any two elements w_i, $i = 1, 2$, of V. In this case we call V *2-tame*.

Clearly being 2-tame is stronger than being tame and likely to be weaker than being compatible. It is also clear from the definition that one may define an n-tame N-group where n is any cardinal.

If a near-ring N has a faithful 2-tame (compatible) N-group, then we call N *2-tame* (*compatible*).

7. N-endomorphisms of 2-tame N-groups

Although the N-endomorphisms of an N-group V need not form a near-ring the situation is simplified in the case that V is 2-tame by the fact that if δ is an N-endomorphism then so is $1-\delta$.

The following proposition is basic.

Proposition 7.1. *If V is a 2-tame N-group and μ an N-endomorphism of V, then*

$$(v+w\mu)\alpha - v\alpha = (v\mu + w\mu)\alpha - v\mu\alpha$$

for all v and w in V and α in N.

Proof. Since V is 2-tame, there exists β in N such that

$$(v+w)\alpha - v\alpha = w\beta; \quad \text{and}$$
$$(v+w\mu)\alpha - v\alpha = w\mu\beta.$$

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Now

$$w\mu\beta = w\beta\mu$$
$$= [(v+w)\alpha - v\alpha]\mu$$
$$= (v\mu + w\mu)\alpha - (v\mu\alpha)$$

and the proposition follows.

Let V be an N-group and δ an N-endomorphism of V. By $1-\delta$ we mean the map of V into V taking v in V to $v - v\delta$.

Proposition 7.2. *If V is a 2-tame N-group and δ an N-endomorphism of V, then $u(1-\delta)$ commutes with $w\delta$ for all u and w in V.*

Proof. In 7.1 take $\alpha = 1$, $\mu = \delta$ and $v = -u$. It follows that

$$-u + w\delta + u = -u\delta + w\delta + u\delta$$

for all w in V (i.e. $-u(1-\delta) + w\delta + u(1-\delta) = w\delta$). The proposition follows.

Theorem 7.3. *If V is a 2-tame N-group and δ an N-endomorphism of V, then $1-\delta$ is an N-endomorphism.*

Proof. Let v_1 and v_2 be in V. We have

$$(v_1 + v_2)(1-\delta) = v_1 + v_2 - v_2\delta - v_1\delta.$$

Since by 7.2, $v_2 - v_2\delta$ commutes with $v_1\delta$, it follows that

$$(v_1 + v_2)(1-\delta) = v_1 - v_1\delta + v_2 - v_2\delta$$
$$= v_1(1-\delta) + v_2(1-\delta).$$

Thus $1-\delta$ is a group homomorphism of V into V. Now take $\mu = \delta$ and $w = -v$ in 7.1. We see that

$$(v - v\delta)\alpha - v\alpha = -(v\delta\alpha)$$

for all v in V and α in N. Thus

$$v(1-\delta)\alpha = -(v\alpha\delta) + v\alpha.$$

By 7.2, $v\alpha\delta$ commutes with $v\alpha - v\alpha\delta$ and

$$-(v\alpha\delta) + v\alpha = v\alpha - v\alpha\delta.$$

It follows that

$$v(1-\delta)\alpha = v\alpha - v\alpha\delta$$
$$= v\alpha(1-\delta).$$

The theorem is now proved.

If V is an N-group we denote the centre of V by $Z(V)$. Theorem 7.3 has the following corollary.

Corollary. *If V is a 2-tame N-group and δ an N-endomorphism of V, then $V\delta$ and $V(1-\delta)$ are submodules of V, $V = V(1-\delta) + V\delta$ and $V\delta \cap V(1-\delta) \leqq Z(V)$.*

Proof. Since $1-\delta$ is an N-endomorphism, $V\delta$ and $V(1-\delta)$ are N-subgroups of V and thus submodules. Now if v is in V, $v=v-v\delta+v\delta$ and therefore $V(1-\delta)+V\delta=V$. By 7.2, $V(1-\delta)\cap V\delta$ is in the centre of $V\delta$ and $V(1-\delta)$. Therefore $V\delta\cap V(1-\delta)\leqq Z(V)$ and the corollary follows.

An interesting special case of the above corollary is where $Z(V)=\{0\}$. In this case if δ is non-trivial, then neither $V(1-\delta)$ nor $V\delta$ is $\{0\}$ and $V=V(1-\delta)\oplus V\delta$. Thus in this case every N-endomorphism of V induces a unique split of V and a strong version of the Krull-Schmitt theorem holds.

We now look at the case where $Z(V)\neq\{0\}$. A non-zero N-group V will be called *indecomposible* if V is not the direct sum of two non-zero submodules. If V is a 2-tame N-group, the proof of Lemma 14.4 p. 82 of (**2**) goes through without change. From this the following theorem is easily deduced.

Theorem 7.4. *Let N be a near-ring with a 2-tame N-group V where V has maximal and minimal condition on submodules. If*

$$V=U_1\oplus\ldots\oplus U_r=W_1\oplus\ldots\oplus W_s$$

where the U_i, $i=1,\ldots,r$, and W_i, $i=1,\ldots,s$, are indecomposible, then $r=s$ and there exists a permutation p of $1,\ldots,r$ such that U_i is N-isomorphic to $W_{(i)p}$, $i=1,\ldots,r$.

8. Primitive 2-tame near-rings

We call N a *primitive 2-tame near-ring*, if it has a 2-tame N-group which is minimal and faithful.

Let us denote the near-ring of all zero-fixing maps of a group V by $M_0(V)$.

It is the purpose of this section to look more closely at primitive 2-tame near-rings. Indeed in the case where N has minimal condition these near-rings are either of the form $M_0(V)$ (where V is a finite group) or a complete matrix ring over a division ring.

Lemma 8.1. *Let N be a near-ring, V a minimal N-group which is 2-tame,* $\mathrm{Aut}_N V$ *the group of all N-automorphisms of V and* 0 *the zero N-endomorphism of V. If* $\mathrm{Aut}_N V\neq\{1\}$ *then* $\mathrm{Aut}_N V\cup\{0\}$ *is a division ring under composition and pointwise addition.*

Proof. Since V is minimal, any non-trivial N-endomorphism of V is in $\mathrm{Aut}_N V$. Let $\delta\neq 1$ be in $\mathrm{Aut}_N V$. By 7.3, $1-\delta$ and $1-\delta^{-1}$ are in $\mathrm{Aut}_N V$. But $\delta(1-\delta^{-1})=\delta-1$. Thus $\delta-1$ is in $\mathrm{Aut}_N V$. Hence $(1-\delta)(\delta-1)^{-1}$ is in $\mathrm{Aut}_N V$. Now

$$\begin{aligned}v(1-\delta)(\delta-1)^{-1}&=-v(\delta-1)(\delta-1)^{-1}\\&=-v\end{aligned}$$

for all v in V, and the map $-1: v\rightarrow -v$ is in $\mathrm{Aut}_N V$. It follows that V is abelian.

We shall show that $\mathrm{Aut}_N V\cup\{0\}$ is an abelian group under addition. Clearly 0 is an additive identity of $\mathrm{Aut}_N V\cup\{0\}$ and, if μ is in $\mathrm{Aut}_N V\cup\{0\}$, then $-\mu=(-1)\mu$, is an additive inverse of μ. Suppose δ_1 and δ_2 are in $\mathrm{Aut}_N V\cup\{0\}$. If either δ_1 or δ_2 is zero then it is certainly true that $\delta_1-\delta_2$ is in $\mathrm{Aut}_N V\cup\{0\}$. If $\delta_1=(-1)\delta_2=-\delta_2$, then $\delta_1+\delta_2=0$, and if $\delta_1\neq(-1)\delta_2$ then $1-\delta_1^{-1}(-1)\delta_2$ is in $\mathrm{Aut}_N V$ by 7.3. In this case

$$\delta_1(1-\delta_1^{-1}(-1)\delta_2)=\delta_1-(-1)\delta_2=\delta_1+\delta_2$$

is in $\mathrm{Aut}_N V$. Hence, under addition, $\mathrm{Aut}_N V \cup \{0\}$ is a group. This group is abelian since V is abelian. The left distributive law clearly holds and since the elements of $\mathrm{Aut}_N V \cup \{0\}$ are distributive in $M_0(V)$ the right distributive law holds. Also a non-zero element of $\mathrm{Aut}_N V \cup \{0\}$ has a multiplicative inverse and the lemma is proved.

Lemma 8.2. *If N is a near-field with a minimal 2-tame N-group V, then N is a division ring.*

Proof. By (**6**, 8.3, p. 237) N has no proper right N-subgroups $(0: V) = \{0\}$, and N is faithful on V. Clearly if V has only one non-zero element, then N is the field of order two. Now if u and v are two non-zero elements of V, then $(0: u) = 0 = (0: v)$ and there exists an N-isomorphism μ of $uN(= V)$ onto $vN(= V)$ such that $u\mu = v$. Now, by 8.1, $1 + \mu$ is an N-endomorphism of V. If α is in N, then

$$\begin{aligned}(u+v)\alpha &= (u+u\mu)\alpha \\ &= u\alpha(1+\mu) \\ &= u\alpha + u\mu\alpha \\ &= u\alpha + v\alpha\end{aligned}$$

and the elements of N distributive over V. Now $V \neq \{0\}$ and, since V is a unitary N-group, $(0: V) = \{0\}$. The elements of N are distributive in $M_0(V)$ and the lemma follows.

Before we state our main result on primitive 2-tame near-rings we need a proposition.

A submodule U of an N-group V will be called *abelian* if addition in U^+ is commutative and elements of N distribute over U (i.e. if $N/(0: U)(= A)$ is a ring and U is an A-module in the ring sense).

Proposition 8.3. *If an N-group V is a direct sum $V_1 \oplus V_2$ of two minimal N-groups and if V_1 is non-abelian, then V_1 and V_2 are the only proper submodules of V.*

Proof. If W is a proper submodule of V and $W \neq V_i$, $i = 1, 2$, then by (**6**, 2.23, p. 48)

$$(V_1 + W) \cap (V_2 + W)/W$$

is abelian. But

$$V_1 + W = V_2 + W = V$$

and, since V/W is N-isomorphic to

$$V_1/V_1 \cap W = V_1/\{0\},$$

V_1 is abelian. This contradiction establishes the proposition.

If N is a near-ring and V a faithful minimal N-group we may regard N as a subnear-ring of $M_0(V)$. We say N is *dense* in $M_0(V)$, if it is dense with respect to the finite topology (see (**6**, 4.26, p. 111)).

Theorem 8.4. *Let N be a near-ring with a faithful minimal N-group V. The N-group V is 2-tame if, and only if, N is either dense in $M_0(V)$ or N is a ring.*

Proof. Suppose N is a ring. Since for $v \neq 0$ in V, $vN = V$, it is evident that V is a ring module. Thus if v and w are in V and α is in N, then $(v+w)\alpha - v\alpha = w\alpha$. Clearly V is 2-tame.

Suppose N is dense in $M_0(V)$. It follows that if w_1 and w_2 are any two distinct elements of V, then there exists β in V such that $w_1\beta = v_1$ and $w_2\beta = v_2$ for any two elements v_1 and v_2 of V. Let v be in V and α in N and take

$$v_i = (v+w_i)\alpha - v\alpha, \quad i = 1, 2.$$

The fact that V is 2-tame follows from the existence of β.

Now suppose that V is 2-tame and that N is not a ring. The result will follow from (**6**, 4.54, p. 129) if we can show that an N-automorphism μ of V must in fact be the identity.

Let v be a non-zero element of V. If for all non-zero elements u of V, $(0: v) = (0: u)$, then $(0: v) = \cap(0: u)$ where u ranges over all elements of V. In this case $(0: v) = 0$, since $(0: V) = \{0\}$. Therefore $vN(=V)$ is *N-isomorphic to N* and N has no proper right N-subgroups. By (**6**, 8.3, p. 237) N is a near-field and by 8.2, N is a division ring.

Thus we may assume that there exists a non-zero element u in V such that $(0: u) \neq (0: v)$. If α is in $(0: v)$, then $v\alpha = v\mu\alpha = 0$ and, by 7.1 it follows that $(v + w\mu)\alpha = (v\mu + w\mu)\alpha$ for all w in V. Thus

$$(v+w_1)\alpha = (v\mu + w_1)\alpha \tag{1}$$

for all w_1 in V and α in $(0: v)$. Now take w_1 such that $v + w_1 = u$ and set $x = v\mu + w_1$. Since $v \neq 0$, $(0: v)$ is, by (**6**, p. 103), a maximal right ideal of N. Also $(0: u) \neq (0: v)$. Thus $(0: u) + (0: v) = N$ and $1 = e_1 + e_2$ where e_1 is in $(0: u)$ and e_2 is in $(0: v)$. Now

$$u = ue_1 + ue_2 = ue_2 = xe_2 \tag{2}$$

by (1). Since $xe_2 \neq 0$, $x \neq 0$. We shall show that $(0: u) \cap (0: v) \leqq (0: x)$. If γ is in $(0: u) \cap (0: v)$ then, since V is 2-tame, there exists λ in N such that

$$(v+w_1)\gamma - v\gamma = w_1\lambda$$

and

$$(v + w_1\mu^{-1})\gamma - v\gamma = w_1\mu^{-1}\lambda.$$

Now $v + w_1 = u$ and, since γ is in $(0: u) \cap (0: v)$, it follows that $w_1\lambda = 0$. Therefore $w_1\mu^{-1}\lambda = w_1\lambda\mu^{-1} = 0$. Thus

$$(v + w_1\mu^{-1})\gamma - v\gamma = 0$$

and, since $v\gamma = 0$, we see that $(v + w_1\mu^{-1})\gamma = 0$. It now follows that

$$\begin{aligned} 0 &= (v + w_1\mu^{-1})\gamma\mu \\ &= (v\mu + w_1)\gamma \\ &= x\gamma. \end{aligned}$$

Hence γ is in $(0: x)$ and $(0: u) \cap (0: v) \leqq (0: x)$.

Now $x \neq 0$ and $(0: x)$ is a maximal right ideal of N. By (2) $xe_2 \neq 0$ and $(0: x) \neq (0: v)$. Thus $(0: x) + (0: v) = N$ and with $R = (0: u) \cap (0: v)$, it follows that

$$N/R = (0: x)/R + (0: v)/R = (0: u)/R \oplus (0: v)/R.$$

Now $(0: u)/R$ is N-isomorphic to $N/(0: v)$ which is N-isomorphic to $vN(=V)$. Similarly $(0: v)/R$ is N-isomorphic to V. Since $(0: x)/R$ is a proper submodule of N/R, it follows by

8.3 that $(0:x)/R$ is equal to either $(0:u)/R$ or $(0:v)/R$. But since $(0:x)\neq(0:v)$, $(0:u)/R=(0:x)/R$ and $(0:u)=(0:x)$. Thus $x=xe_2$. By (2) $u=x$. Thus $v+w_1=v\mu+w_1$ and $v=v\mu$. This is true for any v in V and μ is therefore the identity. The proof is complete.

In Section 5 we saw that for tame near-rings minimal condition on right ideals implies maximal condition on right ideals. However, if 2-tame near-rings depart far enough from rings, minimal condition implies finiteness.

We shall call a near-ring N *ring-free* if no non-zero homomorphic image of N is a ring.

With the aid of 8.4 it is possible to deduce the following theorem.

Theorem 8.5. *If N is a 2-tame ring-free near-ring with minimal condition on right ideals then N is finite.*

This is proved by considering right ideals

$$\{0\}=R_0<R_1<\ldots<R_k=N$$

as in lemma 5.6 and, out of all minimal factors of N, N-isomorphic to R_{i+1}/R_i (i in $\{0,\ldots,k-1\}$) choosing one R/H with R minimal. Then, as in the proof of 4.2, one finds an element v of a faithful 2-tame N-group V such that $vR>vH$. The proof is completed by observing that the 2-tame N-group vR/vH is N-isomorphic to R_{i+1}/R_i from which it follows, by 8.4 and (**6**, 4.61, p. 132), that R_{i+1}/R_i is finite.

9. Centralisers of abelian submodules

In this section we prove certain specialised results to be used in proving the final theorem in the next section.

Let V be a group and H_1 and H_2 subsets of V. By $[H_1, H_2]$ we shall mean the subset of V consisting of all $-h_1-h_2+h_1+h_2$ where h_i is in H_i, $i=1,2$.

Let U be a submodule of a tame N-group V, then $C_V(U)$ will denote the set of all elements v in V such that $[vN, U]=\{0\}$.

Proposition 9.1. *If U is a submodule of a tame N-group V, then $C_V(U)$ is a submodule.*

Proof. Since the set of all v in V such that $[v, U]=\{0\}$ forms a subgroup H of V, $C_V(U)$ is simply the sum of all submodules of V contained in H.

From now on we shall be dealing with the following situation.

(a) N is a near-ring with minimal condition on right ideals;
(b) V is a faithful compatible N-group;
(c) U is an abelian minimal N-subgroup of V; and
(d) $(U:V)\not\leq(0:U)$.

Proposition 9.2. *There exists γ in $(U:V)$ such that $u\gamma=u$ for all u in U.*

Proof. Since U is minimal and abelian and N has minimal condition $N/(0:U)$ is a simple ring. Thus $(0:U)$ is a maximal ideal of N and by (d), $(0:U)+(U:V)=N$. Let

$1=\gamma_1+\gamma_2$ where γ_1 is in $(0: U)$ and γ_2 is in $(U: V)$. If we take $\gamma=\gamma_2$ the proposition follows.

Lemma 9.3. *Let v_i, $i \in I$, be a system of coset representatives of V/U in V. For each k in I there exists ξ_k in N such that*

(*i*) $\xi_k \equiv 1 \bmod (U: V)$;

(*ii*) $(v_k+u)\xi_k=(v_k)\xi_k$ *for all u in U; and*

(*iii*) $u\xi_k=0$ *for all u in U.*

Proof. Let γ be as in 9.2. There exists α_k in N such that

$$(-v_k+v)\gamma-(-v_k)\gamma=v\alpha_k$$

for all v in V. Since γ is in $(U: V)$, it follows that for all v in V, $v\alpha_k$ is in U and α_k is in $(U: V)$. Set $\rho_k=1-\alpha_k$. Clearly $\rho_k \equiv 1 \bmod (U: V)$. Now

$$\begin{aligned}(v_k+u)\rho_k &= v_k+u-(v_k+u)\alpha_k\\ &= v_k+u-[(-v_k+v_k+u)\gamma-(-v_k)\gamma]\\ &= v_k+u+(-v_k)\gamma-u\gamma\end{aligned}$$

for all u in U. Since $(-v_k)\gamma$ is in U, U is abelian and $u\gamma=u$ for all u in U, it follows that

$$(v_k+u)\rho_k=v_k+(-v_k)\gamma$$

for all u in U. Taking $u=0$ we see that $v_k+(-v_k)\gamma=v_k\rho_k$ and

$$(v_k+u)\rho_k=v_k\rho_k$$

for all u in U. Suppose, without loss of generality, that 0 is in I and v_0 represents the coset U. Take $\xi_k=\rho_k\rho_0$. Now $u\rho_0=v_0\rho_0$ for all u in U and since $0\rho_0=0$, it follows that $u\rho_0=0$ for all u in U. Thus $u\xi_k=u\rho_k\rho_0=0$ for all u in U since $u\rho_k$ is in U. We have shown (iii) holds. Also, since $\rho_k \equiv \rho_0 \equiv 1 \bmod (U: V)$, $\xi_k \equiv 1 \bmod (U: V)$ and (*i*) holds. Finally

$$\begin{aligned}(v_k+u)\xi_k &= v_k\rho_k\rho_0\\ &= v_k\xi_k\end{aligned}$$

for all u in U and (*ii*) holds. The lemma is proved.

Lemma 9.4. *Let v_i, $i \in I$ be a system of coset representatives of V/U in V. Let S be the subset of I consisting of all s in I such that $v_s \equiv 0 \bmod C_V(U)$. If S is a proper subset of I and k is in $I-S$, then there exists λ_k in $(U: V)$ such that*

(*i*) $(v_k+u_1)\lambda_k \neq v_k\lambda_k$ *for some u_1 in U; and*

(*ii*) $u\lambda_k=0$ *for all u in U.*

Proof. Let ξ_k be as in 9.3. Set $\sigma_k=-\xi_k+1$. Clearly α_k is in $(U: V)$ and

$$\begin{aligned}(v_k+u)\alpha_k &= -v_k\xi_k+v_k+u\\ &= v_k\alpha_k+u \qquad (1)\end{aligned}$$

for all u in U. Since v_k is not in $C_V(U)$, there exists η in N such that

$$[v_k\eta, U]\neq 0. \qquad (2)$$

Take $\lambda_k=[\eta, \alpha_k]$. Clearly λ_k is in $(U\colon V)$. If u is in U, then

$$\begin{aligned} u\lambda_k &= -u\eta - u\alpha_k + u\eta + u\alpha_k \\ &= 0 \end{aligned}$$

since U is abelian. Thus (ii) holds. If u is in U then, since U is a submodule of V,

$$(v_k+u)\eta = u' + (v_k)\eta$$

where u' is in U. Thus

$$\begin{aligned} (v_k+u)\lambda_k &= -(v_k)\eta - u' - (v_k+u)\alpha_k + u' + v_k\eta + (v_k+u)\alpha_k \\ &= [v_k\eta, (v_k+u)\alpha_k], \end{aligned} \tag{3}$$

since $(v_k+u)\alpha_k$ is in U and U is abelian. From (1) and the fact that $v_k\alpha_k$ is in U, $(v_k+u)\alpha_k$ takes all values of U as u ranges over U. From (2) and (3)

$$(v_k+u)\lambda_k \neq 0 \tag{4}$$

for some u in U. If in (3) we take $u_2=-(v_k)\alpha_k$, then from (1), $(v_k+u_2)\alpha_k=0$ and

$$\begin{aligned} (v_k+u_2)\lambda_k &= [v_k\eta, 0] \\ &= 0. \end{aligned} \tag{5}$$

Now (i) follows as a consequence of (4) and (5). The lemma is proved.

Lemma 9.5. *Let v_i, $i\in I$. be a system of coset representatives of V/U in V. Let S be the subset of I consisting of all s in I such that $v_s\equiv 0 \bmod C_V(U)$.*

If S is a proper subset of I and k is in $I-S$, then there exists γ_k in $(U\colon V)$ such that

(i) $(v_k+u)\gamma_k = v_k\gamma_k$ for all u in U; and
(ii) $u_1\gamma_k\neq 0$ for some u_1 in U.

Proof. Let v_i be the representative of the coset $-v_k+U$ of U. Let λ_i be as in 9.4. Take γ_k to be an element of N such that

$$(v_i+v)\lambda_i - v_i\lambda_i = v\gamma_k$$

for all v in V. Since λ_i is in $(U\colon V)$ so is γ_k. Now

$$(v_i+u_1)\lambda_i \neq v_i\lambda_i$$

for some u_1 in U and thus $u_1\gamma\neq 0$. Hence (ii) holds. Also

$$(v_k+u)\gamma_k = (v_i+v_k+u)\lambda_i - v_i\lambda_i$$

for all u in U. Since v_i+v_k is in U, $(v_i+v_k+u)\lambda_i=0$ and

$$(v_k+u)\gamma_k = -v_i\lambda_i$$

for all u in U. On taking $u=0$ we see that $v_k\gamma_k=-v_i\lambda_i$. Thus

$$(v_k+u)\gamma_k = v_k\gamma_k$$

for all u in U. Thus (i) follows and the lemma is proved.

Lemma 9.6. *Let v_i, $i \in I$, be a system of coset representatives of V/U in V. Let S be the subset of I consisting of all s in I such that $v_s \equiv 0 \bmod C_V(U)$. If S is a proper subset of I and k is in $I-S$, then there exists δ_k in N such that*

(*i*) $\delta_k \equiv 1 \bmod (U: V)$;
(*ii*) $(v_k+u)\delta_k \equiv v_k\delta_k$ *for all u in U; and*
(*iii*) $u\delta_k = u$ *for all u in U.*

Proof. Let γ_k be as in 9.5 and $\bar{\gamma}_k$ be the image of γ_k in $N/(0: U)$. By (*ii*) of 9.5, $u_1\bar{\gamma}_k \neq 0$ for some u_1 in U. Since U is abelian, $N/(0: U)$ is a ring which is primitive on U and has minimal condition. Thus $N/(0: U)$ is simple and there exists $\bar{\alpha}_i$, $i=1,\ldots,r$, and $\bar{\beta}_j$, $j=1,\ldots,s$, in $N/(0\,;U)$ such that

$$\sum_{i,j} \bar{\alpha}_i\bar{\gamma}_k\bar{\beta}_j = \bar{1} \tag{1}$$

(where $\bar{1}$ is the identity of $N/(0: U)$). By assumption (d), $(0: U)+(U: V)=N$ and there exists α_i and β_j in $(U: V)$ such that $u\bar{\alpha}_i = u\alpha_i$ and $u\bar{\beta}_j = u\beta_j$ for $i=1,\ldots,r$, and $j=1,\ldots,s$. Let ξ_k be as in 9.3. Set

$$\rho_{ij} = (\xi_k+\alpha_i)\gamma_k\beta_j$$

for $i=1,\ldots,r$, and $j=1,\ldots,s$. Clearly each ρ_{ij} is in $(U: V)$ which has an abelian additive group. The sum $\sum_{i,j}\rho_{ij}$ is therefore well defined. Take

$$\delta_k = \xi_k + \sum_{i,j}\rho_{ij}.$$

Clearly $\delta_k \equiv 1 \bmod (U: V)$. Now for u in U

$$(v_k+u)\rho_{ij} = [v_k\xi_k+(v_k+u)\alpha_i]\gamma_k\beta_j$$

and $v_k\xi_k = v_k+u'$ $(u' \in U)$, since $\xi_k \equiv 1 \bmod (U: V)$. Thus

$$(v_k+u)\rho_{ij} = [v_k+u'+(v_k+u)\alpha_i]\gamma_k\beta_j$$

and, since $u'+(v_k+u)\alpha_i$ is in U, it follows from (*i*) of 9.5 that

$$(v_k+u)\rho_{ij} = v_k\gamma_k\beta_j = v_k\rho_{ij}$$

as can be seen on taking $u=0$. Thus for all u in U

$$\begin{aligned}(v_k+u)\delta_k &= v_k\xi_k + \sum_{i,j} v_k\rho_{ij}\\ &= v_k\delta_k.\end{aligned}$$

Hence (*ii*) follows. Now for u in U

$$\begin{aligned}u\rho_{ij} &= (u\xi_k+u\alpha_i)\gamma_k\beta_j\\ &= u\alpha_i\gamma_k\beta_j\end{aligned}$$

and by (1), $\sum_{i,j} u\rho_{ij} = u$. Thus for u in U

$$\begin{aligned}u\delta_k &= u\xi_k + \sum_{i,j} u\rho_{ij}\\ &= u\end{aligned}$$

and (*iii*) follows. The lemma is proved.

Lemma 9.7. *Let v_i, $i \in I$, be a system of coset representatives of V/U in V. Let S be the subset of I consisting of all s in I such that $v_s \equiv 0 \bmod C_V(U)$. If S is a proper subset of I, then there exists ρ in N such that*

(*i*) $\rho \equiv 1 \bmod (U: V)$;
(*ii*) $(v_k + u)\rho = v_k\rho$ *for all k in $I - S$ and u in U; and*
(*iii*) $u\rho = u$ *for all u in U.*

Proof. Let S_1 be the set of all δ_k as in 9.6. Let S_2 be the set of all finite products of elements of S_1. Take ρ in S_2 such that ρN is minimal (this is possible by 5.7). Clearly $\rho \equiv 1 \bmod (U: V)$ and $u\rho = u$ for all u in U. Suppose for i in $I - S$

$$(v_i + u_1)\rho \neq v_i\rho \tag{1}$$

for some u_1 in U. Consider $\rho\delta_i$ where δ_i is as in 9.6. Now for u in U

$$(v_i + u)\rho = v_i + u'$$

where u' is in U, because $\rho \equiv 1 \bmod (U: V)$. Thus

$$(v_i + u)\rho\delta_i = v_i\delta_i = v_i\rho\delta_i$$

for all u in U. Hence

$$(v_i + u)\rho\delta_i\beta = v_i\rho\delta_i\beta$$

for all β in N and

$$(v_i + u)\eta = v_i\eta$$

for all η in $\rho\delta_i N$. Now $\rho\delta_i$ is in S_2 and $\rho\delta_i N \leqq \rho N$. Since ρ is in ρN we see from (1) that $\rho\delta_i N < \rho N$. This contradiction to the minimality of ρN establishes that ρ satisfies (*ii*). The lemma is now proved.

Lemma 9.8. *Let v_i, $i \in I$, be a system of coset representatives of V/U in V. For k in I there exists π_k in N such that*

(*i*) $(v_k + u_1)\pi_k \neq v_k\pi_k$ *for some u_1 in U; and*
(*ii*) $(v_i + u)\pi_k = v_i\pi_k$ *for all u in U and* $v_i \not\equiv v_k \bmod C_V(U)$.

Proof. Let v_j represent the coset $-v_k + U$. Let ρ be as in 9.7. Take π_k to be an element of N such that

$$(-v_k + v)\rho - (-v_k)\rho = v\pi_k$$

for all v in V. Now taking $v = v_k + u$ where u is in U we see

$$\begin{aligned}(v_k + u)\pi_k &= u\rho - (v_j)\rho \\ &= u - (v_j)\rho.\end{aligned}$$

Clearly $(v_k + u)\pi_k \neq v_k\pi_k$ for some u in U and (*i*) follows. Now if i in I is such that $v_i + u \not\equiv v_k \bmod C_V(U)$ where u is in U, then, since

$$(v_i + u)\pi_k = (-v_k + v_i + u)\rho - (-v_k)\rho$$

and

$$-v_k+v_i+u=v_r+u'$$

for some u' in U and $v_r \not\equiv 0 \bmod C_V(U)$, it follows that

$$(v_i+u)\pi_k = v_r\rho-(-v_k)\rho$$
$$= v_i\pi_k$$

and the lemma follows.

We are now in a position to state the main result of this section in the form of a theorem.

Theorem 9.9. *If conditions* (*a*), (*b*), (*c*) *and* (*d*) *are satisfied, then the index of* $C_V(U)$ *in* V *is finite.*

Proof. We may assume that $C_V(U) < V$. Let v_i, $i \in I$, be a system of coset representatives of V/U in V. We rename some of the indices in I with the ordinals $1, 2, \dots$ in such a way that the v_b, b an ordinal, are coset representatives of $C_V(U)$ in V.

Let r be a positive integer and let M_r be the set of all α in N such that

$$(v_i+u)\alpha = v_i\alpha$$

for all u in U and all $i \in I$, where

$$v_i \not\equiv v_j \bmod C_V(U)$$

for any j in $\{1, \dots, r\}$. Clearly the M_r, $r=1, 2, \dots$ are right N-subgroups of N and

$$M_1 \leqq M_2 \leqq M_3 \leqq \dots .$$

Let $\pi_2, \pi_3, \dots$ be as in 9.8. We have π_2 is in M_2 but not in M_1; π_3 is in M_3 but not in M_2 etc. Thus, if the set of indexing ordinals is infinite, we have a contradiction to 5.7. It follows that the number of cosets of $C_V(U)$ in V is finite and the theorem holds.

10. Minimal condition and the near-ring generated by inner automorphisms

In this section we prove that if the near-ring generated by the inner automorphisms of a group has minimal condition, then it is finite.

We state a lemma for tame N-groups where N has minimal condition.

Lemma 10.1. *Let* N *be a near-ring with minimal condition on right ideals and* V *a faithful tame* N*-group. If* U *is a minimal* N*-subgroup of* V *and* $(U: V) \neq \{0\}$, *then* $(U: V)$ *is a finite direct sum of right ideals* N*-isomorphic to* U.

Proof. Since $V(U: V) \subseteqq U$, it follows that $V(U: V)J(N)=\{0\}$, $(U: V)J(N)=\{0\}$ and, by 5.3, $(U: V) \leqq \operatorname{soc} N$. Now $\operatorname{soc} N$ is absolutely reducible in N. Thus $(U: V)$ is a direct sum of minimal right ideals (cf. (**2**, 15.2, p. 86)). By minimal condition this direct sum is finite. Thus

$$(U: V) = R_1 \oplus \dots \oplus R_k$$

where R_i, $i=1, \dots, k$, are minimal right ideals of N. For any R_i, i in $\{1, \dots, k\}$, there exists v in V such that $vR_i \neq \{0\}$. But since $R_i \leqq (U: V)$, $vR_i = U$. Clearly $(0: v) \cap R_i = \{0\}$ and it follows that R_i is N-isomorphic to U. The proof is complete.

If V is a group, then $I(V)$ will denote the near-ring generated by the inner automorphisms of V. In Section 6 we saw that V is a compatible $I(V)$-group. Clearly the $I(V)$-subgroups of V are precisely the normal subgroups.

Proposition 10.2 *If V is a group and U a normal subgroup of V, then*

$$I(V/U) \cong I(V)/(U: V).$$

Let N be a near-ring with a faithful compatible N-group V and a minimal N-subgroup U. In Section 9 we were considering the situation where $(U: V) \not\leq (0: U)$. These considerations are now required.

Lemma 10.3. *Let V be a group with a minimal normal subgroup U. If $I(V)$ has minimal condition on right ideals and $(U: V) \not\leq (0: U)$, then U is finite.*

Proof. Suppose U^+ is non-abelian. We have U is a minimal compatible $I(V)/(0: U)$-group. In particular U is 2-tame. Also for u in U, $uN' = U$ where $N' = I(V)/(0: U)$. Thus $[N']^+$ is non-abelian and a non-ring. Since N' has minimal condition, it follows by 8.4 that $N' \cong M_0(U)$ and U is finite (see (**6**, 7.19, p. 198)).

If U^+ is abelian, then the inner automorphisms of V distribute over U and $I(V)/(0: U)(= N')$ is a ring and U an N'-module in the ring sense. The conditions (a) to (d) of Section 9 are satisfied. Thus the index $|V: C_V(U)|$ of $C_V(U)$ in V is finite by 9.9. Now all v in V such that $[v, U] = \{0\}$ form a normal subgroup C of V containing $C_V(U)$. Thus $C_V(U) \leq C$ and, since C is an $I(V)$-submodule, $C = C_V(U)$, and $|V: C|$ is finite. Let $v_1, \ldots, v_n$. be a system of coset representatives of V/C in V. Let u_i be a non-zero element of U. Clearly U is the normal subgroup of V generated by u_1, and thus U is generated by all elements of the form

$$v_i + v + u_1 - v - v_i$$

$i = 1, \ldots, n$, with v in C. Thus U is generated by the elements of the form

$$v_i + u_1 - v_i \quad i = 1, \ldots, n,$$

of V. Hence U is a finitely generated abelian group and is a finite direct sum $A_1 \oplus \ldots \oplus A_k$ of cyclic groups. Also being a minimal normal subgroup of V, it is characteristically simple and therefore it is a finite elementary abelian p-group. The lemma is proved.

Theorem 10.4. *Let V be a group. If $I(V)$ has minimal condition on right ideals, then it is finite.*

Proof. By 5.6 there exists a positive integer $k(V)$ and right ideals

$$R_0 = \{0\} < R_1 < \ldots < R_{k(V)} = I(V)$$

of $I(V)$ such that for $i = 0, \ldots, k(V) - 1$, R_{i+1}/R_i is a minimal factor of $I(V)$. We proceed by induction on $k(V)$. If $k(V) = 0$, then $I(V) = \{0\}$, the identity map of V onto V is zero and, $V = \{0\}$. It may therefore be assumed that $k(V) > 0$ and that $I(V) \neq \{0\}$.

Out of all ideals of the form $(W: V)$ where W is a submodule of V, choose one $(H: V)$ such that $(H: V)$ is minimal for being non-zero. We also assume that H is minimal since, if

H_i, $i \in I$, are such that $(H_i: V) = (H: V)$ and $W = \bigcap_{i \in I} H_i$, then $(W: V) = (H: V)$. Take R to be a minimal right ideal of $I(V)$ contained in $(H: V)$. By 4.2, R is a minimal $I(V)$-group. Set

$$K = \sum_{v \in V} vR.$$

For each v in V, $vR = \{0\}$ or vR is $I(V)$-isomorphic to R. Thus K is, by 5.1, absolutely reducible in V. Also $vR \leqq H$, $R \leqq (H: V)$, and we see that $K \leqq H$. But $R \leqq (K: V) \neq \{0\}$. From the minimality of H, $H = K$. From the absolute reducibility of K, $H = K_0 \oplus K_1$, where K_0 is a minimal $I(V)$-subgroup of V. Set $Y = V/K_1$ and $X = K/K_1$. It follows that X is a minimal submodule of Y and by 10.2

$$I(Y)/(X: Y) \cong I(Y/X) \cong I(V/K) \cong I(V)/(K: V). \tag{1}$$

Now $Y = V/K_1$ and thus

$$I(Y) \cong I(V)/(K_1: V)$$

by 10.2. Since, by the minimality of K, $(K_1: V) = \{0\}$ it follows that

$$I(Y) \cong I(V). \tag{2}$$

But $(K: V) > \{0\}$ and $k(V) > k(V/K)$ so that by (1) we may assume $I(Y)/(X: Y)$ is finite. From (2) it will then follow that $I(V)$ is finite if it is shown that $(X: Y)$ is finite. By 10.1 this will follow if it is shown that X is finite. It follows from 10.3 that we may assume $(X: Y) \leqq (0: X)$. Now $I(Y)/(0: X)(= N)$ has X as a minimal N-group. Also

$$[I(Y)/(X: Y)]/[(0: X)/(X: Y)] \cong N$$

and N is finite. Since for $x \neq 0$ in X, $xN = X$, it follows that X is finite. The theorem is proved.

11. Comments

Although it has been assumed throughout that the tame N-groups considered are unitary this assumption is not always required. Indeed the results of Section 1 do not need the existence of an identity and the results of Sections 3 and 4 hold if N has an identity although the N-groups considered may not be unitary.

One defect in the theory developed is that homomorphic images of tame (2-tame, compatible) near-rings need no longer be tame (2-tame, compatible). This defect can, to a certain extent, be overcome by generalising the notion of tame.

It seems hopeful that a worthwhile theory of tame near-rings with maximal condition on right ideals can be developed and that results resembling those due to Goldie (see (**3**, Ch. 3)) for rings, also hold.

In the considerations of Section 9 we dealt with the situation where V was a faithful compatible N-group with a minimal submodule U such that $(U: V) \not\leqq (0: U)$. In a more detailed study of the structure of a compatible near-ring N with minimal condition this situation is of special importance. In this case the near-ring N has a wreath product structure.

The final theorem (10.4) has as a consequence that if the near-ring generated by the inner automorphisms of a group V has a minimal condition on right ideals, then V is centre by finite. If maximal condition on right ideals is assumed it is still possible to obtain information about V. A question that would seem to be of importance in this direction is whether 9.9 holds with the weaker assumption of maximal condition.

As was noted in Section 6 the near-ring of zero-symmetric polynomial maps over an Ω-group forms a compatible near-ring. Dr G. Baird and the author have considered this situation. It turns out that finite simple Ω-groups are, apart from certain interesting exceptions, polynomially complete (cf. 8.4; for a definition of polynomial completeness see (**6**, 7.74, p. 219) or (**4**) or (**5**)). The exceptions have been bound to be Ω-groups bearing an interesting similarity to Lie-algebras. We hope to publish these results shortly.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF AUCKLAND
NEW ZEALAND