THE CENTRE AND THE DEPTH OF THE CENTRE OF A TREE MAP

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Let X be a tree and f be a continuous map from X into itself. Denote by $P(f)$ and $R(f)$ the set of periodic points and the set of recurrent points of f respectively. We show in this note that the centre is $\overline{P(f)}$ and the depth of the centre is at most 3. Furthermore we have $\overline{P(f)} = \overline{R(f)}$.

1. INTRODUCTION

Let X be a compact metric space and $f \in C(X, X)$. A point $x \in X$ is said to be *a nonwandering point of* f if for any neighbourhood V_x of x there exists $n \in \mathbb{N}$ such that $f^n(V_x) \cap V_x \neq \emptyset$. The set of nonwandering points of f will be denoted by $\Omega(f)$. Let $\Omega_1(f) = \Omega(f)$, $\Omega_n(f) = \Omega(f \mid_{\Omega_{n-1}})$, $n \geqslant 2$. Then $\Omega_\infty(f) = \bigcap_1^\infty \Omega_n(f)$ is called *the centre* of f. The minimal $n \in \mathbb{N} \cup \{\infty\}$ such that $\Omega_n(f) = \Omega_\infty(f)$ will be called the *depth of the centre of* f.

We say that $x \in X$ is *a periodic point of* f *of period* n if there exists $n \in \mathbb{N}$ such that $f^n(x) = x$ but $f^i(x) \neq x$, $1 \leqslant i \leqslant n - 1$. If $n = 1$ then x will be called *a fixed point of* f. A point $x \in X$ is *a recurrent point of* f if for every neighbourhood V_x of x there is $n \in \mathbb{N}$ such that $f^n(x) \in V_x$. The sets of fixed points, periodic points and recurrent points of f will be denoted by $F(f)$, $P(f)$ and $R(f)$ respectively.

The notion of periodic point, recurrent point and centre of a continuous map f from a compact metric space into itself is one of the most important notions in dynamical systems. In general it is difficult to determine the centre and the depth of the centre of f and it is not always true that $\overline{P(f)} = \overline{R(f)}$. For example, let f be an irrational rotation from a circle into itself. Then we have that $P(f) = \emptyset$ and $R(f)$ is the circle.

If $X = [0, 1]$ it is known that $\Omega_2(f) = \overline{P(f)}$ and the centre is $\overline{P(f)}$ and the depth of the centre is at most 2 [1] (for a simple proof see [3]). In [2] this result is extended to continuous maps from n-od to itself having zero as a fixed point, where $n\text{-od} = \{z \in \mathbb{C} : z^n \in [0, 1]\}$. In this note we shall generalise these results to general continuous tree maps. To state the results we start with some more notion.

By *a tree* we mean a compact connected one-dimensional branched manifold without cycles. A point $x \in X$ is *a branched point of* f if there is a neighbourhood of x

which is homeomorphic to $\{z \in \mathbf{C} : z^k \in [0, 1]\}$ for some $k \geqslant 3$. The set of branched points of X is a finite subset of X and will be denoted by $O = \{o_1, o_2, \ldots, o_n\}$. For $x \in X \setminus O$, there is a neighbourhood V_x of x such that V_x is homeomorphic to $(0, 1)$ or $(0, 1]$. In the latter case we say that x is *an end point of X*. We assume that the set of end points of X is the finite set $\{e_1, e_2, \ldots, e_t\} \subset X$.

Let ρ be a metric on a tree X. If X_1, X_2 are two subsets of X then we define $\rho(X_1, X_2) = \inf\{\rho(x_1, x_2) : x_i \in X_i, i = 1, 2\}$. Furthermore we assume that $\rho(\emptyset, X_1) > 0$ for every subset X_1 of X.

Now we are ready to state the main results of this note.

THEOREM 2.6. *Let X be a tree and $f \in C(X, X)$. Then $\overline{R(f)} = \overline{P(f)}$.*

THEOREM 2.7. *Let X be a tree, $f \in C(X, X)$ and O be the set of branched points of X. Then $\Omega_2(f) = \overline{P(f)} \cup O'$, where $O' \subset O$ and $\rho\left(O', \overline{P(f)}\right) > 0$. Hence the depth of the centre is at most 3 and the centre is $\overline{P(f)}$.*

2. THE PROOFS

Through out this section we assume that X is a tree and $f \in C(X, X)$. It is well known that $F(f) \neq \emptyset$. In the following we give a partial order on the tree X which is useful in dealing with continuous tree maps.

DEFINITION 2.1: Let X be a tree. Then for every pair $x, y \in X$, $[x; y]$ will be the unique segment connecting x and y.

DEFINITION 2.2: Let e be an end point of the tree X. A partial order $<_e$ on X is defined by: $x <_e y$ if and only if $x \in [e : y]$ and $x \neq y$.

DEFINITION 2.3: Let e be an end point of the tree X and $X \setminus O = \bigcap_1^{n_0} I_j$. Define $<_r = <_{r(j_0, e)}$ on X by: If $\overline{I}_{j_0} = [y_1; y_2]$, $y_1 <_e y_2$ and $X_2 = \{x \in X : y_2 <_e x\} \cup [y_1; y_2]$, $X_1 = (X \setminus X_2) \cup [y_1; y_2]$ then $z_1 <_r z_2$ if and only if either $z_1 \in X_1 \setminus [y_1; y_2]$, $z_2 \in X_2$ or $z_2 <_{y_2} z_1$ if $z_1, z_2 \in X_1$; $z_1 <_{y_1} z_2$ if $z_1, z_2 \in X_2$ (we consider X_1, X_2 to be two subtrees of X).

LEMMA 2.4. *Let X be a tree, $f \in C(X, X)$ and $X \setminus O = \bigcap_1^{n_0} I_j$. If $x <_r y \in [y_1; y_2] = I_{j_0}$ with $x <_r f(x)$, $f(y) <_r y$ then $F(f) \cap [x; y] \neq \emptyset$.*

PROOF: Because $f([x; y])$ is a connected subset of X we know that there exist $x_1 <_r y_1 \in [x; y]$ such that $x_1 <_r f(x_1)$, $f(y_1) <_r y_1$ and $f(x_1), f(y_1) \in I_{j_0}$. By the continuity of f we can find $x_2 <_r y_2 \in [x_1; y_1]$ such that $x_2 <_r f(x_2)$, $f(y_2) <_r y_2$ and $f([x_2; y_2]) \subset I_{j_0}$. It is well known that $F(f) \cap [x_2; y_2] \neq \emptyset$ [1]. $\square$

LEMMA 2.5. *Let X be a tree and $f \in C(X, X)$. If $f([x; y]) \supset [x; y]$ for some $x, y \in X$ and $[x; y] \cap O = \emptyset$ then $F(f) \cap [x; y] \neq \emptyset$.*

PROOF: From $f([x;y]) \supset [x;y]$ we know that there exist $x_1, y_1 \in [x;y]$ such that $f(x_1) = x$, $f(y_1) = y$. By the connectedness of $f([x_1, y_1])$ we can choose $x_2, y_2 \in [x_1; y_1]$ such that $f([x_2; y_2]) \subset [x_1, y_1]$. So we have reduced our case to the interval case. The lemma follows from [1]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\Box$

THEOREM 2.6. *Let X be a tree and $f \in C(X, X)$. Then $\overline{P(f)} = \overline{R(f)}$.*

PROOF: Let $x \in X \setminus \overline{P(f)}$ and V_x be the open connected component of $X \setminus \overline{P(f)}$ such that $x \in V_x$.

Suppose $X \setminus O = \bigcap_1^{n_0} I_j$.

We claim that $(V_x \cap I_{j_0}) \cap R(f) = \emptyset$, $\forall 1 \leqslant j_0 \leqslant n_0$.

If it is not the case let $y \in (V_x \cap I_{j_0}) \cap R(f)$. Then there exists $n_1 \in \mathbf{N}$ such that $f^{n_1}(y) \in V_x \cap I_{j_0}$. Let $g = f^{n_1}$. Without loss of generality we assume $y <_e g(y)$ for some end point e.

Suppose that $V_x \cap I_{j_0} = [a; b]$, $a <_e b$. We have $a <_r y <_r g(y) <_r b$. It follows that $z <_r g(z)$, $\forall z \in [a; b]$. Otherwise if there exists $y_1 \in V_x \cap I_{j_0}$ such that $g(y_1) >_r y_1$ then $[y_1; y] \subset f([y_1; y])$ if $y_1 <_r y$ or it is the case of Lemma 2.4. By Lemma 2.4 and 2.5 it is a contradiction.

Now suppose $z <_r g^i(z)$, $\forall z \in [a; b]$, $1 \leqslant i \leqslant n$. It follows that $y <_r g(y) <_r g^{i+1}(y)$, $1 \leqslant i \leqslant n$. By the same reasoning $z <_r g^{n+1}(z)$, $\forall z \in [a; b]$. So we conclude that $z <_r g^m(z)$, $\forall z \in [a; b]$ and $m \in \mathbf{N}$. Hence $y <_r g(y) <_r g^m(y) \forall m \in \mathbf{N}$. That is to say $y \notin R(g) = R(f)$.

From $(V_x \cap I_{j_0}) \cap R(f) = \emptyset$. $\forall I_{j_0} \cap V_x \neq \emptyset$ we get $V_x \cap R(f) = \emptyset$ because if there is $i_0 \in \mathbf{N}$ such that $o_{i_0} \in V_x$ then $o_{i_0} \notin R(f)$ (an isolated recurrent point is a periodic point).

To sum up $\overline{P(f)} = \overline{R(f)}$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\Box$

THEOREM 2.7. *Let X be a tree and $f \in C(X, X)$. Then $\Omega_2(f) = \overline{P(f)} \cup O'$ where $O' \subset O$ and $\rho\left(O', \overline{P(f)}\right) > 0$. Hence the depth of the centre is at most 3 and the centre is $\overline{P(f)}$.*

For this purpose we show the following lemma

LEMMA 2.8. *Let $f \in C(X, X)$ for a tree X and $X \setminus O = \bigcup_1^{n_0} I_j$. If an open interval $J \subset I_{j_0}$ for some $1 \leqslant j_0 \leqslant n_0$ satisfies $J \cap P(f) = \emptyset$, then $J \cap f^n(J \cap \Omega(f)) = \emptyset$, $\forall n \in \mathbf{N}$.*

PROOF: If it is not the case then there exists $x \in J \cap \Omega(f)$ and $n \in \mathbf{N}$ such that $f^n(x) \in J$, $f^i(x) \notin J$, $1 \leqslant i < n$. It is easy to see that $f^i(x) \neq f^j(x)$ if $0 \leqslant i < j \leqslant n$.

Without loss of generality we assume that $x <_r f^n(x)$. Choose the neighbourhood V_i of $f^i(x)$ for $0 \leqslant i \leqslant n$ such that $V_i \cap V_j = \emptyset$ for $0 \leqslant i < j \leqslant n$ and $V_n \subset J$. Put

$V = \bigcap_0^n f^{-i}(V_i)$, then V is a neighbourhood of x. Since $x \in \Omega(f)$, there exists $l > 0$ such that $V \cap f^l(V) \neq \emptyset$. Obviously $l > n$. As $f^n(V) \subset V_n$ and $f^{l-n}(V_n) \cap V \neq \emptyset$, there is $y \in V_n$ such that $f^{l-n}(y) \in V$. Hence $f^{l-n}(y) <_r y$. By the proof of Theorem 2.6, $z <_r f^{nm_1}(z)$, $f^{m_2(l-n)}(z) <_r z$, $\forall z \in J$, $\forall m_1, m_2 \in \mathbf{N}$. It is a contradiction if we choose $m_1 = l - n$, $m_2 = n$. $\qquad\square$

PROOF OF THEOREM 2.7: Clearly $\overline{P(f)} \subset \Omega_2(f)$.

Let C be a connected component of $X \setminus \overline{P(f)}$ and $X \setminus O = \bigcup_1^{n_0} I_j$. By the above Lemma 2.8 for every nonnegative integer n we have $(C \cap I_j) \cap f^n(C \cap I_j \cap \Omega(f)) = \emptyset$ for all $1 \leqslant j \leqslant n_0$. Hence $C \cap I_j \cap \Omega(f|_{\Omega(f)}) = \emptyset$. To sum up we have proved that $\Omega(f|_{\Omega(f)}) \subset \overline{P(f)} \cup O'$, where $O' \subset O$ and $\forall x \in O'$, x is an isolated point of $\Omega_2(f)$ and $x \notin \overline{P(f)}$. So $\rho\left(O', \overline{P(f)}\right) > 0$ and $\Omega_3(f) = \overline{P(f)}$. $\qquad\square$

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