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On the One-Level Density Conjecture for Quadratic Dirichlet L-Functions

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Abstract. In a previous article, we studied the distribution of "low-lying" zeros of the family of quadratic Dirichlet L-functions assuming the Generalized Riemann Hypothesis for all Dirichlet L-functions. Even with this very strong assumption, we were limited to using weight functions whose Fourier transforms are supported in the interval $(-2, 2)$. However, it is widely believed that this restriction may be removed, and this leads to what has become known as the One-Level Density Conjecture for the zeros of this family of quadratic L-functions. In this note, we make use of Weil's explicit formula as modified by Besenfelder to prove an analogue of this conjecture.

1 Introduction

The distribution of zeros near $s = 1/2$ of certain families of L-functions has been studied in the work of Katz–Sarnak, see [7, 8]. One of the conjectures appearing in their work relates the one-level normalized spacings of "low-lying" zeros of certain families of L-functions, when ordered by their conductors, to classical symmetry groups associated with each family. In the case of the family of quadratic Dirichlet L-functions, partial results suggest that the symmetry group should be symplectic. See [7] for the details. See also [10] for an interesting related unconditional result.

In a previous work [9] we obtain the same distribution results for the low-lying zeros of the quadratic L-functions as in [7], but our approach depends on using the "form factor" in contrast to the approach in [7]. We now recall our method. For positive real numbers x and D, let the so-called "form factor" be defined as

$$F(x, D) = \frac{1}{\sqrt{x}} \sum_{d} e^{-\pi d^2/D^2} \sum_{\rho(\chi_d)} \frac{x^\rho}{\rho},$$

where d ranges over the integers, $\chi_d = (d/\cdot)$, and the sum over $\rho(\chi_d)$ ranges over the nontrivial zeros of $L(s, \chi_d)$. Assuming the Generalized Riemann Hypothesis (GRH) for all Dirichlet L-functions, it has been shown that, *cf.* [9],

$$\frac{1}{D} F(D^\alpha, D) = \begin{cases} -1 + D^{-\alpha/2} \log D + o(1) & \text{if } |\alpha| < 1, \\ o(1) & \text{if } 1 < |\alpha| < 2. \end{cases}$$

Given the present state of knowledge about zeros, we cannot extend the above result beyond $\alpha = 2$, even assuming GRH. This is due to the following rather coarse

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estimate (based on GRH):

$$\sum_{p\le x}\left(\frac{m}{p}\right)\log p \ll x^{1/2}\log^2 x,$$

for $m \ll x$, m any non-square integer. However, there is wide-spread belief that

$$\frac{1}{D}F(D^{\alpha}, D) = o(1)$$

for all $\alpha > 1$. This is an equivalent version of the so-called One-Level Density Conjecture for the family of quadratic Dirichlet L-functions. Refer to [6, 7] for a more thorough account of this and related conjectures. An analysis of the proof of the main theorem in [9] shows that this density conjecture is equivalent to the estimate:

$$\sum_{m\neq\square}\sum_{p\le x}\frac{\log p}{\sqrt{p}}\left(\frac{m}{p}\right)e^{-\pi m^2 D^2/p^2} = -\frac{1}{2}\frac{x}{D} + o(\sqrt{x}), \tag{1}$$

as $D \to \infty$, for any $D = o(x)$. Here m ranges over nonsquare integers.

(As an aside, we note that modifying (1) by removing some of the weights suggests the following:

$$\frac{1}{y}\sum_{\substack{m\neq\square\\ |m|\le y}}\sum_{p\le x}\left(\frac{m}{p}\right)\log p = -\sqrt{x} + o\left(\frac{x}{y}\right), \tag{2}$$

for $y = o(x)$, as $y \to \infty$.)

The purpose of this note is to give some theoretical evidence in favor of the Density Conjecture in the version given by (1). We make use of explicit formulas of A. Weil for Hecke L-functions, as applied in special cases by Besenfelder (who also used the name Bentz); see [3, 4] and also [1].

2 Results

We start with a special case of Weil's explicit formula for Hecke L-functions given by Besenfelder, *cf.* [3], for Dirichlet L-functions associated with primitive Dirichlet characters. This has the following special form.

Let $\chi = \chi_{d_0} = (d_0/\cdot)$ be a primitive quadratic Dirichlet character of modulus (conductor) $|d_0|$. Let $\varepsilon_0 = \varepsilon_0(\chi)$ be 1 or 0 according as χ is principal or not; let $\delta = \delta(\chi)$ be 0 or 1 according as χ is even or odd. Let y be a positive real number and s complex, let C be Euler's constant. Then

$$\mathcal{L}(y,\chi) = \mathcal{R}(y,\chi), \tag{3}$$

where

$$\mathcal{L}(y,\chi) = 2\sqrt{\pi y}\sum_{\rho(\chi)} e^{y(\rho-1/2)^2}$$

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and

$$\mathcal{R}(y,\chi) = \varepsilon_0(\chi) 4\sqrt{\pi y} e^{y/4} + \log \frac{|d_0|}{\pi} - C + 2\int_0^\infty \frac{e^{-\frac{x^2}{4y} + (\frac{3}{2}-\delta)x} - 1}{1-e^{2x}}\, dx$$
$$- 2\sum_{p,n} p^{-n/2} \log p\ \chi(p^n) e^{-\frac{\log^2(p^n)}{4y}},$$

where the sum over ρ represents the sum over the zeros of the Dirichlet L-function, $L(s,\chi)$, whose real parts lie in the interval $(0,1)$, and the sum over p,n ranges over all prime powers p^n.

We have the following facts.

Lemma 1 *For all $y > 0$,*

$$\Bigl|\sum_{\substack{p,n\\ n\geq 3}} p^{-n/2} \log p\ \chi(p^n) e^{-\frac{\log^2(p^n)}{4y}}\Bigr| \leq -3\frac{\zeta'}{\zeta}\Bigl(\frac{3}{2}\Bigr) < \infty.$$

Proof See [4, Lemma 3]. ■

Lemma 2 *For $1 \ll y$*

$$\Bigl|\int_0^\infty \frac{e^{-\frac{x^2}{4y} + (\frac{3}{2}-\delta)x} - 1}{1-e^{2x}}\, dx\Bigr| \leq 5.$$

Proof Use the argument in the proof [4, Lemma 7]. ■

We now need to replace equation (3) by an equality which holds for Dirichlet characters which are no longer assumed to be primitive. To this end, let $\chi = \chi_d = (d/\cdot)$ be a quadratic character modulo $|d|$ and let $\chi' = \chi_{d_0} = (d_0/\cdot)$ be the primitive character which induces χ. Hence d_0 is a fundamental discriminant and $d = d_0 f^2$ for some positive integer f. Define $\mathcal{L}(y,\chi)$ as above and the same for $\mathcal{R}(y,\chi)$ but where we keep the conductor $|d_0|$ in the term $\log(|d_0|/\pi)$. Since $L(s,\chi)$ and $L(s,\chi')$ differ by trivial Euler factors, we see that

$$\mathcal{L}(y,\chi) = \mathcal{L}(y,\chi').$$

On the other side, notice that

$$\mathcal{R}(y,\chi') - \mathcal{R}(y,\chi) = 2\sum_{p,n} p^{-n/2} \log p (\chi(p^n) - \chi'(p^n)) e^{-\frac{\log^2(p^n)}{4y}}$$
$$\ll \sum_{p|d} \Bigl(\frac{\log p}{\sqrt{p}} + \frac{\log p}{p}\Bigr) \ll \sum_{p|d} 1,$$

by Lemma 1. Hence we have proved:

Lemma 3 *For any quadratic character χ of modulus d, not necessarily primitive,*

$$\mathcal{L}(y,\chi) = \mathcal{R}(y,\chi) + O\Big(\sum_{p|d} 1\Big),$$

where the implied constant is independent of all χ and y.

Lemma 4 *Let χ be a quadratic character modulo $|d|$. Then*

$$\sum_p \frac{\log p}{p}\,\chi(p^2)e^{-\frac{\log^2(p)}{4y}} = \sqrt{\pi y} + O\Big(\sum_{p|d} 1\Big),$$

as $y \to \infty$.

Proof Since χ is a quadratic character, $\chi(p^2) = 1$ except when $p|d$. The result follows by [4, Lemma 5]. ■

We now wish to average the explicit formula over all quadratic characters. But, as usual, we pick a smooth averaging function, for example,

$$\frac{1}{D}\sum_d e^{-\pi d^2/D^2},$$

where D is any positive real number and the sum is over all integers d. (Anytime an arbitrary d yields something undefined, we set the value equal to 0.)

Then we have the following proposition:

Proposition 5

$$\begin{aligned}\frac{1}{D}\sum_d e^{-\pi d^2/D^2}\sum_{\rho(\chi_d)} e^{y(\rho-1/2)^2} &= \frac{2}{D}e^{y/4}\sum_{d=\square} e^{-\pi d^2/D^2} - \frac{1}{2D}\sum_{d\neq 0} e^{-\pi d^2/D^2}\\ &\quad + \frac{1}{2D\sqrt{\pi y}}\sum_{d\neq 0} e^{-\pi d^2/D^2}\log|d_0|\\ &\quad - \frac{1}{D\sqrt{\pi y}}\sum_p \frac{\log p}{\sqrt{p}}e^{-\frac{\log^2 p}{4y}}\sum_d e^{-\pi d^2/D^2}\Big(\frac{d}{p}\Big) + O\Big(\frac{\log\log D}{\sqrt{y}}\Big),\end{aligned}$$

as $D, y \to \infty$. Here, $\sum_{d=\square}$ means the sum over the non-zero square integers and d_0 is the fundamental discriminant corresponding to d.

Proof Much of this equality is easily seen. First, we have divided by $2\sqrt{\pi y}$; the sum over squares follows since $\varepsilon_0(\chi_d) = 1$ if and only if d is a perfect square. The second term on the right-hand side follows from Lemma 4.

We are left with the O-term. It suffices to show

$$\frac{1}{D}\sum_{d\neq 0} e^{-\pi d^2/D^2}\sum_{p|d} 1 \ll \log\log D.$$

To this end, we use Riemann–Stieltjes integration along with the asymptotic formula $\sum_{d\leq D}\sum_{p|d} 1 \sim D\log\log D$ (see, [2]) as follows:

$$\begin{aligned}\sum_{d\neq 0} e^{-\pi d^2/D^2}\sum_{p|d} 1 &= 2\int_{2-}^{\infty} e^{-\pi u^2/D^2}\, d\Big(\sum_{d\leq u}\sum_{p|d} 1\Big)\\ &= 2\Big(\sum_{d\leq u}\sum_{p|d} 1\Big) e^{-\pi u^2/D^2}\Big]_{2-}^{\infty}\\ &\quad + \frac{4\pi}{D^2}\int_{2}^{\infty}\Big(\sum_{d\leq u}\sum_{p|d} 1\Big) e^{-\pi u^2/D^2} u\, du\\ &\ll \frac{1}{D^2}\int_{2}^{\infty} (u\log\log u) e^{-\pi u^2/D^2} u\, du\\ &= D\int_{2/D}^{\infty}\log\log(Dv) e^{-\pi v^2} v^2\, dv\\ &\ll D\log\log D,\end{aligned}$$

as desired. ■

We shall use the following results to simplify Proposition 5.

Lemma 6

$$\sum_{d} e^{-\pi d^2/D^2} = D + O(e^{-D^2}),$$

as $D \to \infty$.

Proof We use the transformation formula for theta functions to obtain

$$\sum_{d} e^{-\pi d^2/D^2} = D\sum_{m} e^{-\pi m^2 D^2} = D + 2D\sum_{m=1}^{\infty} e^{-\pi m^2 D^2} = D + O(De^{-\pi D^2}),$$

as desired. ■

Lemma 7

$$\sum_{d\neq 0} e^{-\pi d^2/D^2}\log|d_0| = D\log D + O(D),$$

as $D \to \infty$.

Proof We may rewrite

$$\sum_{d\neq 0} e^{-\pi d^2/D^2}\log|d_0|$$

as

$$2\int_{1-}^{\infty} e^{-\pi u^2/D^2}\, d\alpha(u),$$

where

$$\alpha(u)=\alpha_1(u)+\alpha_2(u)+\alpha_3(u),$$

with

$$\alpha_1(u)=\sum_{\substack{m,n\geq 1\\ m^2n\leq u\\ n\equiv 1(2)}}\mu^2(n)\log n,\qquad \text{corresponding to } d_0\equiv 1(4),$$

$$\alpha_2(u)=\sum_{\substack{m,n\geq 1\\ 4m^2n\leq u\\ n\equiv 1(2)}}\mu^2(n)\log 4n,\qquad \text{corresponding to } d_0\equiv 4(8),$$

$$\alpha_3(u)=2\sum_{\substack{m,n\geq 1\\ 8m^2n\leq u\\ n\equiv 1(2)}}\mu^2(n)\log 8n,\quad \text{corresponding to } d_0\equiv 0(8),$$

with μ the Möbius function.

First, we claim that

$$\alpha(x)=x\log x+O(x).$$

From

$$\sum_{n\leq x}\mu^2(n)=\frac{1}{\zeta(2)}x+O(x^{1/2}),$$

(see [2]), it is easy to show that

$$\sum_{\substack{n\leq x\\ n\equiv 1(2)}}\mu^2(n)=\frac{2}{3\zeta(2)}x+O(x^{1/2}).$$

But then it follows easily by partial summation that

$$\sum_{\substack{n\leq x\\ n\equiv 1(2)}}\mu^2(n)\log n=\frac{2}{3\zeta(2)}x\log x+O(x^{1/2}\log x).$$

But then for $m^2\leq x$, we have

$$\sum_{\substack{n\leq x/m^2\\ n\equiv 1(2)}}\mu^2(n)\log n=\frac{2}{3\zeta(2)}\frac{x}{m^2}\log\Bigl(\frac{x}{m^2}\Bigr)+O\bigl((x/m^2)^{1/2}\log(x/m^2)\bigr).$$

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Thus,

$$\alpha_1(x) = \sum_{m\le\sqrt{x}} \sum_{\substack{n\le x/m^2\\ n\equiv 1(2)}} \mu^2(n)\log n$$

$$= \frac{2}{3\zeta(2)} x \sum_{m\le\sqrt{x}} \frac{1}{m^2}\log\Big(\frac{x}{m^2}\Big) + O\Big(x^{1/2}\sum_{m\le\sqrt{x}}\frac{1}{m}\log\Big(\frac{x}{m^2}\Big)\Big).$$

But notice that

$$\frac{2}{3\zeta(2)} x \sum_{m\le\sqrt{x}} \frac{1}{m^2}\log\Big(\frac{x}{m^2}\Big) = \frac{2}{3\zeta(2)}\, x\log x \sum_{m\le\sqrt{x}}\frac{1}{m^2} - \frac{2}{3\zeta(2)} x \sum_{m\le\sqrt{x}}\frac{1}{m^2}\log(m^2)$$

$$= \frac{2}{3\zeta(2)}\, x\log x \sum_{m=1}^{\infty}\frac{1}{m^2} + O(x) = \frac{2}{3}x\log x + O(x).$$

On the other hand, it is clear that

$$x^{1/2}\sum_{m\le\sqrt{x}}\frac{1}{m}\log\Big(\frac{x}{m^2}\Big) \ll x.$$

Hence

$$\alpha_1(x) = \frac{2}{3}x\log x + O(x).$$

Moreover, it is not hard to see that

$$\alpha_2(x) = \alpha\Big(\frac{x}{4}\Big) + O(x),$$

and

$$\alpha_3(x) = 2\alpha\Big(\frac{x}{8}\Big) + O(x).$$

From this is follows that

$$\alpha(x) = x\log x + O(x),$$

as claimed.

Now,

$$2\int_{1^-}^{\infty} e^{-\pi u^2/D^2}\, d\alpha(u) = 2e^{-\pi u^2/D^2}\alpha(u)\Big]_{1^-}^{\infty} + \frac{4\pi}{D^2}\int_1^{\infty} e^{-\pi u^2/D^2} u\alpha(u)\, du$$

$$= \frac{4\pi}{D^2}\int_1^{\infty} e^{-\pi u^2/D^2} u^2\log u\, du + O\Big(\frac{1}{D^2}\int_1^{\infty} e^{-\pi u^2/D^2} u^2\, du\Big).$$

First, we have

$$\frac{4\pi}{D^2}\int_1^{\infty} e^{-\pi u^2/D^2} u^2\log u\, du = 4\pi D\int_{1/D}^{\infty} e^{-\pi v^2} v^2\log(Dv)\, dv$$

$$= 4\pi D\log D\int_0^{\infty} e^{-\pi v^2} v^2\, dv + O(D)$$

$$= D\log D + O(D).$$

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Second,

$$\frac{1}{D^2}\int_1^\infty e^{-\pi u^2/D^2}u^2\,du \ll D.$$

All of the above establish the lemma. ■

Lemma 8

$$\sum_{d=\square} e^{-\pi d^2/D^2} = I\sqrt{D} - \frac{1}{2} + O(e^{-D^{2/3}}),$$

as $D \to \infty$, *where* $I = (1/4)\pi^{-1/4}\Gamma(1/4)$.

Proof By the Poisson summation formula we have

$$\begin{aligned}\sum_n e^{-\pi n^4/y^4} &= \sum_n \int_{-\infty}^\infty e^{-2\pi i n u}e^{-\pi u^4/y^4}\,du = \sum_n y\int_{-\infty}^\infty e^{-2\pi i n y u}e^{-\pi u^4}\,du\\ &= y\int_{-\infty}^\infty e^{-\pi u^4}\,du + \sum_{n\neq 0} y\int_{-\infty}^\infty e^{-2\pi i n y u}e^{-\pi u^4}\,du\\ &= (1/2)\pi^{-1/4}\Gamma(1/4)y + \sum_{n\neq 0} y\int_{-\infty}^\infty e^{-2\pi i n y u}e^{-\pi u^4}\,du.\end{aligned}$$

Thus,

$$\sum_{n\neq 0} e^{-\pi n^4/y^4} = 2Iy - 1 + \sum_{n\neq 0} y\int_{-\infty}^\infty e^{-2\pi i n y u}e^{-\pi u^4}\,du.$$

Next, consider

$$\int_{-\infty}^\infty e^{-\pi f(u,x)}\,du$$

where $f(u,x) = u^4 + 2ixu$. Now let $u = z + a$ where a is chosen so that $\frac{\partial f}{\partial u}(a,x) = 0$. Then a calculation shows that we may choose

$$a = -e^{\pi i/6}(x/2)^{1/3} = -\frac{\sqrt{3}+i}{2}\cdot(x/2)^{1/3}$$

and a further calculation yields

$$f(u,x) = b_0x^{4/3} - c_0x^{4/3}i + b_2x^{2/3}z^2 - b_3x^{1/3}z^3 + z^4 + c_2x^{2/3}iz^2 - c_3x^{1/3}iz^3,$$

where the b_j and c_j are positive real numbers, namely,

$$b_0 = 3\times 2^{-7/3},\quad c_0 = 3^{3/2}\times 2^{-7/3},\quad b_2 = 3\times 2^{-2/3},\quad b_3 = 3^{1/2}\times 2^{2/3},$$
$$c_2 = 3^{3/2}\times 2^{-3/2},\quad c_3 = 2^{2/3}.$$

Next we may write $z = u + c + di$ where $c = (\sqrt{3}/2)(x/2)^{1/3}$ and $d = (1/2)(x/2)^{1/3}$. But then

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-\pi f(u,x)}\,du &= e^{-\pi b_0 x^{4/3}} e^{\pi c_0 x^{4/3} i} \int_{-\infty}^{\infty} e^{-\pi[(b_2+c_2 i)x^{2/3}z^2-(b_3+c_3 i)x^{1/3}z^3+z^4]}\,du\\ &= e^{-\pi b_0 x^{4/3}} e^{\pi c_0 x^{4/3} i} \int_{-\infty+di}^{\infty+di} e^{-\pi[(b_2+c_2 i)x^{2/3}z^2-(b_3+c_3 i)x^{1/3}z^3+z^4]}\,dz.\end{aligned}$$

Now let $b > 0$ and define the contour of integration γ_b, given as the path over the rectangle from $-b$ to b, then b to $b + di$, then $b + di$ to $-b + di$, and then back to $-b$. Notice that if $g(z, x) = (b_2 + c_2 i)x^{2/3}z^2 - (b_3 + c_3 i)x^{1/3}z^3 + z^4$, then

$$\int_{\gamma_b} e^{-\pi g(z,x)}\,dz = 0$$

by the residue theorem, since the integrand is entire. On the other hand,

$$\int_{\pm b}^{\pm b+di} e^{-\pi g(z,x)}\,dz = o(1),$$

as $b \to \infty$; for

$$\int_{\pm b}^{\pm b+di} e^{-\pi g(z,x)}\,dz = i\int_0^d e^{-\pi g(\pm b+ti,x)}\,dt = i\int_0^d e^{-\pi b^4(1+o(b))}\,dt = o(1).$$

Therefore,

$$\int_{-\infty}^{\infty} e^{-\pi f(u,x)}\,du = e^{-\pi b_0 x^{4/3}} e^{\pi c_0 x^{4/3} i} \int_{-\infty}^{\infty} e^{-\pi g(z,x)}\,dz.$$

We need only consider the size of

$$\int_{-\infty}^{\infty} e^{-\pi h(z,x)}\,dz,$$

where h is the real part of g. A calculation shows

$$h(z, x) = z^2(z - \alpha)^2,$$

with $\alpha = \sqrt{3}(x/2)^{1/3}$. But then

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-\pi h(z,x)}\,dz &= \int_{-\infty}^{\infty} e^{-\pi z^2(z-\alpha)^2}\,dz\\ &\ll \int_{-\infty}^{\infty} e^{-\pi z^2}\,dz + \int_{|z-\alpha|\le 1} e^{-\pi z^2(z-\alpha)^2}\,dz = O(1).\end{aligned}$$

Therefore,

$$\int_{-\infty}^{\infty} e^{-\pi f(u,x)} du = O\left(e^{-3\pi\cdot 2^{-7/3} x^{4/3}}\right) = O(e^{-1.5x^{4/3}}).$$

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But now

$$\sum_{n\neq 0} y \int_{-\infty}^{\infty} e^{-2\pi i n y u} e^{-\pi u^4}\, du \ll \sum_{n=1}^{\infty} y e^{-1.5 n^{4/3} y^{4/3}}$$

$$\ll y e^{-1.5 y^{4/3}} \ll e^{-y^{4/3}},$$

as $y \to \infty$.

Letting $y = D^{1/2}$ yields our result. ■

Using Lemmas 6, 7, 8 in Proposition 5, we obtain the following proposition.

Proposition 9

$$\frac{1}{D}\sum_d e^{-\pi d^2/D^2} \sum_{\rho(\chi_d)} e^{y(\rho-1/2)^2} = \frac{2I}{D^{1/2}} e^{y/4} - \frac{e^{y/4}}{D} - \frac{1}{2} + \frac{1}{2D}$$

$$+ \frac{\log D}{2\sqrt{\pi y}} - \frac{1}{D\sqrt{\pi y}} \sum_p \frac{\log p}{\sqrt{p}} e^{-\frac{\log^2 p}{4y}} \sum_d e^{-\pi d^2/D^2} \left(\frac{d}{p}\right)$$

$$+ O\left(e^{-D^{2/3}} e^{y/4} + \frac{\log\log D}{\sqrt{y}}\right),$$

as $D, y \to \infty$.

Next, let

$$A = -\frac{1}{D\sqrt{\pi y}} \sum_p \frac{\log p}{\sqrt{p}} e^{-\frac{\log^2 p}{4y}} \sum_d e^{-\pi d^2/D^2} \left(\frac{d}{p}\right).$$

Then by the transformation formula for theta functions:

$$\sum_d e^{-\pi d^2/D^2} \left(\frac{d}{p}\right) = \frac{D}{p^{1/2}} \sum_m e^{-\pi m^2 D^2/p^2} \left(\frac{m}{p}\right),$$

we have

$$A = -\frac{1}{\sqrt{\pi y}} \sum_p \frac{\log p}{p} e^{-\frac{\log^2 p}{4y}} \sum_m e^{-\pi m^2 D^2/p^2} \left(\frac{m}{p}\right).$$

Now, decompose A as

$$A = A_1 + A_2 + A_3 + O,$$

where

$$A_1 = \frac{1}{2\sqrt{\pi y}} \sum_p \frac{\log p}{p} e^{-\frac{\log^2 p}{4y}},$$

$$A_2 = -\frac{I}{\sqrt{\pi y D}} \sum_p \frac{\log p}{\sqrt{p}} e^{-\frac{\log^2 p}{4y}},$$

$$A_3 = -\frac{1}{\sqrt{\pi y}} \sum_p \frac{\log p}{p} e^{-\frac{\log^2 p}{4y}} \sum_{m\neq\square} e^{-\pi m^2 D^2/p^2} \left(\frac{m}{p}\right),$$

and finally, where

$$O = O\Big(\frac{1}{\sqrt{y}}\sum_{p}\frac{\log p}{p}e^{-\frac{\log^2 p}{4y}}\Big(e^{-(p/D)^{2/3}}+\sum_{\substack{m\neq 0\\ p|m}}e^{-\pi m^2D^2/p^2}\Big)\Big).$$

Concerning A_1, A_2, and O, we have the following lemmas.

Lemma 10

$$A_1 = \frac{1}{2} + O\Big(\frac{1}{\sqrt{y}}\Big),$$

as $y, D \to \infty$.

Proof The result follows from [4, Lemma 5]. ■

Lemma 11 *Assuming the Riemann Hypothesis (R.H.),*

$$A_2 = -\frac{2I}{\sqrt{D}}\,e^{y/4} + O\Big(\sqrt{\frac{y}{D}}\Big),$$

as $D, y \to \infty$ *and* I *is as in Lemma 8.*

Proof Write

$$A_2 = -\frac{I}{\sqrt{\pi y D}}\int_1^{\infty} e^{-\frac{\log^2 u}{4y}}\,d\beta(u),$$

where $\beta(u) = \sum_{p\le u} p^{-1/2}\log p$. Then under R.H. $\beta(u) = 2\sqrt{u}+E(u)$, with $E(u) \ll \log^2 u$. But then

$$A_2 = -\frac{I}{\sqrt{\pi y D}}\int_1^{\infty} e^{-\frac{\log^2 u}{4y}}\,\frac{du}{\sqrt{u}} - \frac{I}{\sqrt{\pi y D}}\int_1^{\infty} E(u)e^{-\frac{\log^2 u}{4y}}\Big(-\frac{\log u}{2yu}\Big)\,du + O\Big(\frac{1}{\sqrt{yD}}\Big).$$

Now by a change of variables $z = (\log u)/(2\sqrt{y})$ and then $v = z - \sqrt{y}/2$,

$$\begin{aligned}
-\frac{I}{\sqrt{\pi y D}}\int_1^{\infty} e^{-\frac{\log^2 u}{4y}}\,\frac{du}{\sqrt{u}} &= -\frac{I}{\sqrt{\pi y D}}\int_0^{\infty} e^{-z^2+\sqrt{y}z}2\sqrt{y}\,dz\\
&= -\frac{2I}{\sqrt{\pi D}}e^{y/4}\int_0^{\infty} e^{-(z-\sqrt{y}/2)^2}\,dz\\
&= -\frac{2I}{\sqrt{\pi D}}e^{y/4}\int_{-\sqrt{y}/2}^{\infty} e^{-v^2}\,dv\\
&= -\frac{2I}{\sqrt{\pi D}}\,e^{y/4}\Big(\int_{-\infty}^{\infty} e^{-v^2}\,dv - \int_{\sqrt{y}/2}^{\infty} e^{-v^2}\,dv\Big)\\
&= -\frac{2I}{\sqrt{D}}\,e^{y/4} + \frac{2I}{\sqrt{\pi D}}\,e^{y/4}\int_{\sqrt{y}/2}^{\infty} e^{-v^2}\,dv.
\end{aligned}$$

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Then, using the estimate

$$\int_x^\infty e^{-u^2}\,du \le \frac{e^{-x^2}}{2x},$$

we obtain

$$\frac{2I}{\sqrt{\pi D}} e^{y/4} \int_{\sqrt{y}/2}^\infty e^{-v^2}\,dv \ll \frac{1}{\sqrt{yD}}.$$

Finally, (again, assuming R.H.) letting $z = (\log u)/(2\sqrt{y})$

$$\frac{I}{\sqrt{\pi y D}} \int_1^\infty E(u) e^{-\frac{\log^2 u}{4y}} \frac{\log u}{y} \frac{du}{u} \ll \frac{1}{y^{3/2} D^{1/2}} \int_1^\infty e^{-\frac{\log^2 u}{4y}} \log^3 u \frac{du}{u} \ll \frac{\sqrt{y}}{\sqrt{D}}.$$

This establishes the lemma ■

Now, we come to the O-term.

Lemma 12 *As above, let*

$$O = O\left(\frac{1}{\sqrt{y}} \sum_{p \ge D} \frac{\log p}{p} e^{-\frac{\log^2 p}{4y}} \left(e^{-(p/D)^{2/3}} + \sum_{\substack{m \ne 0 \\ p \mid m}} e^{-\pi m^2 D^2 / p^2} \right) \right).$$

Then

$$O \ll 1,$$

as $y, D \to \infty$.

Proof Write $O = O_1 + O_2$ with

$$O_1 \ll \frac{1}{\sqrt{y}} \sum_p \frac{\log p}{p} e^{-\frac{\log^2 p}{4y}} \sum_{n \ne 0} e^{-\pi n^2 D^2}$$

and

$$O_2 \ll \frac{1}{\sqrt{y}} \sum_p \frac{\log p}{p} e^{-\frac{\log^2 p}{4y}} e^{-(p/D)^{2/3}}.$$

But now

$$O_1 \ll \frac{1}{\sqrt{y}} \sum_p \frac{\log p}{p} e^{-\frac{\log^2 p}{4y}} e^{-D^2} \ll 1.$$

Moreover,

$$O_2 \ll \frac{1}{\sqrt{y}} \sum_p \frac{\log p}{p} e^{-\frac{\log^2 p}{4y}} \ll 1,$$

as desired. ■

From all of this we have established the following theorem:

Theorem 13 *Assuming R.H.*

$$\frac{1}{D}\sum_{d} e^{-\pi d^2/D^2} \sum_{\rho(\chi_d)} e^{y(\rho-1/2)^2} = -\frac{1}{\sqrt{\pi y}} \sum_{p} \frac{\log p}{p} e^{-\frac{\log^2 p}{4y}} \sum_{m \neq \Box} e^{-\pi m^2 D^2/p^2} \left(\frac{m}{p}\right)$$
$$- \frac{e^{y/4}}{D} + \frac{\log D}{2\sqrt{\pi y}}$$
$$+ O\left(e^{-D^{2/3}} e^{y/4} + 1 + \frac{\sqrt{y}}{\sqrt{D}} + \frac{\log\log D}{\sqrt{y}}\right),$$

as $D, y \to \infty$.

We are interested in obtaining an asymptotic expression for the first term on the right-hand side of the equation in the theorem above. To this end, we have the following result:

Proposition 14 *Assuming GRH,*

$$\frac{1}{D}\sum_{d} e^{-\pi d^2/D^2} \sum_{\rho(\chi_d)} e^{y(\rho-1/2)^2} \ll \log D,$$

as $y, D \to \infty$.

Proof Using Riemann–Stieltjes integration, we have

$$\frac{1}{D}\sum_{d} e^{-\pi d^2/D^2} \sum_{\rho(\chi_d)} e^{y(\rho-1/2)^2} = \frac{1}{D}\sum_{d} e^{-\pi d^2/D^2} \int_0^{\infty} e^{-yu^2}\, dN(u, \chi_d),$$

where $N(T, \chi)$ is the number of nontrivial zeros of $L(s, \chi)$ with imaginary part between $-T$ and T. But

$$N(T, \chi) = \frac{T}{\pi} \log \frac{qT}{2\pi} - \frac{T}{\pi} + O(\log T + \log q),$$

for $T \geq 2$, where q is the conductor of χ, and

$$N(1, \chi) \ll \log q,$$

where the implicit constants are independent of χ (see [5]). Hence,

$$\begin{aligned}
&\frac{1}{D}\sum_{d} e^{-\pi d^2/D^2}\int_1^\infty e^{-yu^2}\,dN(u,\chi_d)\\
&\qquad\ll \frac{1}{D}\sum_{d} e^{-\pi d^2/D^2} e^{-yu^2}N(u,\chi_d)\Big]_1^\infty\\
&\qquad\qquad + \frac{2y}{D}\sum_{d} e^{-\pi d^2/D^2}\int_1^\infty N(u,\chi_d)e^{-u^2y}u\,du\\
&\qquad\ll \frac{y}{D}\sum_{d} e^{-\pi d^2/D^2}\int_1^\infty u^2\log|d_0u|e^{-u^2y}\,du\\
&\qquad\ll \frac{y}{D}\sum_{d} e^{-\pi d^2/D^2}\log|d_0|\int_1^\infty u^2\log|u|e^{-u^2y}\,du\\
&\qquad\ll \frac{y}{D}\sum_{d} e^{-\pi d^2/D^2}\log|d_0|\int_1^\infty \frac{v^2}{y}\log\Big(\frac{|v|}{\sqrt{y}}\Big)e^{-v^2}\,\frac{dv}{\sqrt{y}}\\
&\qquad\ll \frac{\log y}{D\sqrt{y}}\sum_{d} e^{-\pi d^2/D^2}\log|d_0| \ll \frac{\log y}{\sqrt{y}}\log D,
\end{aligned}$$

by Lemma 7.

But,

$$\frac{1}{D}\sum_{d} e^{-\pi d^2/D^2}\sum_{\substack{\rho(\chi_d)\\ |\gamma|<1}} e^{-y\gamma^2} \ll \frac{1}{D}\sum_{d} e^{-\pi d^2/D^2}\log|d_0| \ll \log D. \qquad \blacksquare$$

By this proposition and the previous theorem, we obtain:

Theorem 15 *Assuming GRH,*

$$\frac{1}{\sqrt{\pi y}}\sum_{m\neq\square}\sum_{p}\frac{\log p}{p}\,e^{-\frac{\log^2 p}{4y}}e^{-\pi m^2D^2/p^2}\Big(\frac{m}{p}\Big) = -\frac{e^{y/4}}{D}+O\Big(e^{-D^{2/3}}e^{y/4}+\frac{\sqrt{y}}{\sqrt{D}}+\log D\Big),$$

as $y, D \to \infty$.

3 Conclusion

First notice that the difference between equation (1) and Theorem 15 is the introduction of a certain smooth weight function. Of course, the method of proof given above depends crucially on the particular shape of the weight.

We now compare the result in Theorem 15 to equation (2). One thing to notice immediately is that both statements have a negative main term which shows in some

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rough sense that there is a preponderance of primes which are quadratic non-residues modulo non-square integers.

Next, multiply equation (2) by y and then replace y by x/D to obtain:

$$\sum_{\substack{m\neq\square \\ |m|\leq x/D}} \sum_{p\leq x} \Big(\frac{m}{p}\Big) \log p = -\frac{x^{3/2}}{D} + o(x).$$

On the other hand, replacing the sum over non-square m by squares yields:

$$\sum_{\substack{m=\square \\ |m|\leq x/D}} \sum_{p\leq x} \log p \sim \frac{x^{3/2}}{\sqrt{D}}.$$

Hence, in absolute value the main terms of the two expressions differ by a factor of $\sqrt{D}$.

On the other hand, replacing the sum over non-square m in Theorem 15 by squares yields,

$$\frac{1}{\sqrt{\pi y}} \sum_{m=\square} \sum_{p} \frac{\log p}{p}\, e^{-\frac{\log^2 p}{4y}} e^{-\pi m^2 D^2/p^2} \sim \frac{e^{y/4}}{\sqrt{D}},$$

showing the same relation as above.

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