Can. J. Math., Vol. XXXIX, No. 4, 1987, pp. 938-955

DECOMPOSABLE FREE LOOP SPACES

J. AGUADÉ

In this paper we study the spaces X having the property that the space of free loops on X is equivalent in some sense to the product of X by the space of based loops on X. We denote by ΛX the space of all continuous maps from S^1 to X, with the compact-open topology. ΩX denotes, as usual, the loop space of X, i.e., the subspace of ΛX formed by the maps from S^1 to X which map 1 to the base point of X.

If G is a topological group then every loop on G can be translated to the base point of G and the space of free loops ΛG is homeomorphic to $G \times \Omega G$. More generally, any H-space has this property up to homotopy. Our purpose is to study from a homotopy point of view the spaces X for which there is a homotopy equivalence between ΛX and $X \times \Omega X$ which is compatible with the inclusion $\Omega X \subset \Lambda X$ and the evaluation map $\Lambda X \to X$. We call these spaces *T-spaces*, where T stands for translation. We develop the theory of T-spaces in a way which is reminiscent of the homotopy theory of H-spaces as in Stasheff's monograph (**[12]**).

T-spaces are defined in Section 1. In Section 2 we introduce the concept of a T-map between T-spaces, which is useful in order to construct examples of T-spaces. In Section 3 we characterize the rational homotopy type of T-spaces in the same well-known way as it is done for H-spaces. In Section 4 we define T_n-spaces for $n \geqq 1$ as intermediate stages between H-spaces and T-spaces. In the last section we study the existence of T-structures on spaces with polynomial mod p cohomology and we prove that, for any odd prime p, any mod p T-space with polynomial mod p cohomology is mod p equivalent to the classifying space of a torus. This generalizes a well-known result of Hubbuck (**[7]**).

Throughout this paper, space means space of the homotopy type of a CW complex of finite type. We assume also that spaces have a non-degenerate base point. According to a well-known result of Milnor (**[10]**) if X is a space of this kind, then ΛX and ΩX have the homotopy type of CW complexes.

I want to thank A. Zabrodsky for some conversations on the subject of this paper.

1. *T*-spaces. Let X be a 0-connected space and let us consider the fibration

---

Received September 16, 1985 and in revised form June 23, 1986.

$$\Omega X \to \Lambda X \xrightarrow{p} X$$

where p is the evaluation at 1. As in **[11]**, observe that this fibration is the pullback of the fibration

$$\Omega X \to X^I \to X \times X$$

by the diagonal map $\Delta: X \to X \times X$.

Definition. We say that X is a T-space if the fibration

$$\Omega X \to \Lambda X \to X$$

is trivial, in the sense of fibre homotopy type.

With our definition of space, the couple $(\Lambda X, \Omega X)$ has the same homotopy type as a CW pair, but more is true:

PROPOSITION 1.1. *The inclusion $i:\Omega X \to \Lambda X$ is a cofibration.*

Proof. By a result of Strøm (**[13]**) we have to show that there are maps

$$u:\Lambda X \to \mathbf{R}^+$$

$$\Phi:\Lambda X \times I \to \Lambda X,$$

such that:

i) $u|\Omega X = 0;$

ii) $\Phi(\omega, s) = \omega \quad \text{if } s = 0 \text{ or } \omega \in \Omega X;$

iii) $\Phi(\omega, s) \in \Omega X \quad \text{if } s > u(\omega).$

Since we assume that the inclusion of the base point into X is a cofibration, we have maps

$$u:X \to \mathbf{R}^+$$

$$\varphi:X \times I \to X$$

which satisfy properties analogous to i) ii) iii) above. Moreover, since $*$ is closed in X, it follows easily that $* = u^{-1}(0)$. Let us define $u:\Lambda X \to \mathbf{R}^+$ as the composition

$$\Lambda X \xrightarrow{p} X \xrightarrow{u} \mathbf{R}^+.$$

Let us consider also the map $g:\Lambda X \to \mathbf{R}^+$, $g = \min(u, 1/3)$. We define the map

$$\Phi:\Lambda X \times I \to \Lambda X$$

by the following rather complicated formula:

$$\Phi(\omega, s)(t) = \begin{cases} \omega(t), & \text{if } g(\omega) = 0, \\ \varphi(\omega(0), s - (t/g(\omega))), & 0 \leqq t \leqq sg(\omega), g(\omega) \neq 0, \\ \omega((t - sg(\omega))/(1 - 2sg(\omega))), & sg(\omega) \leqq t \leqq 1 - sg(\omega), g(\omega) \neq 0, \\ \varphi(\omega(0), s - (1 - t)/g(\omega)), & 1 - sg(\omega) \leqq t \leqq 1, g(\omega) \neq 0. \end{cases}$$

One can check that Φ is well-defined, continuous and has the desired properties.

The homotopy extension property of the pair $(\Lambda X, \Omega X)$ can be used to give some equivalent definitions of a T-space.

PROPOSITION 1.2. *The following conditions are equivalent*:
i) *X is a T-space*;
ii) *ΩX is a retract of ΛX*;
iii) *there is a map $r:\Lambda X \to \Omega X$ such that $ri \simeq \mathrm{id}$, where $i:\Omega X \to \Lambda X$ is the inclusion.*

Proof. The equivalence between i) and iii) follows from the work of Dold on fibre homotopy type (**[5]**). The equivalence between ii) and iii) is an immediate consequence of Proposition 1.1.

Remarks. 1. If X is an H-space, it is easy to see that X is a T-space (see [**15**], p. 10). If $m:X \times X \to X$ is a multiplication with strict unit, one can define a homotopy equivalence

$$h:X \times \Omega X \to \Lambda X$$

by

$$h(x, \omega)(t) = m(x, \omega(t)).$$

Since h commutes with the projections over X, the results of Dold (**[5]**) imply that h is a fibre homotopy equivalence and so the fibration

$$\Omega X \to \Lambda X \to X$$

is fibre homotopy trivial.

2. There are T-spaces which are not H-spaces. A specific example will be constructed in the next section. However, it is not known if there is any finite T-space which fails to be an H-space.

3. Obviously, a necessary condition for being a T-space is that

$$H^*(\Lambda X) \approx H^*(X \times \Omega X).$$

It is not difficult to show that this is not sufficient. For instance, since S^{2n+1} is a mod p H-space for all n and for any odd prime p, we have

$$H^*(\Lambda S^{2n+1}; \mathbf{F}_p) \approx H^*(S^{2n+1} \times \Omega S^{2n+1}; \mathbf{F}_p), \quad p \text{ odd}.$$

https://doi.org/10.4153/CJM-1987-047-9 Published online by Cambridge University Press

A straightforward computation shows that also

$$H^*(\Lambda S^{2n+1}) \approx H^*(S^{2n+1} \times \Omega S^{2n+1}).$$

However, it is proven in [2] that only S^1, S^3 and S^7 are T-spaces.

A T-structure on a space X will be a retraction

$$r:\Lambda X \to \Omega X.$$

We could include the T-structure in the definition of T-space by defining a T-space as a couple (X, r). It is clear that a T-space will in general admit many different T-structures. The set of homotopy classes of T-structures on a space X can be identified to the coset of the group $[X \times \Omega X, \Omega X]$ formed by those elements represented by maps

$$f:X \times \Omega X \to \Omega X$$

such that $f|\Omega X \simeq \mathrm{id}$. This group can be computed in some cases. For example, an Eilenberg-MacLane space $K(G, n)$, $n > 1$, has up to homotopy one T-structure.

2. T-maps. Let $f:X \to Y$ be a map and assume that X, Y are T-spaces with retractions

$$r_X:\Lambda X \to \Omega X, \quad r_Y:\Lambda Y \to \Omega Y,$$

respectively. We say that f is a *T-map* if the following diagram is homotopy commutative:

$$\begin{tikzcd}
\Omega X \arrow[r, "\Omega f"] & \Omega Y \\
\Lambda X \arrow[u, "r_X"] \arrow[r, "\Lambda f"] & \Lambda Y \arrow[u, "r_Y"']
\end{tikzcd}$$

Since ΩY is a homotopy associative H-space, the homotopy set $[\Lambda X, \Omega Y]$ is a group and we can define the *deviation* of a map $f:X \to Y$ as the element

$$\delta(f) = [(\Omega f) r_X] \quad [r_Y(\Lambda f)]^{-1} \in [\Lambda X, \Omega Y].$$

Then $\delta(f) = 1$ if and only if f is a T-map. Since ΩX is a retract of ΛX, we have an exact sequence of groups

$$[\Lambda X/\Omega X, \Omega Y] \rightarrowtail [\Lambda X, \Omega Y] \twoheadrightarrow [\Omega X, \Omega Y].$$

Since $\delta(f)$ is trivial when restricted to ΩX, we can consider the deviation of f as an element in the group $[\Lambda X/\Omega X, \Omega Y]$.

T-maps are useful because they can be used to construct T-spaces as in the following result.

PROPOSITION 1.2. *Let $f:X \to Y$ be a T-map and let P_f be the homotopy fibre of f. Then P_f admits a T-structure such that the map $p:P_f \to X$ is a T-map.*

Proof. We have a commutative diagram

$$\begin{tikzcd}
\Omega P_f \arrow[d] & \approx & P_{\Omega f} \arrow[r] \arrow[d, "i"] & \Omega X \arrow[r, "\Omega f"] \arrow[d, "j"] & \Omega Y \arrow[d, "k"] \\
\Lambda P_f & \approx & P_{\Lambda f} \arrow[r] & \Lambda X \arrow[r, "\Lambda f"] & \Lambda Y
\end{tikzcd}$$

There are retractions $r_1:\Lambda X \to \Omega X$ and $r_2:\Lambda Y \to \Omega Y$ such that

$$r_1 j = \text{id}, \quad r_2 k = \text{id}.$$

We want to construct a map

$$R:P_{\Lambda f} \to P_{\Omega f}$$

such that $Ri \simeq \text{id}$. Since f is a T-map, there is a homotopy

$$h:\Lambda X \times I \to \Omega Y,$$

$$h(\omega, 1) = (\Omega f)r_1(\omega),$$

$$h(\omega, 0) = r_2(\Lambda f)(\omega).$$

If $\omega \in \Omega X$, we have

$$h(\omega, 1) = h(\omega, 0) = (\Omega f)(\omega).$$

We can now use a theorem of James (**[8]**) and assume, without loss of generality, that

$$h(\omega, t) = (\Omega f)(\omega) \quad \text{for all } (\omega, t) \in \Omega X \times I.$$

A point in $P_{\Lambda f}$ is a couple (ω, Ω) where

$$\omega \in \Lambda X, \quad \Omega:I \to \Lambda Y, \quad \Omega(0) = *, \quad \Omega(1) = (\Lambda f)(\omega).$$

We define $R:P_{\Lambda f} \to P_{\Omega f}$ by

$$R(\omega, \Omega) = (r_1(\omega), r_2 \circ \Omega * h(\omega, -)).$$

(Here $*$ denotes the usual composition of paths $I \to \Omega Y$.) One sees easily that R is well-defined. We have also

$$Ri(\omega, \Omega) = R(j\omega, k\Omega) = (r_1 j(\omega), r_2 k(\omega) * h(j\omega, -)) = (\omega, \Omega * \epsilon)$$

where $\epsilon:I \to \Lambda Y$ is the constant path

$$\epsilon(t) = (\Omega f)(\omega).$$

It is clear also that $Ri \simeq \text{id}$. Hence,

$$R:P_{\Lambda f} \rightarrow P_{\Omega f}$$

gives a T-structure on P_f and moreover $p:P_f \rightarrow X$ is a T-map.

Let us consider the special case in which $Y = K(k, n)$ where k is a field and $n > 1$. Then, a map $f:X \rightarrow Y$ is given by a cohomology class $x \in H^n(X; k)$ and the deviation of f is a class

$$\delta(x) \in H^{n-1}(\Lambda X, \Omega X; k).$$

If X is a simply connected H-space then the deviation of a map $f:X \rightarrow K(k, n)$ can be computed in a rather explicit way. Let $m:X \times X \rightarrow X$ be the multiplication of X. It induces a homomorphism

$$m^*:H^*(X; k) \rightarrow H^*(X; k) \otimes H^*(X; k)$$

such that

$$m^*(z) = 1 \otimes z + z \otimes 1 + \sum c_j(z) \otimes d_j(z), \quad \deg c_j, d_j > 0.$$

Using m we can define a homotopy equivalence

$$h:X \times \Omega X \rightarrow \Omega X$$

$$h(x, \omega)(t) = m(x, \omega(t)).$$

Let $r:\Lambda X \rightarrow \Omega X$ be a T-structure on X; not necessarily the one which comes from the H-structure of X. The composition

$$rh:X \times \Omega X \rightarrow \Omega X$$

gives a homomorphism

$$(rh)^*:H^*(\Omega X; k) \rightarrow H^*(X; k) \otimes H^*(\Omega X; k)$$

and one sees immediately that

$$(rh)^*(y) = 1 \otimes y + \sum a_i(y) \otimes b_i(y), \quad \deg a_i > 0.$$

If the T-structure that we take on X is obtained from the H-structure, then $(rh)^*$ is just the canonical map $y \mapsto 1 \otimes y$. Let

$$\omega:H^i(X; k) \rightarrow H^{i-1}(\Omega X; k)$$

be the suspension.

PROPOSITION 2.2. *In the situation above we have*

$$h^*\delta(x) = \sum a_i(\omega(x)) \otimes b_i(\omega(x)) - \sum c_j(x) \otimes \omega(d_j(x)).$$

Proof. The only point which needs to be discussed is the computation of the composition

$$X \times \Omega X \xrightarrow{h} \Lambda X \xrightarrow{\Lambda f} \Lambda K(k, n) \xrightarrow{r'} \Omega K(k, n)$$

where r' can be defined in the following way. We take a model for $K(k, n)$ which is a topological group. Then we have a map

$$\alpha:K(k, n) \times K(k, n) \to K(k, n)$$

$$\alpha(x, y) = yx^{-1}$$

and we define

$$r'(\omega)(t) = \alpha(\omega(0), \omega(t)).$$

r' is a T-structure on $K(k, n)$. The following diagram is commutative

$$\begin{tikzcd}
X \times S^1 \times \Omega X \arrow[r] \arrow[d, "\Delta \times e"] & \Sigma(X \times \Omega X) \arrow[r, "g"] & K(k, n) \\
X \times X \times X \arrow[d, "1 \times m"] & & \\
X \times X \arrow[rr, "f \times f"] & & K(k, n) \times K(k, n) \arrow[uu, "\alpha"']
\end{tikzcd}$$

where g is the adjoint of $r'(\Lambda f)h$ and $e:S^1 \times \Omega X \to X$ is the evaluation map. The proposition follows if we notice that α is classified by $1 \otimes \iota - \iota \otimes 1$, where

$$\iota \in H^n(K, k, n); k)$$

is the fundamental class.

Example. Let $f:K(\mathbf{Z}, 2) \to K(\mathbf{Z}_2, 12)$ be classified by

$$\iota^6 \in H^{12}(K(\mathbf{Z}, 2); \mathbf{F}_2).$$

We have

$$m^*(\iota^6) = \iota^6 \otimes 1 + \iota^4 \otimes \iota^2 + \iota^2 \otimes \iota^4 + 1 \otimes \iota^6.$$

On the other side, $K(\mathbf{Z}, 2)$ has a unique T-structure r and

$$(rh)^*(y) = 1 \otimes y \quad \text{for all } y \in H^*(\Omega K(\mathbf{Z}, 2); \mathbf{F}_2).$$

Since $\omega(\iota^2) = \omega(\iota^4) = 0$, we see that the deviation is zero and so the fibre P_f of f is a T-space. Since ι^6 is not primitive in $H^*(K(\mathbf{Z}, 2); \mathbf{F}_2)$, P_f is not an H-space and we have an example of a simply connected T-space which is not an H-space.

3. Postnikov systems and rational T-spaces. It is well-known that the Postnikov invariants of an H-space are H-maps. In this section we will prove a similar result for T-spaces and T-maps and we will determine the rational homotopy type of T-spaces.

Let X be a simply connected space. A Postnikov system for X consists of a sequence of fibrations

$$\ldots \to X_{n+1} \xrightarrow{p_{n+1}} X_n \xrightarrow{p_n} X_{n-1} \to \ldots \to X_2 \to X_1 = *$$

and cohomology classes

$$k^n \in H^{n+1}(X_{n-1}; \pi_n X),$$

together with maps $j_n : X \to X_n$ such that

i) $p_n j_n = j_{n-1}$;
ii) $\pi_i X_n = 0$ for $i > n$;
iii) $j_{n*} : \pi_i X \approx \pi_i X_n$ for $i < n + 1$;
iv) p_n is induced by

$$k^n : X_{n-1} \to K(\pi_n X, n + 1).$$

Assume we have a Postnikov system for X.

PROPOSITION 3.1. *If X is a T-space and $n \geqq 1$, then there exists a T-structure on X_n such that $j_n : X \to X_n$ is a T-map.*

Proof. Let $r : \Lambda X \to \Omega X$ be the T-structure of X and let us consider the diagram

$$\begin{tikzcd}
\Lambda X_n \arrow[drr, dashed, "r'"] & & \\
\Lambda X \vee \Omega X_n \arrow[u, "\Lambda j_n \vee i_n"] \arrow[rr, "(\Omega j_n) r \vee 1"] & & \Omega X_n
\end{tikzcd}$$

where $i_n : \Omega X_n \to \Lambda X_n$ is the inclusion. It is obvious that the proposition will follow from the existence of the extension r'. The obstructions to this extension problem are classes in

$$H^i(\Lambda X_n, \Lambda X \vee \Omega X_n; \pi_i X_n), \quad i \leqq n.$$

Let us consider the diagram

$$\begin{tikzcd}
\Lambda X \arrow[r, "\Lambda j_n"] \arrow[drr, dashed, "r''"] & \Lambda X_n & \\
\Lambda X \vee \Omega X \arrow[u, "1 \vee i"] \arrow[r, "1 \vee \Omega j_n"] & \Lambda X \vee \Omega X_n \arrow[u] \arrow[r] & \Omega X_n
\end{tikzcd}$$

The map of pairs $(\Lambda j_n, 1 \vee \Omega j_n)$ maps the obstructions to the existence of r'

https://doi.org/10.4153/CJM-1987-047-9 Published online by Cambridge University Press

to the obstructions to the existence of r'' which are zero because $r'' = (\Omega j_n)r$ makes the diagram commutative. Hence, the proof is complete if we show that

$$(\Lambda j_n, 1 \vee \Omega j_n)^*: H^i(\Lambda X_n, \Lambda X \vee \Omega X_n; \pi_i X_n) \rightarrow H^i(\Lambda X, \Lambda X \vee \Omega X; \pi_i X_n)$$

is an isomorphism for $i < n$ and a monomorphism for $i = n$. This is a straightforward consequence of the fact that $j_n: X \rightarrow X_n$ is an $(n + 1)$-equivalence.

THEOREM 3.2. *If X is a T-space, then the spaces of any Postnikov system for X are T-spaces and the Postnikov invariants are T-maps.*

Proof. Let us consider a stage of some Postnikov system for X:

$$X \xrightarrow{j_n} X_n \xrightarrow{k^{n+1}} K(G, n + 2).$$

The above proposition shows that X_n is a T-space and j_n is a T-map. We only need to show that the deviation of k^{n+1} is zero. Since j_n is a T-map, the class

$$(\Lambda j_n, \Omega j_n)^* \delta(k^{n+1}) \in H^{n+1}(\Lambda X, \Omega X; G)$$

is the deviation of $k^{n+1} j_n$ which is obviously trivial. Hence, it suffices to show that

$$(\Lambda j_n, \Omega j_n)^*: H^{n+1}(\Lambda X_n, \Omega X_n; G) \rightarrow H^{n+1}(\Lambda X, \Omega X; G)$$

is a monomorphism. Since j_n is a T-map, this is equivalent to showing that

$$(j_n \times \Omega j_n, \Omega j_n)^*: H^{n+1}(X_n \times \Omega X_n, \Omega X_n; G) \rightarrow H^{n+1}(X \times \Omega X, \Omega X; G)$$

is a monomorphism. This follows easily from the fact that $j_n: X \rightarrow X_n$ is an $(n + 1)$-equivalence and X and X_n are simply connected, by using the naturality of the Künneth sequences of

$$(X_n \times \Omega X_n, \Omega X_n) = (X_n, *) \times (\Omega X_n, \emptyset) \quad \text{and}$$
$$(X \times \Omega X, \Omega X) = (X, *) \times (\Omega X, \emptyset),$$

respectively.

We want now to determine the rational homotopy type of the T-spaces. This problem is completely solved in the case of H-spaces: a rational H-space is a product of Eilenberg-MacLane spaces. We will prove that in the rational category any T-space is an H-space. This result was also obtained in [**14**] by Vigué-Poirrier using Sullivan's minimal models. From now on, we assume that X is a simply connected space whose rational

https://doi.org/10.4153/CJM-1987-047-9 Published online by Cambridge University Press

homotopy groups are finite dimensional $\mathbf{Q}$-vector spaces.

We say that X is a rational T-space if the rationalisation of X is a T-space.

THEOREM 3.3. *X is a rational T-space if and only if X has the same rational homotopy type as a product of Eilenberg-MacLane spaces.*

Proof. We assume that all spaces are localized at zero. Since a product of Eilenberg-MacLane spaces is an H-space and also a T-space, we only need to prove the converse. Let X be a T-space and let us consider a Postnikov system for X. We know from Theorem 3.2 that X_n, $n \geqq 1$ is a T-space and k^n, $n \geqq 2$ are T-maps. We prove by induction that each space X_n is a product of Eilenberg-MacLane spaces. This is obviously true for X_1 and X_2 and the next proposition provides the inductive step.

PROPOSITION 3.4. *Let X be a rational space and an H-space with $\pi_i X = 0$, $i > N > 1$, and let*

$$f:X \to K(V, N + 2)$$

be a map, where V is a finite dimensional $\mathbf{Q}$-vector space. If f is a T-map with respect to some T-structure of X then the fibre of f is a rational H-space.

Proof. We have

$$\begin{aligned} H^*(X; V) &\approx H^*(X; \mathbf{Q}) \otimes V \\ &\approx (\mathbf{Q}[x_1, \dots, x_n] \otimes E(y_1, \dots, y_m)) \otimes V. \end{aligned}$$

Let $x \in H^{N+2}(X; V)$ be the class corresponding to the map f. We will show that $x = 0$ and so the fibre of f is homotopy equivalent to $X \times K(V, N + 1)$. Since we assume $\pi_i X = 0$ for $i > N$, x can be written as

$$x = z_1 \otimes v_1 + \dots + z_t \otimes v_t,$$

where $z_i \in H^*(X; \mathbf{Q})$, $i = 1, \dots, t$ are decomposable and $v_1, \dots, v_t$ is a basis of V as a $\mathbf{Q}$-vector space.

Let

$$r:X \times \Omega X \to \Omega X$$

be the composition of the T-structure of X and the homotopy equivalence $X \times \Omega X \simeq \Lambda X$ provided by any H-structure on X such that the generators $x_1, \dots, x_n, y_1, \dots, y_m$ are primitive. Since f is a T-map, $\delta(f) = 0$. On the other side, it is clear that $\omega(x) = 0$ in $H^*(\Omega X; V)$. Using Proposition 2.2 to compute $\delta(f)$, we obtain

$$(*) \qquad \sum a_{ij} \otimes \omega(b_{ij}) \otimes v_i = 0$$

where a_{ij}, b_{ij} are such that

https://doi.org/10.4153/CJM-1987-047-9 Published online by Cambridge University Press

$$m^*(z_i) = z_i \otimes 1 + 1 \otimes z_i + \sum a_{ij} \otimes b_{ij}, \deg a_{ij}, b_{ij} > 0.$$

But it is clear that since z_i, $i = 1, \ldots, t$, is decomposable, the equation (*) can only hold if $z_1 = \ldots = z_t = 0$.

4. T_n-spaces. We have seen that any H-space is a T-space and also that the converse is not true. The relationship between H-spaces and T-spaces resembles in some way the relationship between loop spaces and H-spaces. In order to get a better understanding of the distinction between H-spaces and loop spaces, Stasheff introduced a sequence of intermediate properties which led to the definition of A_n-spaces (see [**12**], p. 50). In this section we do a similar construction for T-spaces.

For any X, the space ΩX has the homotopy type of an associative H-space and so we can consider X as filtered by the projective spaces of ΩX.

$$\Sigma\Omega X = (\Omega X)P^1 \to (\Omega X)P^2 \to \ldots \to (\Omega X)P^m \to \ldots \to X.$$

These spaces are defined in the following way: If Y is an H-space, then we set $YP^1 = \Sigma Y$. The projective plane of Y, YP^2 is defined as the mapping cone of the Hopf fibration $Y * Y \to \Sigma Y$. If Y is associative, this fibration can be extended inductively to fibrations

$$Y * \overset{n}{\ldots} * Y \to YP^{n-1}$$

and we define YP^n as the mapping cone of this map. In the limit, we obtain BY, the classifying space of the associative H-space Y. We refer to [**12**] for more details on this construction.

Let

$$i_m:(\Omega X)P^m \to X \quad \text{and} \quad j_m:\Sigma\Omega X \to (\Omega X)P^m$$

be the canonical maps.

Definition. We say that X is a T_n-space, $\infty \geqq n \geqq 1$, if

$$\Sigma\Omega X \vee X \xrightarrow{i_1 \vee 1} X$$

can be extended up to homotopy to $(\Omega X)P^m \times X$.

The most immediate consequences of this definition are contained in the following proposition.

PROPOSITION 4.1. i) *Any H-space is a T_∞-space;*

ii) *any T_∞-space is an H-space;*

iii) *any T_1-space is a T-space;*

iv) *any T-space is a T_1-space;*

v) *any T_n-space is a T_m-space, $n \geqq m$;*

vi) *if G is a discrete group, then BG is a T-space if and only if G is* *abelian*;

vii) *if* X *is a* T*-space, then all Whitehead products vanish on* X;

viii) *if* X *is a* T*-space, then* ΩX *is a homotopy commutative* H*-space*;

ix) *the fundamental group of a* T*-space is abelian.*

Proof. i) and v) are trivial. vii) follows easily from iv). vi) and ix) follow from vii). viii) follows from a result of James-Thomas ([**9**]) who proved that a loop space Y is homotopy commutative if and only if the map $\Sigma Y \vee \Sigma Y \to BY$ can be extended to $\Sigma Y \times \Sigma Y$. iii) follows from the fact that if

$$f:\Sigma\Omega X \times X \to X$$

is an extension of $i_1 \vee 1$, then the adjoint to the map

$$S^1 \times \Omega X \times X \to \Sigma\Omega X \times X \xrightarrow{f} X$$

is a fibre homotopy equivalence between the fibration

$$\Omega X \to \Lambda X \to X$$

and the trivial fibration.

Let us prove ii). If X is a T_∞-space, then we have a homotopy commutative diagram

$$\begin{tikzcd}
X \times X \arrow[r, "m"] & X \\
\Sigma\Omega X \vee X \arrow[u] \arrow[ur, "i_1 \vee 1"'] &
\end{tikzcd}$$

Let $f:X \to X$ be the map given by $f(x) = m(x, *)$. Then $fi_1 \sim i_1$. By passing to the adjoint maps, we see that $\Omega f \sim 1$ and so f is a homotopy equivalence. If g is a homotopy inverse of f, then $\bar{m} = m(g \times 1)$ is a multiplication on X with two-sided homotopy unit.

It remains only to prove iv). If X is a T-space, we have a map

$$h:S^1 \times X \times \Omega X \to X$$

such that

$$h(1, x, \omega) = x; \quad h(t, *, \omega) = \omega(t).$$

However, $h(t, x, *)$ could be different from $*$. We will modify h to obtain a map with this property. For each $x \in X$, let λ_x be the path on X given by

$$\lambda_x(t) = h(t, x, *).$$

Define

$$\bar{h}:S^1 \times X \times \Omega X \rightarrow X$$

$$\bar{h}(\ ,x,\omega) = h(\ ,x,\omega)^*\lambda_x^{-1}$$

(composition of paths). Then $\bar{h}$, restricted to the "fat wedge" $S^1 \mathbin{\text{\rotatebox[origin=c]{180}{$\bar{\wedge}$}}} X \mathbin{\text{\rotatebox[origin=c]{180}{$\bar{\wedge}$}}} \Omega X$ is homotopic to the map

$$g:S^1 \mathbin{\text{\rotatebox[origin=c]{180}{$\bar{\wedge}$}}} X \mathbin{\text{\rotatebox[origin=c]{180}{$\bar{\wedge}$}}} \Omega X \mathbin{\text{\rotatebox[origin=c]{180}{$\bar{\wedge}$}}} X$$

such that

$$g(t,x,*) = x$$

$$g(1,x,\omega) = x$$

$$g(t,*,\omega) = \omega(t).$$

Using the homotopy extension property, we can assume that $\bar{h} = g$ on $S^1 \mathbin{\text{\rotatebox[origin=c]{180}{$\bar{\wedge}$}}} X \mathbin{\text{\rotatebox[origin=c]{180}{$\bar{\wedge}$}}} \Omega X$. Hence $\bar{h}$ factorizes through $X \times \Sigma\Omega X$ and so X is a T_1-space.

We ask now if the conditions of being a T_n-space or a T_m-space, $n \neq m$, are really different.

Example. Let $p > 3$ be a prime and let X be the homotopy fibre of the map

$$K(\mathbf{Z}, 2) \rightarrow K(\mathbf{Z}_p, 2n), \quad (n > 9)$$

classified by

$$\iota^n \in H^{2n}(K(\mathbf{Z}, 2); \mathbf{F}_p).$$

We want to know if X is a T_m-space, for some $m \geqq 1$. Let $Y = \Omega X$ which is homotopy equivalent to

$$S^1 \times \mathrm{K}(\mathbf{Z}_p, 2n-2).$$

Let us consider the diagram

$$\begin{tikzcd}
\Sigma Y \vee X \arrow[rr, "i_1 \vee 1"] \arrow[d, "j_m \vee 1"'] & & X \arrow[d, "\pi"] & \\
YP^m \times X \arrow[rr, "g"] \arrow[urr, dashed, "f"] & & K(\mathbf{Z}, 2) \arrow[r] & K(\mathbf{Z}_p, 2n)
\end{tikzcd}$$

where g is classified by

$$(\bar{x} \otimes 1 + 1 \otimes x) \in H^2(YP^m \times X) \approx H^2(\Sigma Y \vee X),$$

where $x \in H^2(X)$ classifies π and

$$j_m^*(\bar{x}) = i_1^*(x).$$

It is easy to see that X is a T_m-space if and only if there exists a lifting f making the diagram commutative. The obstruction to the existence of f is a class in

$$H^{2n}(YP^m \times X, \Sigma Y \vee X; \mathbf{F}_p)$$

which maps to

$$(\bar{x} \otimes 1 + 1 \otimes x)^n \in H^{2n}(YP^m \times X, \mathbf{F}_p).$$

Let us recall that YP^m is obtained from ΣY by a sequence of cofibrations

$$Y * \overset{i}{\ldots} * Y \rightarrow YP^{i-1} \rightarrow YP^i.$$

Since

$$H^{2n-1}(Y * \overset{i}{\ldots} * Y; \mathbf{F}_p) = 0 \quad \text{for } i < n,$$

we conclude that

$$(j_m \vee 1)^*: H^{2n-1}(YP^m \times X; \mathbf{F}_p) \rightarrow H^{2n-1}(\Sigma Y \vee X; \mathbf{F}_p)$$

is onto if $m < n$. This shows that for $m < n$ X is a T_m-space if and only if

$$(\bar{x} \otimes 1 + 1 \otimes x)^n = 0 \quad \text{in } H^{2n}(YP^m \times X; \mathbf{F}_p).$$

The map π induces maps between the projective spaces of Y and S^1. This shows that

$$\bar{x}^{r+1} = 0 \quad \text{in } H^*(YP^r; \mathbf{F}_p).$$

On the other side, one can consider the maps

$$S^1 \xrightarrow{k} \Omega \mathbf{C}P^{n-1} \xrightarrow{\Omega h} Y$$

where k is the adjoint of the inclusion of the bottom cell and h is a lifting of the canonical map

$$\mathbf{C}P^{n-1} \rightarrow K(\mathbf{Z}, 2).$$

By a result of Stasheff ([**12**], p. 34), k is an A_{n-1}-map and so it induces a map between the projective r-spaces of S^1 and $\mathbf{C}P^{n-1}$, for $r < n$. We obtain maps

$$\mathbf{C}P^r \rightarrow (\Omega \mathbf{C}P^{n-1})P^r \rightarrow YP^r$$

for $r < n$. Hence, $\bar{x}^r \neq 0$ in $H^*(YP^r; \mathbf{F}_p)$ for $r < n$.

Let us consider now the case $n = 3p$. We have

$$(\bar{x} \otimes 1 + 1 \otimes x)^n$$

https://doi.org/10.4153/CJM-1987-047-9 Published online by Cambridge University Press

$$= \bar{x}^{3p} \otimes 1 + 3\bar{x}^{2p} \otimes x^p + 3\bar{x}^p \otimes x^{2p} + 1 \otimes x^{3p}$$

and the discussion above shows that X is a T_{p-1}-space but not a T_p-space.

PROPOSITION 4.2. *For any prime $p > 3$ there is a space X such that X is a T_{p-1}-space but not a T_p-space.*

5. T-structures on spaces with polynomial cohomology. Throughout this section, p is an odd prime and X is a simply connected space such that

$$H^*(X; \mathbf{F}_p) \approx \mathbf{F}_p[x_1, \dots, x_n], \quad \deg x_1 \leqq \dots \leqq \deg x_n.$$

According to a result of Hubbuck ([**7**]), X is not an H-space unless $\deg x_1 = \dots = \deg x_n = 2$. We will generalize this to T-spaces by proving that X cannot be a T-space unless the same condition on the degrees of the generators holds. The proof uses only primary cohomology operations and is based on the work of Adams-Wilkerson on cohomology algebras ([**1**]).

Let $Q^{r,s}$, $r > 0$, $s \geqq 0$ be the element of the mod p Steenrod algebra which in the Milnor base is written as

$$P^{\{0,\dots,0,s,0,\dots\}}$$

where the s comes in the rth place. We put $Q^r = Q^{r,1}$, following the notation of [**1**]. We also consider the operation Q^0 (not a Steenrod operation!) defined by $Q^0 x = dx$ for all x of degree $2d$. These operations have the following properties:

i) $\deg Q^{r,s} = 2s(p^r - 1)$;

ii) the Cartan formula for $Q^{r,s}$ is

$$Q^{r,s}(xy) = \sum_{i+j=s} (Q^{r,i}(x))(Q^{r,j}(y));$$

iii) if $\deg x = 2(s + 1)$ then

$$Q^{r+k}Q^{r,s}(x) = (Q^k(x))^{p^r}, \quad r > 0, s \geqq 0, k \geqq 0.$$

i) and ii) are elementary and iii) is proved in [**1**].

PROPOSITION 5.1. *If X is a T-space and r is big enough, then*

$$Q^{r,s}(x_i) \in \mathbf{F}_p[x_1, \dots, x_{n-1}, x_n^p], \quad i = 1, \dots, n, s \geqq 1.$$

Proof. Let r be such that

$$2(p^r - 1) > \deg x_1 + \dots + \deg x_n.$$

This implies that the operations $Q^{r,s}$, $s \geqq 1$ vanish on

$$H^*(\Omega X; \mathbf{F}_p) \approx E(\omega(x_1), \dots, \omega(x_n)).$$

Let x be equal to x_i for some i. We have

$$Q^{r,s}(x) = \sum \lambda_{i_1,\ldots,i_n} x_1^{i_1} \ldots x_n^{i_n}.$$

We want to prove $i_n \equiv 0(p)$. Since X is a T-space, there is a homotopy equivalence $X \times \Omega X \simeq \Lambda X$. If φ is the adjoint of this map, it is easy to see that the homomorphism induced by φ on $H^*(X; \mathbf{F}_p)$ should be as follows

$$\varphi^*(x_j) = x_j \otimes 1 + 1 \otimes u\omega(x_j) + \sum a_{ij} \otimes ub_{ij}$$

where u is the generator of $H^*(S^1; \mathbf{F}_p)$ and $\deg a_{ij} > 0$. Since x_n has maximal degree, the term of $\varphi^* Q^{r,s}(x)$ containing $u\omega(x_n)$ is

$$(*) \qquad \left(\sum \lambda_{i_1,\ldots,i_n} i_n x_1^{i_1} \ldots x_{n-1}^{i_{n-1}} x_n^{i_n - 1}\right) \otimes u\omega(x_n).$$

On the other hand, φ^* commutes with $Q^{r,s}$. Hence,

$$\begin{aligned} \varphi^* Q^{r,s}(x) &= Q^{r,s} \varphi^*(x) \\ &= Q^{r,s}(x \otimes 1 + 1 \otimes x + \sum a_i \otimes ub_i) \\ &= Q^{r,s}(x) \otimes 1 + \sum Q^{r,s}(a_i) \otimes ub_i. \end{aligned}$$

This shows that the term (*) vanishes and the proposition follows.

According to [**3**], we say that an unstable algebra A over the mod p Steenrod algebra satisfies the *Q-condition* if

$$A^p = \bigcap_{r>0} \operatorname{Ker} Q^r$$

where $A^p = \{x^p | x \in A\}$. The importance of this condition comes from the fact that the Q-condition characterizes the polynomial algebras which are isomorphic as algebras over the Steenrod algebra to some subalgebra of invariants of $H^*(BT^n; \mathbf{F}_p)$ under the action of some subgroup of $GL_n(\mathbf{F}_p)$. Here T^n is the n-dimensional torus. This is one of the main results of [**1**]. On the other hand, it is important to notice that, though it is easy to provide examples of polynomial algebras which are algebras over the Steenrod algebra and which do not satisfy the Q-condition, no example is known of such an algebra which is realizable as the cohomology algebra of some space. It has been proved ([**6**]) that such examples cannot exist, because the Q-condition, at least for polynomial algebras, is a consequence of the realizability as a cohomology ring. The following theorem generalizes a result of Hubbuck ([**7**], see also [**4**]).

A mod p T-space is a space such that its localisation at p is a T-space.

THEOREM 5.2. *Let X be a simply connected* mod p *T-space. If*

$$H^*(X; \mathbf{F}_p) \approx \mathbf{F}_p[x_1, \ldots, x_n]$$

then X is mod p *equivalent to BT^n.*

Proof. We will prove that $\deg x_1 = \ldots \deg x_n = 2$. Assume $\deg x_n > 2$ and let r be as large as required by Proposition 5.1. Since $H^*(X; \mathbf{F}_p)$ satisfies the Q-condition, [**1**] shows that the derivations $Q^0, \ldots, Q^{n-1}$ are linearly independent on $\mathbf{F}_p[x_1, \ldots, x_n]$. This implies

$$\det(Q^i(x_j))_{\substack{i=0,\ldots,n-1\\ j=1,\ldots,n}} \neq 0.$$

Hence,

$$\det((Q^i(x_j))^{p^r})_{\substack{i=0,\ldots,n-1\\ j=1,\ldots,n}} \neq 0.$$

Let s_j be such that $\deg x_j = 2(s_j + 1)$. Then

$$(Q^i(x_j))^{p^r} = Q^{r+i}Q^{r_1,s_i}(x_j) = Q^{r+i}(y_j),$$

where $y_j = Q^{r,s_i}(x_j)$. We have:

$$\det(Q^{r+i}(y_j))_{\substack{i=0,\ldots,n-1\\ j=1,\ldots,n}} \neq 0.$$

Notice now that

$$y_j \in \mathbf{F}_p[x_1, \ldots, x_{n-1}, x_n^p], \quad j = 1, \ldots, n.$$

This leads immediately to a contradiction because the derivations $Q^r, Q^{r+i}, \ldots, Q^{r+n-1}$ are linearly dependent on $\mathbf{F}_p[x_1, \ldots, x_n^p]$.

References

1. J. F. Adams and C. Wilkerson, *Finite H-spaces and algebras over the Steenrod algebra*, Ann. of Math. *111* (1980), 95-143.

2. J. Aguadé, *On the space of free loops of an odd sphere*, Publ. Mat. UAB *25* (1981), 87-90.

3. ——— *Invariants of modular representations and polynomial algebras over the Steenrod algebra*, Duke Math. J. *52* (1985), 315-327.

4. J. Aguadé and L. Smith, *On the* mod p *torus theorem of John Hubbuck*, Math. Z. *191* (1986), 325-326.

5. A. Dold, *Partitions of unity in the theory of fibrations*, Ann. of Math. *78* (1963), 223-255.

6. W. Dwyer, H. Miller, and C. Wilkerson, *Uniqueness of classifying spaces* (unpublished).

7. J. R. Hubbuck, *Finitely generated cohomology Hopf algebras*, Topology *9* (1970), 205-210.

8. I. M. James, *On H-spaces and their homotopy groups*, Quart. J. Math. (Oxford) *11* (1960), 161-179.

9. I. M. James and E. Thomas, *On homotopy-commutativity*, Ann. of Math. *76* (1962), 9-17.

10. J. Milnor, *On spaces having the homotopy type of a CW-complex*, Trans. Amer. Math. Soc. *90* (1959), 272-280.

11. L. Smith, *On the characteristic zero cohomology of the free loop space*, Amer. J. Math. *103* (1981), 887-910.

12. J. Stasheff, *H-spaces from a homotopy point of view*, Lecture Notes in Math. *161* (Springer, 1970).

13. A. Strøm, *Note on cofibrations II*, Math. Scand. *22* (1968), 130-142.

14. M. Vigué-Poirrier, *Dans le fibré de l'espace des lacets libres, la fibre n'est pas, en général, totalement non cohomologue à zero*, Math. Z. *181* (1982), 537-542.
15. A. Zabrodsky, *Hopf spaces* (North-Holland Math Studies *22*, North Holland, 1976).

Universitat Autònoma de Barcelona,
Barcelona, Spain