THE EMBEDDING THEOREMS OF MALCEV AND LAMBEK

GEORGE C. BUSH

A given semigroup $\mathfrak{A}$ is said to be embeddable in a group if there exists a group $\mathfrak{G}$ which contains a subsemigroup isomorphic to $\mathfrak{A}$.

It can easily be proved that cancellation is a necessary condition for embeddability. It can also be shown that we can adjoin an identity to a semigroup without identity in such a way that the new semigroup is embeddable if and only if the original was embeddable. Therefore, we can, without loss of generality, restrict our attention to cancellation semigroups with identity, whenever this is convenient.

If $\mathfrak{A}$ satisfies the cancellation law, then it can be proved that in order for $\mathfrak{A}$ to be embeddable it is sufficient that each pair of elements of $\mathfrak{A}$ have a common left multiple. This condition is satisfied by any commutative semigroup. In general a semigroup may or may not be embeddable in a group.

Different necessary and sufficient conditions for the embeddability of a semigroup in a group are due to Malcev **(3)** and to Lambek **(2)**.

Part 1 of this paper describes some results of Malcev, without proofs, but with sufficient details to make the proofs in Part 3 meaningful. The reader who wants more details is referred to Tamari **(5)** whose proofs, although in a more general framework, are more readily grasped than those of Malcev.

Part 2 states the necessary and sufficient conditions as given by Lambek. Instead of proving these results, we show that a closely related set of conditions (in fact, a subset) is necessary and sufficient for the embeddability of a semigroup in a group. This approach uses only algebraic concepts in place of the geometric concepts used by Lambek.

In Part 3 the sets of conditions given by Malcev and by Lambek are compared. We prove that each member of the set of Lambek conditions is a consequence of the set of Malcev conditions and conversely.

1. The Malcev conditions. Let $\mathfrak{A}$ be a semigroup with identity e. With each $x \in \mathfrak{A}$ Malcev associated two symbols, not in $\mathfrak{A}$, which he denoted by x^+ and x^-. These he called, respectively, *formal right* and *formal left elements* associated with x. Let $\mathfrak{S}$ be the set of all finite sequences, or words, consisting

Received July 31, 1961. The material of this paper is taken from the author's Ph.D. thesis at Queen's University. The author wishes to acknowledge his indebtedness to his supervisor, Professor I. Halperin. The work was carried out while the author held a studentship of the National Research Council of Canada.

of elements of $\mathfrak{A}$ and formal elements, and which begin and end with e. The following transformations on a word in $\mathfrak{S}$ are called *elementary*:

(I) Between any two adjacent elements of the word a pair xx^+ or x^-x is inserted where x is any element of $\mathfrak{A}$.

(II) A pair of adjacent elements xx^+ or x^-x is dropped out of the word provided x is not the first or last element of the word (the identity).

(III) A pair of elements of $\mathfrak{A}$ that are adjacent in the word, but neither of which is the first or last element of the word (the identity) is replaced by their product in $\mathfrak{A}$.

(IV) An element $x \in \mathfrak{A}$ appearing in the word, other than in the first or last position, is replaced by a pair of elements of $\mathfrak{A}$ whose product is x.

Let α, β be two words in $\mathfrak{S}$. We say that β is equivalent to α if it is possible to transform α into β by a finite chain of elementary transformations. This equivalence relation partitions $\mathfrak{S}$ into equivalence classes. We write (α) for the equivalence class containing α. We define the product of two classes by $(\alpha)(\beta) = (\alpha\beta)$. Under this multiplication the set of equivalence classes forms a group, called the *formal group of the semigroup* $\mathfrak{A}$. (Malcev called it the quotient group.)

A necessary and sufficient condition for the embeddability of a semigroup in a group is as follows:

LEMMA 1 (Malcev). *$\mathfrak{A}$ is embeddable in a group if and only if two elements of $\mathfrak{A}$ which are equivalent are also equal.*

A formal left (right) element is called *left* (*right*) *normal* in a chain of transformations if no transformations occur to its left (right).* A *chain of transformations* is called *normal* if each of its formal left elements is left normal and each of its formal right elements is right normal. Two chains are called *equivalent* if they have the same initial and final words. The following theorem shows that we can confine our attention to normal chains.

THEOREM 1 (Malcev). *Every chain of transformations whose initial and final words consist only of elements of $\mathfrak{A}$ is equivalent to some normal chain.*

In any chain that carries a word consisting only of elements of $\mathfrak{A}$ into another word consisting only of elements of $\mathfrak{A}$ it is clear that each formal element that is inserted must later be deleted. Each chain of transformations determines some sequence of insertions and deletions of formal left and right elements. For a normal chain this sequence of insertions and deletions has a simple form that enabled Malcev to develop a catalogue of the conditions for embeddability.

In order to develop this catalogue Malcev associated with each sequence of insertions and deletions a chain of transformations in general variables

*If the same formal element appears in more than one position in a chain of transformations we distinguish between these by an index. They are then considered as two different formal elements.

https://doi.org/10.4153/CJM-1963-006-x Published online by Cambridge University Press

which we represent* as $\bar{x}_i$, $\bar{L}_i^*$, $\bar{l}_i^*$. This new chain of transformations has the property that any normal chain with the same sequence of insertions and deletions can be obtained by replacing the $\bar{x}_i$ by elements of $\mathfrak{A}$, $\bar{L}_i^*$ by formal left elements and $\bar{l}_i^*$ by formal right elements.

This chain in the barred letters involves transformations that replace a product of two barred letters by another product of two barred letters. The set of all equations formed by equating these two products is called the *normal system of equations* of the normal chain. The first and last words in the chain of barred letters are products of two barred letters. The equation obtained by equating these two products is called the *completing equation* of the normal chain.

Malcev's embedding theorem is as follows:

THEOREM 2 (Malcev). *In order that a semigroup with identity be embeddable in a group it is necessary and sufficient that the following condition be fulfilled: If certain elements of the semigroup satisfy some normal system of equations, then the corresponding elements must satisfy the completing equation of this system.*

2. The Lambek conditions

2.1. THE POLYHEDRAL CONDITION. Lambek's **(2)** development of necessary and sufficient conditions for the embeddability of a semigroup in a group is in terms of polyhedra on the surface of the sphere.

A given semigroup $\mathfrak{A}$ is said to satisfy *the polyhedral condition* if the following statement is true. If elements of $\mathfrak{A}$ are assigned to all sides (each edge has two sides, one belonging to each of the adjoining faces) and angles of a polyhedron on a sphere (see Figure 1) such that to each half-edge there corresponds an equation $xa = yb$ where x and y have been assigned to the sides and a and b have been assigned to the corresponding angles of the half-edge, then these $2E$ equations, where E is the number of edges, are interdependent, that is, any one of them can be derived from the totality of all others.

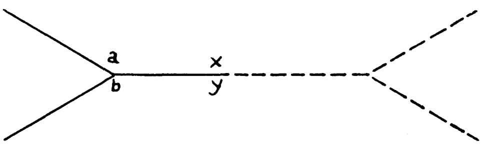

FIGURE 1

Lambek's embedding theorem is:

THEOREM 3 (Lambek). *A semigroup can be embedded in a group if and only if the cancellation laws and the polyhedral condition are satisfied.*

*Malcev's notation is much more involved than this, but this serves our purposes.

https://doi.org/10.4153/CJM-1963-006-x Published online by Cambridge University Press

Clearly the polyhedral condition is really an infinite set of conditions since there is an infinite number of such polyhedra and each leads to $2E$ conditions, according to the choice of equation to be implied by the $2E - 1$ others. Any one such condition obtained from some polyhedron and some choice of equation to be implied will be called *a Lambek polyhedral condition*. The term *the polyhedral condition* will refer to the totality of all Lambek polyhedral conditions.

Lambek proved the necessity of all the Lambek polyhedral conditions, but his sufficiency proof uses only those conditions that serve as diagrammatic representations of the associative laws. The following section outlines a proof of the necessity and sufficiency of this subset of conditions which will be defined later and are called the Lambek associative conditions. This proof does not depend on the properties of polyhedra.

2.2. THE ASSOCIATIVE CONDITIONS. Let $\mathfrak{A}$ be a cancellation semigroup. Let a, b be elements of $\mathfrak{A}$. Let a/b represent the set of all pairs x, y of elements of $\mathfrak{A}$ such that $xa = yb$. If a/b is not empty we call it a *ratio*. If two ratios can be put in the form a/d and d/b, where a/b is a ratio, we call a/b the *contraction** of a/d and d/b.

A finite sequence of ratios of the form $(\ldots, a/b, b/c, \ldots)$ is said to *contract* to the sequence $(\ldots, a/c, \ldots)$. Two finite sequences of ratios are called similar if there is a finite sequence from which both can be obtained by repeated contraction. This similarity can be shown to be an equivalence relation and hence partitions the set of ratios into equivalence classes.

Let S^* denote the class of all sequences that are similar to the sequence S. Define the product of two equivalence classes by $S^*T^* = (ST)^*$. Under this multiplication the equivalence classes form a group, called *the group of ratios* of $\mathfrak{A}$.

Let (a) represent the ratio at/t for any $t \in \mathfrak{A}$, and $(a)^*$ the class of sequences similar to (a). The mapping $a \to (a)^*$ of $\mathfrak{A}$ into the group of ratios can easily be shown to be a homomorphism. We ask under what conditions this mapping will be an isomorphism, that is an embedding. In order for $a \to (a)^*$ to be one-to-one we must show that if two sequences, each consisting of a single ratio, are similar, then the ratios are equal. This is equivalent to the associative law for the contraction of ratios.

The details of the foregoing can be found in Lambek's paper **(2)**. In order to prove the associative law he used the polyhedral condition. Our approach is somewhat different. We represent each associative law for n contractions by a $2n$-tuple of integers. From this $2n$-tuple we obtain $6n$ equations. Necessary and sufficient conditions for the embeddability of a semigroup in a group are given in terms of the interdependence of these equations.

Consider a chain of contractions that reduces a sequence of $n + 1$ ratios

*Under suitable conditions the contraction is in fact a unique product. Lambek used two Lambek polyhedral conditions to prove it here. We need only similarity, not equality of contractions, at this point. Uniqueness of the product follows from Theorem 4.

to a single ratio. We want to represent this by an n-tuple of integers $(\alpha_1, \alpha_2, \ldots, \alpha_n)$. If the first contraction operates on the kth and $(k+1)$th ratios, we set $\alpha_1 = k$. There are now n ratios in the sequence. If the second contraction operates on the mth and $(m+1)$th ratios we set $\alpha_2 = m$. We proceed in this way until values are determined for all the α's. Each associative law is the equality of the results of two repeated contractions on the same sequence. If we include the trivial or identical associative laws (such as $(AB)C = (AB)C$) and distinguish between $A(BC) = (AB)C$ and $(AB)C = A(BC)$, etc., then there is a one-to-one correspondence between the associative laws for $n+1$ factors and the $2n$-tuples $(\alpha_1, \alpha_2, \ldots, \alpha_{2n})$, where

$$\begin{aligned} \alpha_1 &= 1, 2, \ldots, n, \\ \alpha_2 &= 1, 2, \ldots, n-1, \\ &\ldots \\ \alpha_n &= 1, \\ \alpha_{n+1} &= 1, 2, \ldots, n, \\ \alpha_{n+2} &= 1, 2, \ldots, n-1, \\ &\ldots \\ \alpha_{2n} &= 1. \end{aligned}$$

For example, the associative law $A(B(CD)) = (AB)(CD)$ is represented by the 6-tuple $(3, 2, 1 : 1, 2, 1)$.

In order to obtain equations from such a $2n$-tuple we treat the initial n elements $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ first. As an initial step we replace this n-tuple by a $3n$-tuple $(\beta_1, \beta_2, \ldots, \beta_{3n})$. In order to do this we write out the integers

$$1, 2, \ldots, n+1 \tag{1}$$

and define $(\beta_1, \beta_2, \beta_3) = (\alpha_1, \alpha_1 + 1, -1)$. We then rewrite (1), replacing the pair $\alpha_1, \alpha_1 + 1$ by -1, to get

$$1, 2, \ldots, \alpha_1 - 1, -1, \alpha_1 + 2, \ldots, n+1,$$

which we denote by

$$1_2, 2_2, \ldots, n_2. \tag{2}$$

For our example, (1) is 1, 2, 3, 4 and $(\beta_1, \beta_2, \beta_3) = (3, 4, -1)$. Then (2) becomes $1, 2, -1$.

We define $(\beta_4, \beta_5, \beta_6) = ((\alpha_2)_2, (\alpha_2 + 1)_2, -2)$. In (2) we replace the pair $(\alpha_2)_2, (\alpha_2 + 1)_2$ by -2 and denote the result by

$$1_3, 2_3, \ldots, (n-1)_3. \tag{3}$$

For our example $(\beta_4, \beta_5, \beta_6) = (2, -1, -2)$.

We define $(\beta_7, \beta_8, \beta_9) = ((\alpha_3)_3, (\alpha_3 + 1)_3, -3)$ and continue until we have

$$1_n, 2_n. \tag{n}$$

Then we define $(\beta_{3n-2}, \beta_{3n-1}, \beta_{3n}) = (1_n, 2_n, -n)$.

For our example $n = 3$, (3) is 1, -2 and hence $(\beta_7, \beta_8, \beta_9) = (1, -2, -3)$. This $3n$-tuple now represents the ratios involved in the various contractions. For example, the elements $(\beta_{3r-2}, \beta_{3r-1}, \beta_{3r})$ are associated with a single contraction. Here β_{3r} is always negative and $-\beta_{3r}$ is the number of the contraction being performed.

In our example, $(3, 4, -1)$ is associated with the first contraction, $(2, -1, -2)$ with the second contraction, and $(1, -2, -3)$ with the third.

If $\beta_{3r-2} > 0$, then the first ratio entering into the contraction is the (β_{3r-2})th ratio of the original sequence. If $\beta_{3r-2} < 0$, then the first ratio entering into the contraction is the result of the $(-\beta_{3r-2})$th contraction. Similarly β_{3r-1} represents the second ratio entering into the contraction.

In our example $(3, 4, -1)$ indicates that the first contraction involves the third and fourth ratios of the original sequence; $(2, -1, -2)$ indicates that the second contraction involves the second ratio of the original sequence and the result of the first contraction.

From this $3n$-tuple we obtain $3n$ equations. For example, from the three elements $(\beta_{3r-2}, \beta_{3r-1}, \beta_{3r})$ we get

$$\beta^*_{3r-2}\, \bar{c}_r = \beta^{**}_{3r-2}\, \bar{a}_r,$$

$$\beta^*_{3r-1}\, \bar{b}_r = \beta^{**}_{3r-1}\, \bar{c}_r,$$

$$\bar{u}_r \bar{b}_r = \bar{v}_r \bar{a}_r,$$

where

$$\beta^*_i = \begin{cases} \bar{x}_{\beta_i} & \text{if } \beta_i > 0, \\ \bar{u}_{-\beta_i} & \text{if } \beta_i < 0; \end{cases}$$

$$\beta^{**}_i = \begin{cases} \bar{y}_{\beta_i} & \text{if } \beta_i > 0, \\ \bar{v}_{-\beta_i} & \text{if } \beta_i < 0. \end{cases}$$

In our example we have associated with $(3, 4, -1)$ the equations

$$\bar{x}_3 \bar{c}_1 = \bar{y}_3 \bar{a}_1,$$

$$\bar{x}_2 \bar{b}_1 = \bar{y}_2 \bar{c}_1,$$

$$\bar{u}_1 \bar{b}_1 = \bar{v}_1 \bar{a}_1,$$

and with $(2, -1, -2)$ the equations

$$\bar{x}_2 \bar{c}_2 = \bar{y}_2 \bar{a}_2,$$

$$\bar{u}_1 \bar{b}_2 = \bar{v}_1 \bar{c}_2,$$

$$\bar{u}_2 \bar{b}_2 = \bar{v}_2 \bar{a}_2.$$

From $(\alpha_{n+1}, \ldots, \alpha_{2n})$ we obtain $3n$ equations in the same way except that instead of $\bar{a}_i, \bar{b}_i, \bar{c}_i$ we write $\bar{d}_i, \bar{e}_i, \bar{f}_i$, $i = 1, 2, \ldots, n$; and instead of $\bar{u}_j, \bar{v}_j$, we write $\bar{s}_j, \bar{t}_j$, $j = 1, 2, \ldots, n - 1$. We note that $\bar{u}_n, \bar{v}_n$ are not replaced by $\bar{s}_n, \bar{t}_n$.

https://doi.org/10.4153/CJM-1963-006-x Published online by Cambridge University Press

A semigroup $\mathfrak{A}$ is said to satisfy the *Lambek associative condition* $L(\alpha_1, \alpha_2, \ldots, \alpha_n : \alpha_{n+1}, \ldots, \alpha_{2n})$ if whenever elements of $\mathfrak{A}$ are substituted for the symbols in the equations associated with the $2n$-tuple $(\alpha_1, \alpha_2, \ldots, \alpha_{2n})$ in such a way that the first $6n - 1$ equations are satisfied, then the $6n$th equation must also be satisfied.

The proof of the following theorem is omitted. It is somewhat lengthy, but presents no great difficulty. It consists of a consideration of the various equations that must hold if a contraction is to take place and the observation that the set of equations required for two repeated contractions to take place is essentially the set of equations in a Lambek associative condition. The details are given in the author's thesis **(1)**.

THEOREM 4. *If $\mathfrak{A}$ satisfies all the Lambek associative conditions, then all the associative laws hold in the set of ratios, that is, if a sequence of ratios can be contracted to a single ratio, then that ratio is unique.*

From this the sufficiency of all the Lambek associative conditions follows easily. The necessity can be proved by direct calculation in any group in which the given semigroup is embedded. This gives a modified form of Lambek's embedding theorem:

THEOREM 5. *A cancellation semigroup is embeddable in a group if and only if it satisfies all the Lambek associative conditions.*

3. Comparison of the Malcev and Lambek conditions. We shall prove a simple lemma which will be used in both of the theorems in this section.

LEMMA 2. *Let $\mathfrak{A}$ be a cancellation semigroup generated by $s_1, s_2, \ldots$ and with defining relations $g_k(s_1, s_2, \ldots) = h_k(s_1, s_2, \ldots)$, $k = 1, 2, \ldots$. Let $\mathfrak{A}'$ be a cancellation semigroup including among its elements $s_1', s_2', \ldots$ such that $g_k(s_1', s_2', \ldots) = h_k(s_1', s_2', \ldots)$, $k = 1, 2, \ldots$. Let $g(s_1, s_2, \ldots) = h(s_1, s_2, \ldots)$ be any relationship that holds in $\mathfrak{A}$. Then $g(s_1', s_2', \ldots) = h(s_1', s_2', \ldots)$ holds in $\mathfrak{A}'$.*

Proof. If $g(s_1, s_2, \ldots) = h(s_1, s_2, \ldots)$ holds in $\mathfrak{A}$, then it can be obtained from $g_k(s_1, s_2, \ldots) = h_k(s_1, s_2, \ldots)$, $k = 1, 2, \ldots$, by a finite sequence of semigroup operations, that is, multiplication, cancellation, and substitution. These same semigroup operations applied to $g_k(s_1', s_2', \ldots) = h_k(s_1', s_2', \ldots)$, $k = 1, 2, \ldots$, must produce $g(s_1', s_2', \ldots) = h(s_1', s_2', \ldots)$.

Let $\bar{K}$ be a set of equations, each of which equates two products of symbols $\bar{a}, \bar{b}, \bar{c}, \ldots$ and let $\bar{M}$ be a single equation of this same type. We say that the condition $\bar{K} \Rightarrow \bar{M}$ is *satisfied* by a semigroup if for each choice of elements from the semigroup for which the equations $\bar{K}$ hold, the equation $\bar{M}$ must also hold. We say that the condition $\bar{K}_1 \Rightarrow \bar{M}_1$ is a *consequence* of the condition $\bar{K} \Rightarrow \bar{M}$ if $\bar{K}_1 \Rightarrow \bar{M}_1$ is satisfied by every semigroup that satisfies $\bar{K} \Rightarrow \bar{M}$.

https://doi.org/10.4153/CJM-1963-006-x Published online by Cambridge University Press

The following theorem developed out of an attempt to provide more details for one of Malcev's theorems (**4**, Theorem 4a). In his proof he made a statement that is stronger than our theorem, but which seems to be inaccurate and for which he offered no proof.* Our theorem is sufficient for the use that Malcev made of this statement.

THEOREM 6. *Let $\bar{K}$ be a system of equations, each equating two products of the symbols $\bar{a}, \bar{b}, \bar{c}, \ldots$, and let $\bar{M}$ be a single equation of the same form. If the condition $\bar{K} \Rightarrow \bar{M}$ is satisfied by every semigroup that is embeddable in a group, then $\bar{K} \Rightarrow \bar{M}$ is a consequence of some Malcev condition.*

Proof. Let $\mathfrak{H}$ be the cancellation semigroup generated by $a, b, c, \ldots$ with defining relations K, that is, the equations of $\bar{K}$ with $\bar{a}$ replaced by a, etc. Let $\mathfrak{G}$ be the formal group of $\mathfrak{H}$ as defined in § 1. Then the equations K hold in $\mathfrak{G}$.

The condition $\bar{K} \Rightarrow \bar{M}$ is satisfied by every embeddable semigroup, hence by every group, and in particular by $\mathfrak{G}$. But K, a substitution of elements of $\mathfrak{H}$ into $\bar{K}$, holds in $\mathfrak{G}$. Therefore M, the corresponding substitution into $\bar{M}$, holds in $\mathfrak{G}$. But $\bar{M}$ involves only the symbols of $\bar{K}$ and K involves only the elements of $\mathfrak{H}$. Therefore M involves only the elements of $\mathfrak{H}$.

By the definition of the formal group $\mathfrak{G}$, the two sides of the equation M must be connected by a chain of transformations of the forms:

(a) insertion of ll^* or L^*L where l^*, L^* are formal right and left elements,

(b) deletion of ll^* or L^*L,

(c) replacement of a word in $\mathfrak{H}$ by an equal word in $\mathfrak{H}$.

By Theorem 1 we see that we can replace this chain by a normal chain. The sequence of insertions and deletions of formal elements determines a chain of transformations in the barred symbols used in § 1 and this chain in turn determines a normal system of equations. Since the two sides of the equation M form the first and last words of the normal chain in $\mathfrak{G}$, we can choose our notation so that the completing equation is $\bar{M}$. Let $\bar{N}$ represent the normal system of equations in this notation.

We have now associated with the given condition $\bar{K} \Rightarrow \bar{M}$ a Malcev condition $\bar{N} \Rightarrow \bar{M}$. The normal chain in $\mathfrak{G}$ provides a substitution of elements of $\mathfrak{H}$ for the symbols of the normal system $\bar{N}$ such that the corresponding equations N hold in $\mathfrak{H}$.

Let $\mathfrak{A}$ be a cancellation semigroup satisfying the Malcev condition $\bar{N} \Rightarrow \bar{M}$. We want to show that $\mathfrak{A}$ also satisfies $\bar{K} \Rightarrow \bar{M}$. Let K' be some substitution of elements of $\mathfrak{A}$ for the symbols of $\bar{K}$ such that all the equations hold in $\mathfrak{A}$. Now $\mathfrak{H}$ has the defining relations K, and in $\mathfrak{A}$ the corresponding equations K' hold. Also the equations in N hold in $\mathfrak{H}$. Therefore by Lemma 2 the corresponding equations N' hold in $\mathfrak{A}$. But N' is a substitution of elements of $\mathfrak{A}$ into $\bar{N}$. Also $\bar{N} \Rightarrow \bar{M}$ in $\mathfrak{A}$ and hence M', the corresponding substitution into

*See the author's thesis (**1**) for a discussion of this apparently false statement.

$\bar{M}$, holds in $\mathfrak{A}$. Thus $\bar{K} \Rightarrow \bar{M}$ is satisfied in $\mathfrak{A}$, and hence is a consequence of the Malcev condition $\bar{N} \Rightarrow \bar{M}$.

Since every Lambek polyhedral condition (and hence every Lambek associative condition) has the form $\bar{K} \Rightarrow \bar{M}$, we have the following obvious application of the theorem.

COROLLARY 1. *Every Lambek polyhedral (or associative) condition is a consequence of a single Malcev condition.*

The corollary to the following theorem shows that the converse is also true.

THEOREM 7. *Let $\bar{K}$ be a set of equations and $\bar{M}$ a single equation as in Theorem 6. If the condition $\bar{K} \Rightarrow \bar{M}$ is satisfied by every semigroup that is embeddable in a group, then $\bar{K} \Rightarrow \bar{M}$ is a consequence of some Lambek associative condition.*

Proof. Let $\mathfrak{H}$ be the cancellation semigroup as defined in the proof of Theorem 6. Let $\mathfrak{P}$ be the group of ratios of $\mathfrak{H}$ as defined in § 2. The equations K, that is, $\bar{K}$ with $\bar{a}$ replaced by a, etc., hold in $\mathfrak{H}$ and hence in $\mathfrak{P}$. The condition $\bar{K} \Rightarrow \bar{M}$ is satisfied in any embeddable semigroup, hence in any group, and in particular in $\mathfrak{P}$. But K is a substitution of elements of $\mathfrak{H}$ for the symbols of $\bar{K}$ and hence the corresponding equation M holds in $\mathfrak{P}$. Also M involves only elements of $\mathfrak{H}$, that is, M equates two ratios. Two elements are equal in $\mathfrak{P}$ if and only if they can both be obtained from the same finite sequence of ratios of elements of $\mathfrak{H}$. The two repeated contractions to produce the two sides of the equation M can be associated with a Lambek associative condition as in § 2. Let $\bar{J} \Rightarrow \bar{M}$ be this Lambek condition.

The particular contractions used to obtain M determine one or more substitutions of elements of $\mathfrak{H}$ for the symbols of $\bar{J}$ such that all the equations hold in $\mathfrak{H}$. Choose any one of the substitutions and let J represent the resulting equations.

Let $\mathfrak{B}$ be a cancellation semigroup satisfying $\bar{J} \Rightarrow \bar{M}$. We want to show that $\mathfrak{B}$ also satisfies $\bar{K} \Rightarrow \bar{M}$. Let K'' be a substitution of elements of $\mathfrak{B}$ into the equations $\bar{K}$ such that all the equations are satisfied. Now $\mathfrak{H}$ has the defining relations K and the corresponding equations K'' hold in $\mathfrak{B}$. Also the equations J hold in $\mathfrak{H}$. Thus by Lemma 2 the corresponding equations J'' hold in $\mathfrak{B}$. But J'' is a substitution of elements of $\mathfrak{B}$ into $\bar{J}$ and $\bar{J} \Rightarrow \bar{M}$ in $\mathfrak{B}$. Hence M'', the corresponding substitution into $\bar{M}$, holds in $\mathfrak{B}$. Thus $\bar{K} \Rightarrow \bar{M}$ is satisfied in $\mathfrak{B}$ and hence is a consequence of the Lambek associative condition $\bar{J} \Rightarrow \bar{M}$.

Since every Malcev condition has the form $\bar{K} \Rightarrow \bar{M}$, the following application of the theorem is obvious.

COROLLARY 2. *Every Malcev condition is a consequence of a single Lambek associative condition.*

https://doi.org/10.4153/CJM-1963-006-x Published online by Cambridge University Press

The Lambek and Malcev conditions have now been shown to be consequences of each other. But the two sets of conditions are not identical. This is shown by means of counterexamples in the author's thesis **(1)**.

References

1. G. C. Bush, *On embedding a semigroup in a group*, Ph.D. thesis, Queen's University, Kingston, Ontario (1961).

2. J. Lambek, *The immersibility of a semigroup into a group*, Can. J. Math., *3* (1951), 34–43.

3. A. Malcev, *Über die Einbettung von assoziativen Systemen in Gruppen*, Mat. Sbornik, *6* (*48*) (1939), 331–336.

4. ——— *Über die Einbettung von assoziativen Systemen in Gruppen, II*, Mat. Sbornik, *8* (*50*) (1940), 251–264.

5. D. Tamari, *Monoïdes préordonnés et chaînes de Malcev*, Bull. Soc. Math. France, *82* (1954), 53–96.

Queen's University