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Liouville property and quasi-isometries on negatively curved Riemannian surfaces

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Kanai proved powerful results on the stability under quasi-isometries of numerous global properties (including Liouville property) between Riemannian manifolds of bounded geometry. Since his work focuses more on the generality of the spaces considered than on the two-dimensional geometry, Kanai's hypotheses in many cases are not satisfied in the context of Riemann surfaces endowed with the Poincaré metric. In this work we fill that gap for the Liouville property, by proving its stability by quasi-isometries for every Riemann surface (and even Riemannian surfaces with pinched negative curvature). Also, a key result characterizes Riemannian surfaces which are quasi-isometric to $\mathbb{R}$.

Keywords: Liouville property; ends; quasi-isometry; Riemann surface; Poincaré metric; pinched negative curvature

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1. Introduction

The classical Liouville Theorem roughly states that the only positive (bounded) harmonic functions on Euclidean spaces are the constant ones. Since this result was announced, several authors have considered whether this property was unique to Euclidean spaces or could be generalized to other contexts, like Riemannian manifolds or Aleksandrov spaces. A manifold will be said to satisfy the Liouville property if it has no non-constant positive harmonic functions.

Yau, using a method based on the maximum principle, showed in [**28**] that complete Riemannian manifolds with nonnegative Ricci curvature satisfied the Liouville property. Later, Moser proved in [**22**] that such property also holds on Riemannian manifolds which are bilipschitz to an Euclidean space. Kanai ([**20**]) added conditions on the manifold and generalized this result by considering quasi-isometries instead of bilipschitz maps; the additional conditions being having bounded geometry, i.e., Ricci curvature bounded from below by a negative constant and a positive radius of injectivity.

On the other hand, some negative results were obtained. There exist bounded harmonic functions on any simply connected manifold with negatively pinched sectional curvature (see [**2**], [**27**], [**3**]). Also, Lyons showed that this Liouville property is not preserved by bilipschitz maps in general ([**21**]).

The aim of this paper is to complete the line of work opened by Kanai discussing under what conditions Riemannian surfaces quasi-isometric to an Euclidean space have the Liouville property. To this end, we will consider pinched negative curvature surfaces (with no bound on the injectivity radius). Even though Riemann surfaces endowed with their Poincaré metrics are a natural context to apply Kanai's results, if they have cusps, then their injectivity radii are equal to zero.

Removing the hypothesis on the injectivity radius from Kanai's results is a problem that has been studied in many previous works. In [**12**] some of the conclusions of Kanai's results regarding isoperimetric inequalities, existence of Green's function and Liouville property are obtained for manifolds (without hypotheses on the injectivity radius) by using appropriate weighted graphs; to this end, it is needed that the Riemannian volumes are doubling measures and the quasi-isometry has an additional property (it quasi-preserves the volume of the balls, see (*iii*) in [**12**, p.688]). Papers [**9**], [**16**] and [**17**] also deal with this problem for isoperimetric inequalities and existence of Green's function on Riemannian surfaces, replacing the additional hypotheses on the quasi-isometry in [**12**] by other hypothesis (for instance, on the genus of the surfaces).

One of the main results in this paper is the following. Note that it does not need any of those extra hypothesis; in particular, it does not require the condition (*iii*) in [**12**, p.688] about the volume of balls.

THEOREM 1.1. *Let X be an orientable complete Riemannian surface with pinched negative curvature quasi-isometric to the Euclidean space $\mathbb{R}^m$ for some $m \geqslant 1$. Then every positive harmonic function on X is constant.*

Kanai proved this theorem for surfaces with lower bounded injectivity radius. Thus, the hypothesis of being quasi-isometric to $\mathbb{R}^m$ is a natural one (quasi-isometries preserve the Liouville property of $\mathbb{R}^m$ under these conditions). In addition, in this type of theorems a hypothesis about the curvature is always necessary; in particular, the hypothesis on pinched negative curvature allows the use of some powerful tools of Geometric Function Theory in several arguments. By Kanai's results it suffices to consider surfaces with zero injectivity radius. We know by [**11**, Corollary 1, p.336] that if there exists a constant C such that

$$A_X(B_X(p,r)) \leqslant Cr^2,$$

for every $p \in X$ and $r > 0$, then X has the Liouville property (A_X and B_X denote Riemannian area and ball on the surface X, respectively). Thus, the heart of this paper is the following surprising result of Theorem 4.2 (which is a consequence of Theorem 3.4): if a surface with pinched negative curvature and zero injectivity radius is quasi-isometric to $\mathbb{R}^m$, then $m = 1$ (in particular, Theorem 4.2 allows to prove proposition 4.9, which gives that X has linear growth rate); Theorem 4.2 also provides some additional properties of the surface. This shows the need of characterizing the Riemannian surfaces which are quasi-isometric to $\mathbb{R}$. This goal is achieved in the following theorem, framed in the more general setting of metric spaces.

THEOREM 1.2. *Let X be a proper geodesic metric space. The following facts are equivalent:*

(1) *X is quasi-isometric to $\mathbb{R}$.*

(2) *There is a $(1,\,0)$-quasi-isometry from $\mathbb{R}$ to X.*

(3) *There exists a positive constant R_1 such that, for all $p \in X$, $X \setminus \overline{B_X(p,\, R_1)}$ has exactly two unbounded connected components $E_1(p,\, R_1)$, $E_2(p,\, R_1)$ and*

$$d_X(x,p) \leqslant 4R_1, \quad \forall\, x \notin E_1(p,R_1) \cup E_2(p,R_1).$$

Next, we would like to highlight the following result, that provides a good property of the Riemannian surfaces with pinched negative curvature quasi-isometric to $\mathbb{R}$: it is possible to decompose these surfaces as union of generalized Y-pieces (or 'pair of pants') in such a way that the length of the boundary of the pieces has an upper bound. This property is potentially interesting in Teichmüller theory of surface of infinite type, i.e. surfaces with infinitely generated fundamental group. Informally, a Y-piece is a compact bordered surface with the shape of a pair of pants whose boundary is the union of three simple closed geodesics; a generalized Y-piece is either a Y-piece or a surface obtained by replacing some closed curves in the boundary of a Y-piece by cusps or 'pseudospheres' (see definition 2.3 for the precise definitions).

THEOREM 1.3. *Let X be an orientable complete Riemannian surface with pinched negative curvature and quasi-isometric to $\mathbb{R}$. There exist positive constants α_1, α_2 and one (at most countable) collection $\{Y_k\}_k$ of generalized Y-pieces, with pairwise disjoint interiors, so that $X = \cup_k Y_k$ and $\alpha_1 \leqslant L_X(\partial Y_k) \leqslant \alpha_2$ for all k.*

Moreover, the elements on the collection $\{Y_k\}_k$ are Y-pieces except for, at most, two.

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By nature, quasi-isometries represent a flexible class of maps that behave well on a global scale, but that produce a large distortion on the local properties of the manifolds involved. Intuitively, two metric spaces are quasi-isometric if their large-scale metric structures are the same, ignoring fine details. Informally, quasi-isometries allow stretching and contracting distances. Note that even though they form a large class of maps which do not need to be continuous, they do have interesting invariance properties.

Quasi-isometries preserve Gromov hyperbolicity of geodesic metric spaces (see, e.g., [**19**], [**15**]); also preserve the parabolic Harnack inequality [**13**] and various estimates on transition probabilities of random walks, such as heat kernel estimates.

Following the idea in [**18**], a function between two metric spaces $f : X \to Y$ is said to be an (a, b)-*quasi-isometric embedding* with constants $a \geqslant 1$, $b \geqslant 0$, if

$$\frac{1}{a}\, d_X(x_1, x_2) - b \leqslant d_Y(f(x_1), f(x_2)) \leqslant a\, d_X(x_1, x_2) + b\,, \quad \text{for every } x_1, x_2 \in X.$$

Such a quasi-isometric embedding f is a *quasi-isometry* if, furthermore, there exists a constant $c \geqslant 0$ such that f is *c-full*, i.e., if for every $y \in Y$ there exists $x \in X$ with $d_Y(y, f(x)) \leqslant c$.

Two metric spaces X and Y are *quasi-isometric* if there exists a quasi-isometry between them. It is well-known that to be quasi-isometric is an equivalence relation (see, e.g., [**20**]).

A quasi-isometry can drastically change the local topology (for example, any compact Riemannian manifold is quasi-isometric to a single point). Furthermore, important global properties, like the dimension, are not preserved by quasi-isometries: if X and Y are Riemannian manifolds and Y is compact, then X and $X \times Y$ are quasi-isometric. Nevertheless, quasi-isometries sometimes preserve a local property: some results on the stability of the injectivity radius for Riemann surfaces with some hypotheses on their genus are shown in [**9**] and [**17**]. In particular, points with small injectivity radius are shown to be mapped onto points with small injectivity radius.

Recall that the *injectivity radius* $\iota(p)$ *of* $p \in X$ is the largest radius for which the exponential map at p is a diffeomorphism. If X has non-positive sectional curvatures, then the injectivity radius can be defined, also, as the supreme of those $r > 0$ such that the ball $B_X(p, r)$ is simply connected or, equivalently, as half the infimum of the lengths of the (homotopically non-trivial) loops based at p in X. The *injectivity radius* $\iota(X)$ *of* X is the infimum over $p \in X$ of $\iota(p)$.

The plan of the paper is as follows. In § 2 some definitions and background are given. Section 3 deals with stability of ends, which will be needed in the proof of the main results, given in § 4 and 5 (see Theorems 1.2, 1.1 and 1.3).

2. Definitions and background

Let us consider $h > 0$, a metric space X, and subsets $Y, Z \subseteq X$. The set $V_h(Y) := \{x \in X \mid d(x, Y) \leqslant h\}$ is called the *h-neighbourhood* of Y in X. The *Hausdorff distance* between Y and Z is defined by $\mathcal{H}(Y, Z) := \inf\{h > 0 \mid Y \subseteq V_h(Z), Z \subseteq V_h(Y)\}$.

A *minimizing geodesic* γ in a metric space X is an isometry from an interval $I \subseteq \mathbb{R}$ onto X, i.e., $L_X(\gamma|_{[t,s]}) = d_X(\gamma(t), \gamma(s)) = |t-s|$ for every $s, t \in I$, where L_X denotes the length in X. We say that X is a *geodesic metric space* if for every $x, y \in X$ there exists a minimizing geodesic joining x and y; we denote by $[xy]$ any of such minimizing geodesics (since we do not require uniqueness of geodesics, this notation is ambiguous, but it is convenient).

A *geodesic ray* in X is a minimizing geodesic with domain $I = [0, \infty)$.

A *geodesic line* in X is a minimizing geodesic with domain $I = \mathbb{R}$.

A *geodesic* is a map $\gamma : I \to X$ such that for every $t \in I$, there exists $\varepsilon > 0$ verifying that the restriction of γ to $I \cap (t-\varepsilon, t+\varepsilon)$ is a minimizing geodesic.

An (a, b)-*quasigeodesic* in X is an (a, b)-quasi-isometric embedding from a real interval on X. Note that any $(1, 0)$-quasigeodesic is a minimizing geodesic.

A *non-exceptional* Riemann surface S is a Riemann surface whose universal covering space is the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, endowed with its Poincaré metric (also called the hyperbolic metric), i.e., the metric obtained by projecting the Poincaré metric of the unit disk

$$ds^2 = \left(\frac{2}{1-|z|^2}\right)^2 (\,\mathrm{d}x^2 + \mathrm{d}y^2).$$

With this metric, S is a complete Riemannian manifold with constant curvature -1. The only Riemann surfaces which are left out (the exceptional Riemann surfaces) are the sphere, the plane, the punctured plane and the tori.

Recall that a set is *doubly connected* if its fundamental group is isomorphic to $\mathbb{Z}$.

Assume now that S is a non-exceptional Riemann surface. A *collar* in S about a simple closed geodesic γ is a doubly connected domain in S 'bounded' by two Jordan curves β_1, β_2, (called the boundary curves of the collar) orthogonal to the pencil of geodesics emanating from γ; such collar can be written as $C_{\gamma,t} = \{p \in S : d_S(p, \gamma) < t\}$, for some positive constant t. The constant t is called the *width* of the collar.

The following result, known as Collar Lemma [**24**], will be used several times along this work. It is generalized for surfaces with variable negative curvature in [**7**].

Lemma 2.1. *If* γ *is a simple closed geodesic in a non-exceptional Riemann surface* S, *then there exists a collar about* γ *of width* t, *for every* $0 < t \leqslant w$, *where* $\cosh w = \coth(L_S(\gamma)/2)$ *or, equivalently,* $\sinh w = 1/\sinh(L_S(\gamma)/2)$.

Remark 2.2. Denote by C_γ the collar about γ of width w given by the above lemma. It is well-known that if γ_1 and γ_2 are disjoint simple closed geodesics, then $C_{\gamma_1} \cap C_{\gamma_2} = \emptyset$ (see [**24**]).

Let us recall now some definitions for complete Riemannian surfaces X with pinched negative curvature (i.e., the Gaussian curvature satisfies $-k_2^2 \leqslant K \leqslant -k_1^2$ for some constants $0 < k_1 \leqslant k_2$).

An end in X is *doubly connected* if it has a neighbourhood whose fundamental group is isomorphic to $\mathbb{Z}$.

A *funnel* in X is a doubly connected bordered Riemannian surface whose boundary is a simple closed geodesic. In the case of non-exceptional Riemann surfaces, given any positive number a, there is a unique (up to isometry) funnel such that its boundary curve has length a.

A *cusp* in X is a neighbourhood of a doubly connected end whose fundamental group is generated by a simple closed curve α and there is no closed geodesic in the homotopy class of α. We know that the infimum of the lengths of non-trivial curves in a cusp is zero (see [**23**, Theorem 3.7]). In the case of a non-exceptional Riemann surface S, a *collar* in S about a cusp q is a doubly connected domain in S 'bounded' both by q and a Jordan curve (called the boundary curve of the collar) orthogonal to the pencil of geodesics emanating from q. It is well-known that the length of the boundary curve in S is equal to the area of the collar (see, e.g., [**5**]). A collar of area β is called a β-collar. Thus, the length of the boundary of a β-collar is also β. For each cusp there exists a 2-collar and 2-collars of different cusps are disjoint. Besides, the collar C_γ of the simple closed geodesic γ does not intersect the 2-collar of a cusp (see [**24**], [**26**] and [**8**, Chapter 4]).

Definition 2.3. *A Y-piece or 'pair of pants' in X is a compact bordered Riemannian surface which is topologically a sphere without three disks and whose boundary is the union of three simple closed geodesics. In the case of non-exceptional Riemann surfaces, given three positive numbers a, b, c, there is a unique (up to isometry) Y-piece such that its boundary curves have lengths a, b, c (see, e.g., [**25**, p.410]). Y-pieces are a standard tool for constructing Riemann surfaces (see [**10**, Chapter X.3] and [**8**, Chapter 1]).*

A generalized Y-piece *in X is a bordered or non-bordered Riemannian surface which is topologically a sphere without three open disks, such that there exist integers $n, m \geqslant 0$ with $n + m = 3$, so that the boundary are n minimizing simple closed geodesics and there are m cusps. Notice that a generalized Y-piece is topologically the union of a Y-piece and m cylinders, with $0 \leqslant m \leqslant 3$.*

Definition 2.4. *Any divergent curve $\sigma : [0, \infty) \longrightarrow Y$, where Y is a non-compact Hausdorff space, determines an end E of Y. Given a compact set F of Y, one defines $E(F)$ to be the arc component of $Y \setminus F$ that contains a terminal segment $\sigma([a, \infty))$ of σ for some $a \geqslant 0$. A set $U \subset Y$ is a* neighbourhood of an end *E if U contains $E(F)$ for some compact set F of Y.*

Recall that a topological space X is said *proper* if every closed ball in X is a compact set.

The following result in [**6**, Proposition 8.29] proves that quasi-isometries of groups preserve the number of ends. A more precise result for cusps in Riemann surfaces is presented in [**9**].

Theorem 2.5. *Let X and Y be proper geodesic metric spaces and $f : X \to Y$ a c-full (a, b)-quasi-isometry. Then, f induces a bijection between the ends of X and Y.*

3. A lower bound for the injectivity radius

The next result deals with collars of geodesics and cusps separately. Recall that w stands for the width of the collar C_γ given by lemma 2.1.

LEMMA 3.1. *Assume that X is a non-exceptional Riemann surface and Y a metric space. Let f be a c-full (a, b)-quasi-isometry from X to Y, and $t > 0$ a constant. Then, there exist positive constants k_1, k_2 and k depending only on a, b, c, t that satisfy the following*:

(i) *Let σ be a simple closed geodesic on X with $L_X(\sigma) < k_2$, Γ a geodesic perpendicular to σ contained in C_σ with $L_X(\Gamma) = 2w$, $\gamma := \{p \in \Gamma : d_X(p, \sigma) \leqslant w/2\}$ and $\gamma_0 := \{p \in \gamma : d_X(p, \sigma) < w/2 - k_1\}$. Then the closed ball $\overline{B_Y(f(p), h+t)}$ is contained in the h-neighbourhood of $f(\gamma)$ for every point $p \in \gamma_0$, with $h := 3a + b + c$.*

(ii) *Let $\mathscr{C}$ be the 2-collar of a cusp in X with boundary curve σ, γ a geodesic ray contained in $\mathscr{C}$ perpendicular to σ and $\gamma_0 := \{p \in \gamma : d_X(p, \sigma) > k\}$. Then the closed ball $\overline{B_Y(f(p), h+t)}$ is contained in the h-neighbourhood of $f(\gamma)$ for every point $p \in \gamma_0$, with $h := 2a + b + c$.*

Proof. Let us prove (1). Set $k_1 := 2a(3a + 2b + 2c + t)$ and $k_2 := 2 \operatorname{arccoth}(\cosh k_1)$. Notice that $k_1 \geqslant 6$, since $a \geqslant 1$. Since $L_X(\gamma) = w$ and $d_X(\gamma, \partial C_\sigma) = w/2$, γ is a minimizing geodesic (not just a geodesic); therefore, $f(\gamma)$ is an (a, b)-quasigeodesic.

Seeking for a contradiction, let us assume that there exists a point $p \in \gamma_0$ such that the ball $B := \overline{B_Y(f(p), h+t)}$ is not contained in the h-neighbourhood of $f(\gamma)$. Denote by T_h such neighbourhood. That is, there exists a point $q \in B \backslash T_h$, and so,

$$d_Y(q, f(\gamma)) > h. \tag{3.1}$$

Since f is c-full, there must exist $p_1 \in X$ such that $d_Y(f(p_1), q) \leqslant c$. Let us assume that $d_X(p_1, \sigma) > w/2 - k_1/2$. Since $p \in \gamma_0$, it means that $d_X(p, p_1) > k_1/2$. Using the fact that f is an (a, b)-quasi-isometry,

$$d_Y(f(p), f(p_1)) \geqslant \frac{1}{a}\, d_X(p, p_1) - b > \frac{k_1}{2a} - b. \tag{3.2}$$

By the triangle inequality, and using that $q \in B$,

$$d_Y(f(p_1), f(p)) \leqslant d_Y(f(p_1), q) + d_Y(q, f(p)) \leqslant 3a + b + 2c + t. \tag{3.3}$$

Combining now (3.2) and (3.3), one deduces $k_1 < 2a(3a + 2b + 2c + t)$, which contradicts the definition of k_1. Therefore, $p_1 \in C_{\sigma, w/2 - k_1/2}$. Then, there exists a point $p_2 \in \gamma$ close enough to p_1, verifying that $d_X(p_1, p_2)$ is upper bounded by the length of one of the boundary curves of $C_{\sigma, w/2 - k_1/2}$. Using Fermi coordinates based on σ, we can easily check that $L_X(\partial C_{\sigma, w/2 - k_1/2})/2 < L_X(\partial C_\sigma)/2 = L_X(\sigma) \cosh w$. Collar Lemma gives $d_X(p_1, p_2) \leqslant L_X(\partial C_\sigma)/2 = L_X(\sigma) \cosh w = L_X(\sigma) \coth(L_X(\sigma)/2) \leqslant 3$ since $k_1 \geqslant 6$ and $L_X(\sigma) < 2 \operatorname{arccoth}(\cosh k_1)$.

On one hand, since f is an (a, b)-quasi-isometry (recall that $p_2 \in \gamma$),

$$d_Y(f(p_1), f(\gamma)) \leqslant d_Y(f(p_1), f(p_2)) \leqslant a d_X(p_1, p_2) + b \leqslant 3a + b. \tag{3.4}$$

On the other hand, taking into account (3.1),

$$d_Y(f(p_1), f(\gamma)) \geqslant d_Y(f(\gamma), q) - d_Y(q, f(p_1)) > 3a + b + c - c = 3a + b. \tag{3.5}$$

Obviously (3.5) contradicts (3.4), so such a point $q \in B \backslash T_h$ cannot exist.

The same arguments work for (2), defining $k := a(2a + 2b + 2c + t)$. □

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As usual, by a *Cartan-Hadamard manifold* we mean a complete simply connected Riemannian manifold with dimension greater than 1 and non-positive sectional curvatures. Then, $Y = \mathbb{R}^m$ (with $m \geqslant 2$) is a Cartan-Hadamard manifold but $\mathbb{R}$ is not, and the following statements do not hold (obviously) for $\mathbb{R}$.

LEMMA 3.2. *Let Y be a Cartan-Hadamard manifold, $h > 0$ and η an (a, b)-quasigeodesic in Y. If for some $z_0 \in \eta$ the ball $B_Y(z_0, r)$ is contained in the h-neighbourhood of η, then*

$$r \leqslant r_0 := a^2 \left(\frac{a^2}{2}(3h+b) + 2b + 5\,h \right) + b + 3\,h.$$

Proof. Let us define J as the least integer satisfying

$$J > \frac{1}{h} \left(a^2 \left(\frac{a^2}{2}(3h+b) + 2b + 5\,h \right) + b + h \right).$$

Assume that the ball $B := B_Y(z_0, r)$ is contained in the h-neighbourhood of η, for some $z_0 \in \eta$. Note that, since Y is a Cartan-Hadamard manifold, B is simply connected. Let I be an interval on the real line and $\eta : I \longrightarrow Y$ a parametrization of the (a, b)-quasigeodesic. Seeking for a contradiction, let us assume that $r > h(J+1)$.

Define j_1 as the least integer satisfying

$$j_1 > \frac{1}{h} \left(\frac{a^2}{2}(3h+b) + b + 2\,h \right).$$

There exists $\delta > 0$ such that

$$\begin{aligned} & r > (h+\delta)(J+1), \\ & j_1 < \frac{1}{h+\delta} \left(\frac{a^2}{2}(3h+3\delta+b) + b + 2h + 2\delta \right) + 2, \\ & j_1 > \frac{1}{h+\delta} \left(\frac{a^2}{2}(3h+3\delta+b) + b + 2h + 2\delta \right) \\ & \quad \text{and} \\ & J > \frac{1}{h+\delta} \left(a^2 \left(\frac{a^2}{2}(3h+3\delta+b) + 2b + 5h + 5\delta \right) + b + h + \delta \right). \end{aligned} \tag{3.6}$$

Since Y is a Cartan-Hadamard manifold, there exist two orthogonal geodesic lines $\gamma_1, \gamma_2 : \mathbb{R} \to Y$ with $\gamma_1(0) = \gamma_2(0) = z_0$. We also have $d_Y(\gamma_1(t), \gamma_2) = |t|$ for every $t \in \mathbb{R}$ and $d_Y(\gamma_2(s), \gamma_1) = |s|$ for every $s \in \mathbb{R}$.

Let us fix points $z_1, z_2, z_3, \ldots \in \gamma_1$ in one of the directions starting at z_0 and $z_{-1}, z_{-2}, z_{-3}, \ldots \in \gamma_1$ in the opposite direction from z_0, such that $d_Y(z_0, z_j) = |j|(h+\delta)$ for every $j \in \mathbb{Z}$ with $|j|(h+\delta) < r$. Analogously, choose points $w_j \in \gamma_2$ with $d_Y(z_0, w_j) = |j|(h+\delta)$ for every $j \in \mathbb{Z}$ with $|j|(h+\delta) < r$.

Since B is contained in the h-neighbourhood of η, for each of these points $z_j, w_j \in B$ there exist points $z_j^*, w_j^* \in \eta$ verifying $d_Y(z_j, z_j^*) \leqslant h + \delta$ and

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$d_Y(w_j,\, w_j^*) \leqslant h+\delta$. Let $t_j,\, s_j \in I$ be real values such that $\eta(t_j) = z_j^*$ and $\eta(s_j) = w_j^*$; in particular, we can choose $s_0 = t_0$ and $\eta(t_0) = z_0 = z_0^*$. Thus,

$$\begin{aligned}|t_j - t_k| &\leqslant a(d_Y(z_j^*, z_k^*) + b) \leqslant a(d_Y(z_j, z_k) + 2h + 2\delta + b)\\ &= a(|j-k|(h+\delta) + 2h + 2\delta + b)\end{aligned}$$

and, in particular,

$$|t_j - t_{j+1}| \leqslant a(3\,h + 3\delta + b). \tag{3.7}$$

Note that z_J^* and z_{-J}^* are both in the ball B:

$$d_Y(z_{\pm J}^*, z_0) \leqslant d_Y(z_{\pm J}^*, z_{\pm J}) + d_Y(z_{\pm J}, z_0) \leqslant h+\delta+J(h+\delta) = (h+\delta)(J+1) < r.$$

For the defined value j_1

$$\begin{aligned}|s_{j_1} - t_0| &\leqslant a(d_Y(w_{j_1}^*, z_0) + b) \leqslant a(d_Y(w_{j_1}^*, w_{j_1}) + d_Y(w_{j_1}, z_0) + b)\\ &\leqslant a(h + \delta + j_1(h+\delta) + b).\end{aligned} \tag{3.8}$$

A similar argument gives

$$|t_J - t_0| \geqslant \frac{1}{a}\,(d_Y(z_0, z_J^*) - b) \geqslant \frac{1}{a}\,(J(h+\delta) - h - \delta - b). \tag{3.9}$$

Using the fourth inequality in (3.6) and $j_1(h+\delta) < a^2/2(3h+3\delta+b) + b + 4h + 4\delta$, we can easily check that

$$\frac{1}{a}\,(J(h+\delta) - h - \delta - b) > a(h+\delta+b+j_1(h+\delta)).$$

Therefore, comparing (3.8) and (3.9), one obtains $|t_J - t_0| > |s_{j_1} - t_0|$. Analogously, $|t_{-J} - t_0| > |s_{j_1} - t_0|$. Hence, by (3.7), there exists some $j_2 \in \mathbb{Z}$ such that $|j_2| \leqslant J$ and

$$|s_{j_1} - t_{j_2}| \leqslant \frac{a}{2}(3h + 3\delta + b).$$

Taking into account the above inequality,

$$\begin{aligned}d_Y(w_{j_1}, z_{j_2}) &\leqslant d_Y(w_{j_1}^*, z_{j_2}^*) + 2h + 2\delta \leqslant a|s_{j_1} - t_{j_2}| + b + 2h + 2\delta\\ &\leqslant \frac{a^2}{2}(3h + 3\delta + b) + b + 2h + 2\delta,\\ d_Y(w_{j_1}, z_{j_2}) &\geqslant d_Y(w_{j_1}, \gamma_1) = d_Y(w_{j_1}, z_0) = j_1(h+\delta).\end{aligned}$$

Thus,

$$j_1(h+\delta) \leqslant \frac{a^2}{2}(3h + 3\delta + b) + b + 2h + 2\delta,$$

which contradicts the third inequality in (3.6). Therefore,

$$r \leqslant hJ + h \leqslant r_0. \qquad \square$$

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THEOREM 3.3. *Let X be a non-exceptional Riemann surface, Y a Cartan-Hadamard manifold and $f : X \to Y$ a c-full (a, b)-quasi-isometry. Then there exists a positive constant c_0, which just depends on a, b, c, such that $\iota(X) \geqslant c_0$. In fact, we can choose*

$$c_0 = \log\coth\left(\frac{a^5}{2}(9a+4b+3c) + a^3(15a+7b+5c) + a(9a+5b+4c)\right).$$

Proof. Let h and r_0 be the constants defined in lemmas 3.1 and 3.2 respectively, namely, $h := 3a + b + c$ and $r_0 := a^2((a^2/2)(3h + b) + 2b + 5\,h) + b + 3\,h$. Let us define $t := r_0 - h + \varepsilon$, for any given $\varepsilon > 0$. Note that $t > 0$ since $r_0 > 3\,h$.

As in the proof of lemma 3.1, let us fix the constants k_1 and k_2 there defined, $k_1 := 2a(3a + 2b + 2c + t)$ and

$$\begin{aligned} k_2 := 2\,\mathrm{arccoth}(\cosh k_1) &= \log\frac{\cosh k_1 + 1}{\cosh k_1 - 1} \\ &= \log\frac{\cosh^2(k_1/2)}{\sinh^2(k_1/2)} = 2\log\coth(k_1/2). \end{aligned}$$

Let us suppose that $\iota(X) < k_2/2$, and let us seek for a contradiction. Let $x \in X$ be a point with $\iota(p) < k_2/2$, thus there exists a non-trivial geodesic loop η based at p with length $L_X(\eta) < k_2$.

Assume first that there exists a simple closed geodesic σ freely homotopic to η on X. Hence, $L_X(\sigma) \leqslant L_X(\eta) < k_2$.

As a consequence of lemma 3.1(1), if Γ is a geodesic perpendicular to σ contained in C_σ with $L_X(\Gamma) = 2w$, $\gamma := \{p \in \Gamma : d_X(p, \sigma) \leqslant w/2\}$ and $\gamma_0 := \{p \in \gamma : d_X(p, \sigma) < w/2 - k_1\}$, then the closed ball $\overline{B_Y(f(p), h + t)}$ is contained in the h-neighbourhood of $f(\gamma)$ for every point $p \in \gamma_0$. Moreover, $f(\gamma)$ is an (a, b)-quasigeodesic in Y. By lemma 3.2, one gets that $h + t \leqslant r_0$, which contradicts the definition of the constant t given above.

Assume now that there is no simple closed geodesic freely homotopic to η. Thus, η surrounds a cusp in X. A similar argument, using now lemma 3.1(2) instead of Lemma 3.1(1), allows to obtain a contradiction in this case.

Hence,

$$\begin{aligned} \iota(X) &\geqslant \frac{1}{2}\,k_2 = \log\coth(k_1/2) \\ &= \log\coth\left(a\left(3a + 2b + 2c + a^2\left(\frac{a^2}{2}(3h+b) + 2b + 5\,h\right) + b + 2h + \varepsilon\right)\right) \\ &= \log\coth\left(\frac{a^5}{2}(9a+4b+3c) + a^3(15a+7b+5c) + a(9a+5b+4c+\varepsilon)\right) \end{aligned}$$

for every $\varepsilon > 0$, and so,

$$\iota(X) \geqslant \log\coth\left(\frac{a^5}{2}(9a+4b+3c) + a^3(15a+7b+5c) + a(9a+5b+4c)\right).$$

□

[16, Theorem 5.5] gives that every orientable complete Riemannian surface with pinched negative curvature is bilipschitz equivalent to a complete surface with constant negative curvature. Furthermore, the bilipschitz constants depend just on the bounds of the curvature. This fact and Theorem 3.3 have the following consequence.

THEOREM 3.4. *Let X be an orientable complete Riemannian surface with pinched negative curvature, Y a Cartan-Hadamard manifold and $f : X \to Y$ a c-full (a, b)-quasi-isometry. Then there exists a positive constant c_0, which just depends on a, b, c and the bounds on the curvature, such that $\iota(X) \geqslant c_0$.*

4. Liouville property

We start this section with a technical result which will be very useful.

LEMMA 4.1. *Let X be a proper geodesic metric space. The following facts are equivalent:*

(1) *X is quasi-isometric to $\mathbb{R}$.*

(2) *There is a $(1, 0)$-quasi-isometry from $\mathbb{R}$ to X.*

Proof. It is clear that (2) implies (1), let us prove the converse. Since X is quasi-isometric to $\mathbb{R}$ and $\mathbb{R}$ is Gromov hyperbolic, we know that X is also Gromov hyperbolic (see e.g. [15, p.88]). Also, there exists a c_1-full (a_1, b_1)-quasi-isometry $F : \mathbb{R} \to X$ for some constants a_1, b_1 and c_1 (see e.g. [20]). Then, $g := F(\mathbb{R})$ is a quasigeodesic in X and, since X is a proper geodesic metric space, there exists a geodesic line $\gamma \subset X$ such that $\mathcal{H}(g, \gamma) \leqslant h$, where $\mathcal{H}$ denotes the Hausdorff distance and $h = h(\delta, a_1, b_1)$ (see [15, p.101]). Thus any arc-length parametrization of γ is also a $(c_1 + h)$-full $(1, 0)$-quasi-isometry from $\mathbb{R}$ to X. □

If X is a complete Riemannian surface with pinched negative curvature, we say that a bordered subsurface $H \subset X$ is a *half-plane* if H is simply connected and ∂H is a geodesic line. Note that any funnel in X contains infinitely many half-planes.

THEOREM 4.2. *Let X be an orientable complete Riemannian surface with pinched negative curvature and injectivity radius is equal to zero and Y a complete simply connected Riemannian manifold (with non-positive sectional curvatures if the dimension of Y is greater than 1). If X and Y are quasi-isometric, then Y is isometric to $\mathbb{R}$, $\sharp Ends(X) = 2$, X has positive genus, has at most two cusps and does not have funnels or half-planes.*

Moreover, X has finite genus if and only if X has two cusps.

Proof. Since the injectivity radius of X is equal to zero, Theorem 3.4 implies that Y has dimension 1. Since Y is a complete simply connected Riemannian manifold, Y is isometric to $\mathbb{R}$.

Since X is quasi-isometric to $\mathbb{R}$, it follows that $\sharp\,\mathrm{Ends}(X) = 2$ (in virtue of Theorem 2.5).

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By lemma 4.1, there exists a (1, 0)-quasi-isometry from $\mathbb{R}$ to X. However, it is not possible to have a full geodesic line in a funnel, which implies that X does not have funnels. The same argument gives that X does not contain any half-planes.

In [**16**, Theorem 5.5] it is shown that every orientable complete Riemannian surface with pinched negative curvature is bilipschitz equivalent to a complete surface with constant negative curvature; thus, without loss of generality we can assume in the remaining part of this proof that X has $K = -1$, and so it is a Riemann surface endowed with its Poincaré metric.

Seeking for a contradiction, assume that X is of genus zero. If this is the case, then X is isometric to a domain $\Omega \subset \mathbb{C}$ with its Poincaré metric. Moreover, since $\sharp\,\mathrm{Ends}(X) = 2$, Ω is a doubly connected set. Now, if the two connected components of $\overline{\mathbb{C}}\backslash\Omega$ (with $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$) were isolated points, then Ω would be conformally equivalent to $\mathbb{C}\backslash\{0\}$ and a contradiction would arise since it could not be endowed with its Poincaré metric. This means that at least one of the connected components of $\overline{\mathbb{C}}\backslash\Omega$ is a continuous set and, therefore, Ω is conformally equivalent either to the punctured disk or to some annulus. Lemma 4.1 gives that there exists a geodesic line $\gamma \subset X$ which is a (1, 0)-quasi-isometry from $\mathbb{R}$ to X. However, it is not possible to have a full geodesic line neither in a funnel nor in $\mathbb{D}\backslash\{0\}$. Nevertheless, this is a contradiction with the two available options for Ω obtained above (i.e., Ω is conformally equivalent either to the punctured disk or to an annulus). This contradiction comes from assuming that X is of genus zero, and thus X, necessarily, has positive genus.

Since each cusp provides an end and $\sharp\, Ends(X) = 2$, it follows that X has at most two cusps.

Assume that X has two cusps. Seeking for a contradiction assume that X has infinite genus. Since the infinite genus provides at least an end to X, and each cusp is an end, we have $\sharp\, Ends(X) \geqslant 3$, a contradiction. Hence, X has finite genus.

Assume now that X has at most one cusp and suppose that X is of finite genus. Since there are neither funnels nor half-planes, it means that X can be obtained as the union of some Y-pieces and one generalized Y-piece with a single cusp (see [**1**, Theorem 1.2]). If there is a finite amount of Y-pieces in the decomposition of X, then $\sharp\,\mathrm{Ends}(X) = 1$, which is a contradiction with the proved fact that there are exactly two ends in this surface. Therefore, there must be an infinite amount of Y-pieces in the decomposition of X. But, since X has finite genus, this surface has also infinitely many ends, which is a contradiction. Thus, X must have infinite genus. □

One could think that surfaces satisfying the hypotheses in Theorem 4.2 must have at least one cusp, since they are quasi-isometric to $\mathbb{R}$ and have injectivity radius equal to zero. However, the following example shows that that is not always the case.

Example 4.3. There exists a surface S quasi-isometric to $\mathbb{R}$ with curvature -1 and injectivity radius equal to zero, which has no cusps.

Let us consider for each $n \geqslant 1$ the right-angled geodesic hexagon H_n in the hyperbolic plane with lengths of three alternate sides equal to $1/2$, $1/2$, $1/(2n)$. Denote by l_n the length of the side connecting the two sides of length $1/2$. The usual hyperbolic trigonometric formulas (see e.g. [8, p.40]) give

$$\cosh\frac{1}{2n} = \sinh^2\frac{1}{2}\cosh l_n - \cosh^2\frac{1}{2}\,,$$
$$l_n = \arg\cosh\frac{\cosh\frac{1}{2n} + \cosh^2\frac{1}{2}}{\sinh^2\frac{1}{2}} \to \arg\cosh\frac{1+\cosh^2\frac{1}{2}}{\sinh^2\frac{1}{2}}$$

as $n \to \infty$. For each $n \geqslant 1$, let Y_n be the Y-piece with boundary closed geodesics of lengths 1, 1, $1/n$ obtained by pasting in an appropriate way two geodesic hexagons isometric to H_n. Then the distance between the two closed geodesics of length 1 is l_n, which has a positive and finite limit as $n \to \infty$. For each $n \geqslant 1$, let X_n be the surface with two boundary closed geodesics of lengths $1/n$, $1/(n+1)$ obtained from Y_n and Y_{n+1} by identifying pairwise the geodesic of length 1. Let S_0 be the surface with a boundary closed geodesic of length 1 obtained from X_1, X_2, ... by identifying for each $n > 1$ the geodesics of length $1/n$ in ∂X_{n-1} and ∂X_n.

If S is the surface without boundary obtained from two isometric copies of S_0 by identifying the geodesics of length 1, then S is a complete Riemannian surface with curvature -1 and injectivity radius 0. Also, X is quasi-isometric to $\mathbb{R}$ and it does not have cusps.

Theorem 4.2 shows the importance of characterizing the Riemannian surfaces which are quasi-isometric to $\mathbb{R}$. It also provides some necessary conditions for an orientable complete Riemannian surface with pinched negative curvature and injectivity radius is equal to zero to be quasi-isometric to $\mathbb{R}$. In Theorem 1.2 the general setting of metric spaces is chosen, since it is more convenient to obtain a characterization. First, we need two technical lemmas.

Lemma 4.4. *Let X be a proper geodesic metric space which is quasi-isometric to $\mathbb{R}$. Then there exists $R_0 > 0$ such that $X\setminus\overline{B_X(p,\,R)}$ has exactly two unbounded connected components for every $p \in X$ and $R \geqslant R_0$.*

Proof. By Theorem 2.5 it follows that $\sharp\,\mathrm{Ends}(X) = 2$. Hence, $X\setminus\overline{B_X(p,\,R)}$ has at most two unbounded connected components for every $p \in X$ and $R > 0$.

lemma 4.1 gives that there exists a c-full $(1,\,0)$-quasi-isometry $h : \mathbb{R} \longrightarrow X$ for some $c \geqslant 0$, and let γ be the curve $\gamma = h(\mathbb{R}) \subseteq X$. Let $g : X \longrightarrow \mathbb{R}$ be an $(a,\,b)$-quasi-isometry such that $g|_\gamma = h^{-1}$, and so, $g|_\gamma$ is an isometry. Let us suppose first that $p \in \gamma$. Let x_1 and x_2 be points in $\gamma\setminus\overline{B_X(p,\,3ab/2)}$ such that p is between them (p belongs to the segment of γ joining x_1 and x_2). Let η be any curve in X joining x_1 and x_2. Given $\varepsilon > 0$ let us observe that if $x,\,y \in \eta$ with $d_X(x,\,y) < \varepsilon$, then $d_{\mathbb{R}}(g(x),\,g(y)) < a\varepsilon + b$. Therefore, $g(\eta)$ is a quasigeodesic in $\mathbb{R}$ with endpoints $g(x_1)$ and $g(x_2)$ having all its 'jumps' of less than $a\varepsilon + b$. Note that $g(p)$ is between $g(x_1)$ and $g(x_2)$ (recall that $g|_\gamma$ is an isometry). If g would be a continuous function, then there would exist $z \in \eta$ with $g(z) = g(p)$. Since the jumps of g are of less than $a\varepsilon + b$, it follows that there exists a point $z \in \eta$ such that $d_{\mathbb{R}}(g(p),\,g(z)) < (a\varepsilon + b)/2$. Hence,

$$d_X(p,\eta) \leqslant d_X(p,z) \leqslant a(d_{\mathbb{R}}(g(p),g(z)) + b) \leqslant a\big((a\varepsilon + b)/2 + b\big)\,.$$

By letting ε tends to zero, we obtain that

$$d_X(p,\eta) \leqslant \frac{3ab}{2} =: R_1.$$

Since $x_1,\, x_2 \notin \overline{B_X(p,\, R_1)}$ and every curve η joining them in X intersects $\overline{B_X(p,\, R_1)}$, we conclude that $\overline{B_X(p,\, R_1)}$ disconnects X, for each $p \in \gamma$. Since x_1 and x_2 are arbitrary points such that p is between them and $d_X(p,\, x_1),\, d_X(p,\, x_2) > R_1$, there exist at least two unbounded connected components of $X \backslash \overline{B_X(p,\, R_1)}$. As h is c-full we conclude that $X \backslash \overline{B_X(p,\, R)}$ has at least two unbounded connected components for every $p \in X$ and $R \geqslant R_0 := R_1 + c$. This finishes the proof. □

Lemma 4.5. *Let X be a proper geodesic metric space which is quasi-isometric to $\mathbb{R}$, and R_0 the constant in Lemma 4.4. Then there exists a constant $R_1 > R_0$ such that, for all $p \in X$, $X \backslash \overline{B_X(p,\, R_1)}$ has exactly two unbounded connected components $E_1(p,\, R_1)$, $E_2(p,\, R_1)$ and*

$$d_X(x,p) \leqslant 4R_1, \quad \forall\, x \notin E_1(p,R_1) \cup E_2(p,R_1).$$

Proof. lemma 4.1 gives that there exists a c-full $(1,\,0)$-quasi-isometry $h : \mathbb{R} \longrightarrow X$ for some $c \geqslant 0$. Let γ be the image $\gamma = h(\mathbb{R})$ in X. We are going to prove that for $R_1 = \max\{R_0,\, c\}$ we have

$$d_X(x,p) \leqslant 4R_1\,, \qquad \forall\, x \notin E_1(p,R_1) \cup E_2(p,R_1)$$

(since $R_1 \geqslant R_0$ that lemma 4.4 gives that $X \backslash \overline{B_X(p,\, R_1)}$ has exactly two unbounded connected components). Seeking for a contradiction assume that there exist $p \in X$ and $x \notin E_1(p,\, R_1) \cup E_2(p,\, R_1)$ such that $d_X(p,\, x) > 4R_1$. Let $p_0 \in \gamma$ such that $d_X(p,\, p_0) \leqslant c$. Then

$$\begin{aligned} &\overline{B_X(p,R_1)} \subseteq \overline{B_X(p_0,R_1+c)}\,, \\ &E_1(p_0,R_1+c) \cup E_2(p_0,R_1+c) \subseteq E_1(p,R_1) \cup E_2(p,R_1)\,, \end{aligned}$$

and therefore

$$x \notin E_1(p_0,R_1+c) \cup E_2(p_0,R_1+c) \quad \text{and} \quad d_X(p_0,x) > 4R_1 - c \geqslant 3R_1.$$

Let now η be a minimizing geodesic joining x and γ, with $d_X(x,\, \gamma) = L_X(\eta)$. Since $p_0 \in \gamma$ and h is a geodesic line, we have that $\gamma \cap \overline{B_X(p_0,\, R_1 + c)}$ is a diameter of $\overline{B_X(p_0,\, R_1 + c)}$ and so,

$$F = B_X(p_0,R_1+c) \cup \overline{E_1(p_0,R_1+c)} \cup \overline{E_2(p_0,R_1+c)}$$

is an open connected set containing γ, and $\partial F \subset \partial B_X(p_0,\, R_1 + c)$. Thus, since $x \notin F$, $\gamma \subset F$ and $\partial F \subset \partial B_X(p_0,\, R_1 + c)$, we conclude that η intersects ∂F. Let $\tilde{\eta}$ be the maximal subcurve of η containing x and contained in $X \backslash F$. If $x' \in \tilde{\eta} \cap \partial B_X(p_0,\, R_1 + c)$, then

$$\begin{aligned} d_X(x,\gamma) &= L_X(\eta) > L_X(\tilde{\eta}) = d_X(x,x') \geqslant d_X(x,p_0) - d_X(p_0,x') \\ &> 3R_1 - (R_1 + c) \geqslant c, \end{aligned}$$

which is a contradiction, since h is c-full.

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Hence, the conclusion of the lemma holds. □

LEMMA 4.6. *Let X be a proper geodesic metric space. Assume that there exists a positive constant R_1 such that, for all $p \in X$, $X \setminus \overline{B_X(p, R_1)}$ has exactly two unbounded connected components $E_1(p, R_1)$, $E_2(p, R_1)$ and*

$$d_X(x,p) \leqslant 4R_1, \quad \forall\, x \notin E_1(p,R_1) \cup E_2(p,R_1).$$

Then for each $p \in X$ there exist two geodesic rays starting at p such that its union is a $(4R_1)$-full $(1, 2R_1)$-quasi-isometry from $\mathbb{R}$ to X.

Proof. Fix $p \in X$ and let us choose a sequence $\{x_n^j\} \subset E_j(p, R_1)$ with $d_X(p, x_n^j) \geqslant n$ for $j = 1, 2$. Since X is a proper geodesic metric space, Arzelá-Ascoli's Theorem provides a subsequence $\{x_{n_k}^j\}$ such that $\{[px_{n_k}^j]\}$ converges to a geodesic ray $\gamma_j : [0, \infty) \to X$ starting at p and 'finishing' in $E_j(p, R_1)$ $(j = 1, 2)$. We define $\gamma : \mathbb{R} \longrightarrow X$ as the curve

$$\gamma(t) = \begin{cases} \gamma_1(t), & \text{if } t \geqslant 0, \\ \gamma_2(-t), & \text{if } t \leqslant 0, \end{cases}$$

and let us check that γ is a quasigeodesic.

If $s, t \geqslant 0$, then $d_X(\gamma(s), \gamma(t)) = d_X(\gamma_1(s), \gamma_1(t)) = |t-s|$. If $s, t \leqslant 0$, then $d_X(\gamma(s), \gamma(t)) = d_X(\gamma_2(-s), \gamma_2(-t)) = |t-s|$. Consider now $s \leqslant 0$ and $t \geqslant 0$. In this case,

$$d_X(\gamma(s),\gamma(t)) \leqslant d_X(\gamma(t),\gamma(0)) + d_X(\gamma(0),\gamma(s)) = t + |s| = |t-s|.$$

If $|s| \leqslant R_1$, then

$$\begin{aligned} d_X(\gamma(s),\gamma(t)) &\geqslant d_X(\gamma(t),\gamma(0)) - d_X(\gamma(0),\gamma(s)) = t - |s| \\ &= |t-s| - 2|s| \geqslant |t-s| - 2R_1. \end{aligned}$$

Similarly, if $t \leqslant R_1$, then $d_X(\gamma(s), \gamma(t)) \geqslant |t-s| - 2R_1$. Finally, assume that $|s|, t > R_1$, and let η be a minimizing geodesic joining $\gamma(s)$ and $\gamma(t)$. Since $\gamma(t) \in E_1(p.R_1)$ and $\gamma(s) \in E_2(p.R_1)$, we have

$$L_X(\eta \cap E_1(p,R_1)) \geqslant t - R_1, \quad L_X(\eta \cap E_2(p,R_1)) \geqslant |s| - R_1,$$

and so,

$$d_X(\gamma(s),\gamma(t)) = L_X(\eta) \geqslant |t-s| - 2R_1.$$

In a similar way, if $s \geqslant 0$ and $t \leqslant 0$, then $|t-s| - 2R_1 \leqslant d_X(\gamma(s), \gamma(t)) \leqslant |t-s|$.

Thus, γ is a $(1, 2R_1)$-quasigeodesic.

For each $t \in \mathbb{R}$ and $T > 0$, let us consider the compact sets (since X is a proper space and γ is a continuous curve)

$$\begin{aligned} B_t &:= \overline{B_X(\gamma(t), 4R_1)} \setminus \big(E_1(\gamma(t), R_1) \cup E_2(\gamma(t), R_1)\big), \\ K_T &:= \bigcup_{|t| \leqslant T} B_t. \end{aligned}$$

The hypothesis of the lemma gives that $B_t = X \setminus (E_1(\gamma(t), R_1) \cup E_2(\gamma(t), R_1))$. For each $p \in X$, denote by $E_j(p, R_1)$ the unbounded connected component of

$X \setminus \overline{B_X(p, R_1)}$ containing a terminal segment of γ_j $(j = 1, 2)$. Thus,

$$K_T = X \setminus \big(E_1(\gamma_1(T), R_1) \cup E_2(\gamma_2(T), R_1) \big)$$

for T large enough, and so,

$$X = \bigcup_{T>0} K_T = \bigcup_{t \in \mathbb{R}} \overline{B_X(\gamma(t), 4R_1)}\,.$$

Hence, γ is $(4R_1)$-full and it is a quasi-isometry. □

lemmas 4.1, 4.5 and 4.6 give Theorem 1.2.

LEMMA 4.7. *Let X be a non-exceptional Riemann surface quasi-isometric to $\mathbb{R}$. Then there exists a constant C such that*

$$L_X(\partial B_X(p, r)) \leqslant C, \quad A_X(B_X(p, r)) \leqslant Cr,$$

for every $p \in X$ and $r > 0$.

Proof. Let R_1 the constant in lemma 4.5. Fix $p \in X$ and a universal covering map $\Pi : \mathbb{D} \to X$ with $\Pi(0) = p$. Let D be the Dirichlet fundamental domain (see e.g. [**4**, p.226])

$$D = \big\{ z \in \mathbb{D} : \, d_{\mathbb{D}}(z, 0) < d_{\mathbb{D}}(z, z_0) \; \forall \, z_0 \in \Pi^{-1}(p) \setminus \{0\} \big\}.$$

We claim that if γ is a minimizing geodesic in X with $p \in \gamma$, then the minimizing geodesic g in $\mathbb{D}$ such that $0 \in g$ and $\Pi(g) = \gamma$ (g is the lift of γ at 0) is contained in $\overline{D}$:

Note that $d_{\mathbb{D}}(z, 0) = d_X(\Pi(z), p)$ for every $z \in g$, since Π is a local isometry and γ is a minimizing geodesic in X. Seeking for a contradiction assume that there exists $z \in g \setminus \overline{D}$. Thus, there exists $z_0 \in \Pi^{-1}(p)$ with $d_{\mathbb{D}}(z, z_0) < d_{\mathbb{D}}(z, 0)$. Since Π is a holomorphic function, we have $d_X(\Pi(z), p) \leqslant d_{\mathbb{D}}(z, z_0) < d_{\mathbb{D}}(z, 0) = d_X(\Pi(z), p)$, a contradiction. Thus, g is contained in $\overline{D}$.

lemma 4.6 gives that there exist two geodesic rays γ_1 and γ_2 starting at p such that its union is a $(4R_1)$-full $(1, 2R_1)$-quasi-isometry from $\mathbb{R}$ to X. Let g_1 and g_2 be the lifts at 0 of γ_1 and γ_2, respectively. We have proved that the geodesic rays g_1 and g_2 are contained in $\overline{D}$.

One can check that

$$\Pi\big(\partial B_{\mathbb{D}}(0, r) \cap \overline{D}\,\big) = \partial B_X(p, r) \quad \text{and} \quad L_{\mathbb{D}}\big(\partial B_{\mathbb{D}}(0, r) \cap \overline{D}\,\big) = L_X\big(\partial B_X(p, r)\big)$$

for every $r > 0$. Let us denote by $N_h(A)$ the closed h-neighbourhood of a set A in a metric space Y, i.e., $N_h(A) = \{y \in Y : \, d_Y(y, A) \leqslant h\}$. Since $\gamma_1 \cup \gamma_2$ is $(4R_1)$-full

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in X, we have that $g_1 \cup g_2$ is $(4R_1)$-full in $\overline{D}$ and so,

$$L_X\big(\partial B_X(p,r)\big) = L_{\mathbb{D}}\big(\partial B_{\mathbb{D}}(0,r) \cap \overline{D}\,\big) \leqslant L_{\mathbb{D}}\big(\partial B_{\mathbb{D}}(0,r) \cap N_{4R_1}(g_1 \cup g_2)\,\big)$$

for every $r > 0$. If I denotes the geodesic line in $\mathbb{D}$ given by the real interval $(-1, 1)$, then

$$L_X\big(\partial B_X(p,r)\big) \leqslant L_{\mathbb{D}}\big(\partial B_{\mathbb{D}}(0,r) \cap N_{4R_1}(g_1 \cup g_2)\,\big) \leqslant L_{\mathbb{D}}\big(\partial B_{\mathbb{D}}(0,r) \cap N_{4R_1}(I)\,\big)$$

for every $r > 0$. The last inequality holds since the further apart g_1 and g_2 are, the larger the set $\partial B_{\mathbb{D}}(0, r) \cap N_{4R_1}(g_1 \cup g_2)$ is.

Fix $r > 4R_1$ and let z_r be the point in $\partial B_{\mathbb{D}}(0, r) \cap \partial N_{4R_1}(I)$ contained in the first quadrant. Since the hyperbolic length of an arc of angle α in $\partial B_{\mathbb{D}}(0, r)$ is $\alpha \sinh r$, we have

$$L_{\mathbb{D}}\big(\partial B_{\mathbb{D}}(0,r) \cap N_{4R_1}(I)\,\big) = 4\,\theta_r \sinh r,$$

where θ_r is the argument of z_r. If $x_r \in (0, 1)$ is the point with $d_{\mathbb{D}}(z_r, x_r) = d_{\mathbb{D}}(z_r, I)$, then $\{0, z_r, x_r\}$ are the vertices of a right-angled triangle in $\mathbb{D}$ with angle θ_r at 0, $d_{\mathbb{D}}(0, z_r) = r$ and $d_{\mathbb{D}}(z_r, x_r) = 4R_1$. The hyperbolic sine theorem (see e.g. [**4**, p.148]) gives

$$\sinh r = \frac{\sinh 4R_1}{\sin \theta_r}. \tag{4.1}$$

Hence,

$$L_{\mathbb{D}}\big(\partial B_{\mathbb{D}}(0,r) \cap N_{4R_1}(I)\,\big) = 4\,\theta_r \sinh r = 4 \sinh 4R_1\,\frac{\theta_r}{\sin \theta_r}$$

and, since $t/\sin t$ is an increasing function on $t \in (0, \pi/2)$ and (4.1) gives that θ_r is a decreasing function on r,

$$L_X\big(\partial B_X(p,r)\big) \leqslant 4 \sinh 4R_1\,\frac{\theta_r}{\sin \theta_r} \leqslant 4 \sinh 4R_1\,\frac{\theta_{4R_1}}{\sin \theta_{4R_1}}$$

for every $r > 4R_1$. Since $L_X(\partial B_X(p, r)) \leqslant L_{\mathbb{D}}(\partial B_{\mathbb{D}}(0, r)) = 2\pi \sinh r$, we have $L_X(\partial B_X(p, r)) \leqslant 2\pi \sinh 4R_1$ for every $r \leqslant 4R_1$. Therefore,

$$L_X\big(\partial B_X(p,r)\big) \leqslant C = \max\left\{4 \sinh 4R_1\,\frac{\theta_{4R_1}}{\sin \theta_{4R_1}}\,,\, 2\pi \sinh 4R_1\right\}$$

for every $r > 0$.

Finally,

$$A_X(B_X(p,r)) = \int_0^r L_X(\partial B_X(p,s))\,ds \leqslant \int_0^r C\,ds = Cr. \qquad \square$$

One can think that the converse of lemma 4.7 holds. However, it does not hold, as the following example shows.

Example 4.8. There exists a non-exceptional Riemann surface X which is not quasi-isometric to $\mathbb{R}$, and verifying the conclusion of lemma 4.7.

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Consider the graph G_0 with $V(G_0) = \mathbb{Z}$ and $E(G_0) = \{(n, n+1) : n \in \mathbb{Z}\}$. For each $k \in \mathbb{N}$, let P_{k+1} be the path graph with $k+1$ vertices and length k. Let $\{x_k\}_{k=1}^{\infty}$ be an increasing sequence of integers such that $x_k - x_{k-1} > k+1$ for every $k \geqslant 2$. Let G be any graph obtained from G_0 and $\{P_k\}_{k=1}^{\infty}$ by identifying a vertex with degree 1 in P_k with x_k for each $k \geqslant 1$. Thus, for every $u \in V(G)$ and $n \in \mathbb{N}$, the set $\{v \in V(G) : d_G(u, v) = n\}$ has, at most, 4 vertices and so, $\{v \in V(G) : d_G(u, v) \leqslant n\}$ has, at most, $4n+1$ vertices. It is clear that G is not quasi-isometric to $\mathbb{R}$.

Let us consider a Y-piece A such that the length of its three simple closed geodesics on its boundary is equal to 1. If A_1, A_2 are Y-pieces isometric to A, let B be the bordered Riemann surface with genus one whose boundary is the union of two simple closed geodesics, obtained from A_1 and A_2 by identifying two simple closed geodesics in ∂A_1 with two simple closed geodesics in ∂A_2. If A_0 is a Y-piece isometric to A, let C be the bordered Riemann surface with genus one whose boundary is a simple closed geodesic, obtained from A_0 by identifying two simple closed geodesics in ∂A_0.

Let us construct now a Riemann surface X by using the graph G as a skeleton. Replace each vertex of degree 3, 2 and 1 in $V(G)$ by bordered Riemann surfaces isometric to A, B and C, respectively, and identify the simple closed geodesics in the boundary of any of these pieces following the combinatorial design of G.

Since X is quasi-isometric to G, it is not quasi-isometric to $\mathbb{R}$.

Recall that we have $|\partial B_G(u, n) = n\}| \leqslant 4$ and $|B_G(u, n) \leqslant n\}| \leqslant 4n+1$ for every positive integer n and $u \in V(G)$, G and X are quasi-isometric and X has bounded geometry.

Kanai's arguments in the proofs of [**20**, Lemmas 3.4 and 3.6] imply that there exists a constant C_1 such that $A_X(B_X(p, r)) \leqslant C_1 r$ for every $p \in X$ and $r > 0$.

A similar argument, using also the arguments in the proof of [**20**, Theorem 4.1 and Lemma 4.2], gives the bound for the length of the boundary.

By [**16**, Theorem 5.5], every orientable complete Riemannian surface with pinched negative curvature is bilipschitz equivalent to a complete surface with constant negative curvature. So, Lemma 4.7 has the following consequence.

Proposition 4.9. *Let X be an orientable complete Riemannian surface with pinched negative curvature quasi-isometric to $\mathbb{R}$. Then there exists a constant C such that*

$$A_X(B_X(p, r)) \leqslant Cr,$$

for every $p \in X$ and $r > 0$.

Proof of Theorem 1.1. If $\iota(X) > 0$ and $m \geqslant 2$, then the result follows straightforwardly from [**20**, Theorem 5.1]. Otherwise, applying Theorem 4.2, we can conclude that $m = 1$ and, therefore, Proposition 4.9 gives that X has polynomial growth rate of degree, at most, 1. Now the conclusion follows from [**11**, Corollary 1, p.336]. □

5. A decomposition of surfaces quasi-isometric to $\mathbb{R}$

In this last section we prove Theorem 1.3, which provides a good property of the Riemannian surfaces with pinched negative curvature quasi-isometric to $\mathbb{R}$: it is possible to decompose these surfaces as union of generalized Y-pieces in such a way that the length of the boundary of the pieces has an upper bound.

Let us start with some technical results.

Lemma 5.1. *Let X be a non-exceptional Riemann surface quasi-isometric to $\mathbb{R}$ and let $g : \mathbb{R} \longrightarrow X$ be a c-full $(1, 0)$-quasi-isometry. Then there exists a positive constant k, which just depends on c, with the following property: if σ is a non-trivial simple closed curve in $\partial B_X(p, r)$ for some $p \in X$, $r > 0$ with $\sigma \cap g(\mathbb{R}) = \emptyset$, then $L_X(\sigma) \geqslant k$.*

Proof. First of all, recall that g exists by Lemma 4.1.

Since the function

$$V(t) := \operatorname{arccosh}\left(\coth\frac{t}{2}\right) - \frac{t}{4}\coth\frac{t}{2}$$

satisfies $\lim_{t\to 0^+} V(t) = \infty$, there exists a positive constant k, which just depends on c, such that $V(t) > c$ when $0 < t < k$.

Assume first that σ surrounds a cusp. Thus, $X\backslash\sigma$ has two unbounded components, since X is quasi-isometric to $\mathbb{R}$. Since g is a continuous quasi-isometry, we conclude that $\sigma \cap g(\mathbb{R}) \neq \emptyset$, a contradiction.

Since σ is non-trivial and it does not surround a cusp, there exists a simple closed geodesic σ_0 freely homotopic to σ, and so, $L_X(\sigma_0) \leqslant L_X(\sigma) =: \ell$.

Seeking for a contradiction assume that $\ell = L_X(\sigma) < k$. Thus, Collar Lemma gives that σ_0 has a collar C_{σ_0} of width $w = \operatorname{arccosh}(\coth(L_X(\sigma_0)/2))$. If μ is a simple closed curve in ∂C_{σ_0}, then $L_X(\mu) = L_X(\sigma_0)\cosh w = L_X(\sigma_0)\coth(L_X(\sigma_0)/2)$ and therefore $L_X(\mu) \leqslant \ell\coth(\ell/2)$.

Let g_0 be a connected subset of $g(\mathbb{R}) \cap C_{\sigma_0}$. If g_0 joins the two connected components of ∂C_{σ_0}, then g_0 intersects σ_0, and since g is a proper curve (the preimage of a compact set in X is a compact in $\mathbb{R}$), it intersects every freely homotopic curve to σ_0. In particular, $g(\mathbb{R}) \cap \sigma \neq \emptyset$, a contradiction. Hence, g_0 either is empty or its two endpoints are in the same closed curve μ of ∂C_{σ_0}. Thus,

$$L_X(g_0) \leqslant \frac{1}{2}L_X(\mu) \leqslant \frac{\ell}{2}\coth\frac{\ell}{2},$$

since g (and so, g_0 when it is nonempty) is a minimizing geodesic in X.

Since g is c-full and $L_X(\sigma) < k$, if $q \in \sigma_0$, then

$$c \geqslant d_X(q, g(\mathbb{R})) \geqslant w - \frac{1}{2}L_X(g_0) \geqslant \operatorname{arccosh}\left(\coth\frac{\ell}{2}\right) - \frac{\ell}{4}\coth\frac{\ell}{2} > c,$$

a contradiction, and so, $L_X(\sigma) \geqslant k$. □

Lemma 5.2. *Let X be a non-exceptional Riemann surface quasi-isometric to $\mathbb{R}$ and let γ be a geodesic line in X admiting an arc-length parametrization which is also an $(1, 0)$-quasi-isometry from $\mathbb{R}$ to X. Then there exists a constant N with the following property: for all $p \in X$ and $r > 0$, there are at most N non-trivial simple closed curves in $\partial B_X(p, r)$ not intersecting γ.*

Proof. It is a direct consequence of the facts that $L_X(\partial B_X(p, r)) \leqslant C$ for every $p \in X$ and $r > 0$, and $L_X(\sigma) \geqslant k$ for every non-trivial simple closed curve in $\partial B_X(p, r)$ not intersecting γ, with appropriate positive constants C and k, by Lemmas 4.7 and 5.1. □

Proof of Theorem 1.3. If the fundamental group of X is finitely generated, [**23**, Proposition 4.1 and Remark 4.2] and Theorem 4.2 give that there exists a finite collection $\{Y_k\}_k$ of generalized Y-pieces, with pairwise disjoint interiors, so that $X = \cup_k Y_k$. Since this collection is finite, there exist positive constants α_1, α_2 with $\alpha_1 \leqslant L_X(\partial Y_k) \leqslant \alpha_2$ for all k. Also, the elements on the collection are Y-pieces except for, at most, two of them (since X has at most two cusps by Theorem 4.2).

Assume now that X has an infinitely generated fundamental group. By [**16**, Theorem 5.5], X can be endowed with a complete metric λ, bilipschitz to the original and with constant curvature $K = -1$, i.e., its Poincaré metric. Thus, for the time being, X will be considered to be a Riemann surface.

Since X is quasi-isometric to $\mathbb{R}$, Theorem 1.2 gives that there exists a geodesic line $\gamma \subset X$ which is an image of $\mathbb{R}$ under a $(1, 0)$-quasi-isometry.

Fix a point $p \in \gamma$. As a first step, a collection of nested geodesic domains will be constructed. To that end, an increasing sequence of positive numbers $\{r_n\}_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} r_n = \infty$ will be taken. For each r_n a particular geodesic domain G_n will be constructed from the closed ball $B(r_n) := \overline{B_X(p, r_n)}$. This construction is inspired by [**23**, Theorem 4.3].

For each $r > 0$, the boundary of $B(r)$ is a finite union of pairwise disjoint simple closed curves except for, possibly, a countable set of radii (see, e.g., [**1**, Theorem 1.2]). Let us choose an appropriate sequence $\{r_n\}$ by induction. As usual, we say that a closed curve in X is non-trivial if it is not homotopic to a point. Since X is of infinite type, r_1 can be chosen so that $B(r_1)$ contains at least two non-trivial curves in X which are not freely homotopic in X, and such that $\partial B(r_1)$ is a finite union of pairwise disjoint simple closed curves. Assume now that we have defined r_n and let us choose r_{n+1}. Let $\{\sigma_j^n\}_{j\in I_n}$ be the set of simple closed non-trivial curves in $\partial B(r_n)$. If we define $I_n^1 := \{j \in I_n : \sigma_j^n \cap \gamma \neq \emptyset\}$, we have $\sharp I_n^1 \leqslant 2$ since γ is a geodesic line. Thus, $\sharp I_n \leqslant N + 2$, where N is the constant given by Lemma 5.2, which is independent on n.

In order to construct the geodesic domain G_n, either a minimizing geodesic γ_j^n or the empty set will be related to each simple closed non-trivial curve $\sigma_j^n \subset \partial B(r_n)$. Since each σ_j^n is a simple closed curve, by [**14**] (see Theorem 2.1 and the comments after it) the minimizing geodesic γ_j^n freely homotopic to it (if it exists), will also be a simple curve. It will be said that σ_j^n is *essential* if for every $i \in I_n$, with $i \neq j$, one has $\sigma_i^n \notin [\sigma_j^n]$ (i.e., σ_i^n and σ_j^n are not freely homotopic).

There are two possibilities:

(i) If σ_j^n is essential, then γ_j^n is the minimizing simple closed geodesic in $[\sigma_j^n]$.

(ii) If either σ_j^n bounds a cusp in X or σ_j^n is not essential, then $\gamma_j^n := \emptyset$.

Since $\{\sigma_j^n\}_j$ are disjoint simple closed curves, the minimizing geodesics $\{\gamma_j^n\}_j$ are also disjoint [**25**, p.405].

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Let us define G_n as the geodesic domain whose boundary is $\cup_{j\in I_n}\gamma_j^n$. One says that G_n is the geodesic domain associated to $B(r_n)$. Note that the number of simple closed geodesics in ∂G_n is at most $\sharp I_n \leqslant N+2$.

Set $r_n' := \inf\{t > r_n : \partial B(t)$ contains at least one non-simple closed curve$\}$. Set $r_{n+1} := r_n' + \varepsilon$ for some $\varepsilon > 0$ small enough so that $\partial B(r_{n+1})$ is a finite union of pairwise disjoint simple closed curves, and all components of the interior of $B(r_{n+1})\backslash B(r_n')$ are doubly connected.

Note that, by construction, $G_n \subset G_{n+1}$.

Now, in order to describe the Y-pieces let us consider the blocks $A_n := \overline{G_n\backslash G_{n-1}}$ for every $n \geqslant 2$. As it was observed above, the number of boundary curves of every A_n is, at most, $2N+4$.

We claim that the genus of A_n is, at most, $[N/2]$, i.e., the lower integer part of $N/2$. By the choice of r_n, the only way G_n could gain genus with respect to G_{n-1} is in the situation when there exist i, $j \in I_n$, with $i \neq j$, such that $\sigma_i^n \in [\sigma_j^n]$. Recall that in this case, by definition, $\gamma_i^n = \gamma_j^n = \emptyset$ and A_n contains a *handle* which is topologically a Y-piece with two geodesic boundary curves (with the same length) which have been identified. Therefore, each pair of boundary curves in a ball could add, at most, one genus.

In what follows, an *outer loop* in A_n is a simple closed geodesic in ∂A_n, and an *inner loop* one that is not contained in ∂A_n.

We have seen that, in each A_n, there exist at most $2N+4$ outer loops, and their lengths are bounded above by C, where C is the constant of Lemma 4.7. In each A_n, by [**5**, Theorem 2], there exists a finite number of disjoint inner loops, that decompose A_n into a finite union of generalized Y-pieces, whose length is bounded above by C', where C' is a constant which only depends on C and on the number of outer loops, cusps and genus of A_n. Hence, C' just depends on C and N (in particular, it does not depend on $n \geqslant 2$). By the argument at the beginning of this proof, G_1 is a finite union of generalized Y-pieces, and so, the length of their boundaries is bounded above by some constant β. Let us define $C'' := \max\{C', \beta\}$.

Since X does not have funnels or half-planes by Theorem 4.2, the argument in the proof of [**1**, Theorem 1.2] gives that the Riemann surface X can be decomposed into a countable union of such pieces, that is, $X = \cup_{k\in\mathbb{N}}\widetilde{Y}_k$ where $L_{(X,\lambda)}(\partial\widetilde{Y}_k) \leqslant 3C''$ for every $k \in \mathbb{N}$; furthermore, the elements in that union are Y-pieces except for, at most, two of them (since X has at most two cusps by Theorem 4.2).

Let us denote by ρ the original Riemannian metric on X with pinched negative curvature. Also, for the sake of simplicity, for any curve $\eta \subset X$, let $L_\rho(\eta) := L_{(X,\rho)}(\eta)$ and $L_\lambda(\eta) := L_{(X,\lambda)}(\eta)$. Since λ and ρ are bilipschitz metrics on X, there exists a positive constant C_1 such that $C_1^{-1} \leqslant L_\rho(\eta)/L_\lambda(\eta) \leqslant C_1$ for any curve $\eta \subset X$.

To finish the proof, a new generalized Y-piece, Y_k, will be associated to each $\widetilde{Y}_k$ as follows. Let $\widetilde{\eta}_i^k$ for $i = 1, 2, 3$ be the boundary geodesic curves of $\widetilde{Y}_k$, and η_i^k for $i = 1, 2, 3$ the simple closed geodesic in (X, ρ) such that $\eta_i^k \in [\widetilde{\eta}_i^k]$. Since $\{\widetilde{\eta}_i^k\}_{i,k}$ are disjoint simple closed curves, $\{\eta_i^k\}_{i,k}$ are also disjoint simple closed curves (see [**14**] and [**23**, Theorem 3.11]). Note that

$$L_\rho(\eta_i^k) \leqslant L_\rho(\widetilde{\eta}_i^k) \leqslant C_1 L_\lambda(\widetilde{\eta}_i^k) \leqslant C_1 C''.$$

Next, define Y_k as the generalized Y-piece having η_i^k for $i = 1, 2, 3$ as its boundary curves (note that it is possible to have $\eta_i^k = \widetilde{\eta}_i^k = \emptyset$ for some i, if $\widetilde{Y}_k$ contains a cusp). By construction, $\{Y_k\}_{k\in\mathbb{N}}$ is a countable collection of generalized Y-pieces in (X, ρ) with pairwise disjoint interiors and the elements on the collection are Y-pieces except for, at most, two of them.

Since Theorem 4.2 gives that (X, ρ) does not contains funnels or half-planes, the argument in the proof of [**23**, Theorem 4.3] gives that X can be decomposed into a countable union of such pieces, that is, $X = \cup_{k\in\mathbb{N}} Y_k$, where $L_\rho(\partial Y_k) \leqslant 3C_1C'' =: \alpha_2$ for every $k \in \mathbb{N}$.

Seeking for a contradiction assume that $\inf_k L_\rho(\partial Y_k) = 0$. Since X is quasi-isometric to $\mathbb{R}$, X has exactly two ends and so, it is not possible to have arbitrarily short curves in ∂Y_k with every connected component of ∂Y_k disconnecting X. Also, if X has arbitrarily short geodesics $\gamma_k \subset \partial Y_k$ with $X\setminus\gamma_k$ connected, then their large collars create 'arbitrarily large genus' which obstruct a quasi-isometry to $\mathbb{R}$. Hence, there exists a positive constant α_1 such that $\alpha_1 \leqslant L_\rho(\partial Y_k)$ for every $k \in \mathbb{N}$. □

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