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Characterizations of Besov-Type and Triebel–Lizorkin–Type Spaces via Averages on Balls

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Abstract. Let $\ell \in \mathbb{N}$ and $\alpha \in (0, 2\ell)$. In this article, the authors establish equivalent characterizations of Besov-type spaces, Triebel–Lizorkin-type spaces, and Besov–Morrey spaces via the sequence $\{f - B_{\ell,2^{-k}} f\}_k$ consisting of the difference between f and the ball average $B_{\ell,2^{-k}} f$. These results lead to the introduction of Besov-type spaces, Triebel–Lizorkin-type spaces, and Besov–Morrey spaces with any positive smoothness order on metric measure spaces. As special cases, the authors obtain a new characterization of Morrey–Sobolev spaces and Q_α spaces with $\alpha \in (0,1)$, which are of independent interest.

1 Introduction

Besov and Triebel–Lizorkin spaces are known to be complicated objects (see [17, 28]). Besov-type and Triebel–Lizorkin-type spaces are counterparts (generalizations) defined by using Morrey spaces instead of Lebesgue spaces, hence even more complicated (see [31, 32, 36]). There exists some need for simple descriptions. Usually one starts with a definition by using tools from Fourier analysis. For a given tempered distribution f, the basic objects one has to study are convolutions $\varphi_k * f$, where $\{\varphi_k\}_k$ is a smooth dyadic decomposition of unity. In this article, we present a possible way to replace the sequence $\{\varphi_k * f\}_k$ by $\{f - B_{\ell,2^{-k}} f\}_k$, where $B_{\ell,2^{-k}} f$ denotes a certain ball average of f.

Of course, the characterization of classes of functions via ball averages has a certain history. We continue with a few rather detailed comments.

Recently, based on a new characterization of Sobolev spaces obtained in [2], there have been some attempts to characterize Sobolev spaces, Besov spaces, and Triebel–Lizorkin spaces on $\mathbb{R}^n$ via ball averages (see, for example, [3, 7–9, 19, 34, 35]). These new characterizations only depend on the metric of $\mathbb{R}^n$ and the Lebesgue measure, and hence provide some possible ways to introduce the corresponding function spaces with positive smoothness $s \in (0, \infty)$ on metric measure spaces.

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In particular, recall that in [35], Yang et al. proved that the Triebel–Lizorkin space $\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)$ with $\alpha \in (0,2)$ and $p, q \in (1,\infty]$ can be characterized via the square function

$$S_{\alpha,q}(f)(x) := \Big\{ \sum_{k\in\mathbb{Z}} 2^{k\alpha q} \Big| f(x) - B_{2^{-k}} f(x) \Big|^q \Big\}^{\frac{1}{q}}, \quad \text{for all } x \in \mathbb{R}^n,$$

where, for all $t \in (0,\infty)$ and $x \in \mathbb{R}^n$,

$$B_t f(x) := \frac{1}{|B(x,t)|} \int_{B(x,t)} f(y)\, dy.$$

A similar characterization for Besov spaces $\dot{B}^s_{p,q}(\mathbb{R}^n)$ was also obtained in [35]. Recently, via introducing a higher order average operator, Dai et al. [8] further established the corresponding characterization of Besov and Triebel–Lizorkin spaces for any smoothness order. Moreover, by combining the ideas used in [21] and [7], Yang and Yuan [34] also characterized Besov and Triebel–Lizorkin spaces with arbitrary positive smoothness on $\mathbb{R}^n$ in terms of some pointwise inequalities involving ball averages. One common advantage of all these new characterizations of Besov and Triebel–Lizorkin spaces, established in [8, 34], lies in the fact that they depend only on the metric of $\mathbb{R}^n$ and the Lebesgue measure, and hence may be used to define these spaces with higher order smoothness on metric measure spaces.

The main purpose of this paper is to characterize Besov-type spaces, Triebel–Lizorkin-type spaces, and Besov–Morrey spaces via ball averages. Recall that the Besov-type space $\dot{B}^{s,\tau}_{p,q}(\mathbb{R}^n)$ and the Triebel–Lizorkin-type space $\dot{F}^{s,\tau}_{p,q}(\mathbb{R}^n)$ were introduced in [31, 32] in order to clarify the relations among Besov spaces, Triebel–Lizorkin spaces, and Q spaces (see [5, 14]). Inhomogeneous Besov-type spaces $B^{s,\tau}_{p,q}(\mathbb{R}^n)$, restricted to Banach cases, were considered for the first time in [11–13], and later investigated in [36] for a full range of parameters. Moreover, the Besov-type and the Triebel–Lizorkin-type spaces have several applications in partial differential equations (see, for example, [29, 30, 37]). On the other hand, the inhomogeneous Besov–Morrey spaces, restricted to Banach cases, originate from the study of Navier–Stokes equations by Kozono and Yamazaki [22]. In 2005, Tang and Xu [27] introduced inhomogeneous Besov–Morrey and Triebel–Lizorkin–Morrey spaces for the full set of parameters, and, later on, Sawano and Tanaka [23] introduced their homogeneous versions in 2007. These spaces form a more general scale of function spaces than the classical Besov and Triebel–Lizorkin spaces, and include Morrey spaces, Morrey–Sobolev spaces, and Q spaces as special cases. The characterizations of these spaces via ball averages could serve as a starting point to develop the theory of these spaces for any positive smoothness on metric measure spaces. However, as we believe, these characterizations are interesting for their own sake.

Before stating the main results of this article, we first recall some basic notation. Let $\mathcal{S}(\mathbb{R}^n)$ denote the set of all *Schwartz functions* on $\mathbb{R}^n$, equipped with the usual topology, and $\mathcal{S}'(\mathbb{R}^n)$ its *topological dual*, namely, the collection of all bounded linear functionals on $\mathcal{S}(\mathbb{R}^n)$ equipped with the weak-$*$ topology. Let $\mathcal{S}_\infty(\mathbb{R}^n)$ be the set of all Schwartz functions φ satisfying $\int_{\mathbb{R}^n} \varphi(x) x^\beta\, dx = 0$ for all $\beta \in \mathbb{Z}^n_+$, equipped with the same topology as $\mathcal{S}'(\mathbb{R}^n)$, and $\mathcal{S}'_\infty(\mathbb{R}^n)$ its topological dual equipped with the weak-$*$ topology. For any $\varphi \in \mathcal{S}(\mathbb{R}^n)$, we use $\widehat{\varphi}$ to denote its *Fourier transform*, namely, for all $x \in \mathbb{R}^n$, $\widehat{\varphi}(\xi) := \int_{\mathbb{R}^n} \varphi(x) e^{-ix\cdot\xi}\,dx$, and $\varphi^\vee$ to denote its *inverse Fourier transform*. Also, we need the notation that, for any $\varphi \in \mathcal{S}(\mathbb{R}^n)$, $j \in \mathbb{Z}$ and $x \in \mathbb{R}^n$, $\varphi_j(x) := 2^{jn}\varphi(2^j x)$. For all $j \in \mathbb{Z}$ and $m \in \mathbb{Z}^n$, denote by Q_{jm} the dyadic cube $2^{-j}([0,1)^n + m)$ on $\mathbb{R}^n$ and let $\mathcal{Q} := \{Q_{jm} : j \in \mathbb{Z},\ m \in \mathbb{Z}^n\}$. For any $Q \in \mathcal{Q}$, we use $l(Q)$ to denote its side length.

Let $\ell \in \mathbb{N}$ and $t \in (0,\infty)$. The *2ℓ-th order average operator* $B_{\ell,t}$ is defined by

$$B_{\ell,t} f(x) := -\frac{2}{\binom{2\ell}{\ell}} \sum_{j=1}^{\ell} (-1)^j \binom{2\ell}{\ell - j} B_{jt} f(x), \quad \text{for all } x \in \mathbb{R}^n,\ f \in L^1_{\mathrm{loc}}(\mathbb{R}^n), \tag{1.1}$$

here and hereafter, for any $k,\ r \in \mathbb{N}$ with $k \geq r$, $\binom{k}{r}$ denotes the *binomial coefficient*. Obviously, $B_{1,t} f = B_t f$. For any $q \in (0,\infty)$, denote by $L^q_{\mathrm{loc}}(\mathbb{R}^n)$ the set of all q-locally integrable functions on $\mathbb{R}^n$.

Recall that in [8, Theorem 1.3], Dai et al. established equivalent characterizations of the Besov space $\dot{B}^\alpha_{p,q}(\mathbb{R}^n)$ and the Triebel–Lizorkin space $\dot{F}^\alpha_{p,q}(\mathbb{R}^n)$ via the differences $\{f - B_{\ell,2^{-k}} f\}_{k\in\mathbb{Z}}$. Notice that the difference $f - B_{\ell,2^{-k}} f$, depending only on the metric of $\mathbb{R}^n$ and the Lebesgue measure, can be easily defined on any metric measure space. Thus, the characterizations of $\dot{B}^\alpha_{p,q}(\mathbb{R}^n)$ and $\dot{F}^\alpha_{p,q}(\mathbb{R}^n)$ in [8, Theorem 1.3] provide a possible way to introduce Besov and Triebel–Lizorkin spaces with arbitrary positive smoothness order on metric measure spaces. Moreover, as it is indicated by [8, Theorem 1.3], the differences $\{f - B_{\ell,2^{-k}} f\}_{k\in\mathbb{Z}}$ play the same role as the convolutions $\{\varphi_k * f\}_{k\in\mathbb{Z}}$ in the definitions of Besov and Triebel–Lizorkin spaces. Therefore, it is natural to ask *whether or not* the differences $\{f - B_{\ell,2^{-k}} f\}_{k\in\mathbb{Z}}$ can be used to characterize more general Besov-type spaces, Triebel–Lizorkin-type spaces and Besov–Morrey spaces.

Next we recall the definitions of homogeneous Besov-type spaces, Triebel–Lizorkin-type spaces, Besov–Morrey spaces, and Triebel–Lizorkin–Morrey spaces; see [23, 31, 32].

Definition 1.1 Let $\alpha \in \mathbb{R}$, $\tau \in [0,\infty)$, $q \in (0,\infty]$, and $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfy that

$$\operatorname{supp}\widehat{\varphi} \subset \{\xi \in \mathbb{R}^n : 1/2 \leq |\xi| \leq 2\} \quad \text{and} \quad |\widehat{\varphi}(\xi)| \geq \text{constant} > 0 \text{ if } 3/5 \leq |\xi| \leq 5/3. \tag{1.2}$$

(i) Let $p \in (0,\infty]$. Then the *homogeneous Besov-type space* $\dot{B}^{\alpha,\tau}_{p,q}(\mathbb{R}^n)$ and the *homogeneous Triebel–Lizorkin-type space* $\dot{F}^{\alpha,\tau}_{p,q}(\mathbb{R}^n)$ ($p \in (0,\infty)$) are, respectively, defined as the collections of all $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$ such that

$$\|f\|_{\dot{B}^{\alpha,\tau}_{p,q}(\mathbb{R}^n)} := \sup_{Q\in\mathcal{Q}} \frac{1}{|Q|^\tau} \Big\{ \sum_{k=-\log_2 l(Q)}^{\infty} 2^{k\alpha q} \big\| \varphi_k * f \big\|^q_{L^p(Q)} \Big\}^{1/q} < \infty,$$

$$\|f\|_{\dot{F}^{\alpha,\tau}_{p,q}(\mathbb{R}^n)} := \sup_{Q\in\mathcal{Q}} \frac{1}{|Q|^\tau} \Big\| \Big\{ \sum_{k=-\log_2 l(Q)}^{\infty} 2^{k\alpha q} |\varphi_k * f|^q \Big\}^{1/q} \Big\|_{L^p(Q)} < \infty$$

with the usual modifications made when $p = \infty$ or $q = \infty$.

(ii) Let $0 < p \leq u < \infty$. Then the *Besov–Morrey spaces* $\dot{\mathcal{N}}^\alpha_{u,p,q}(\mathbb{R}^n)$ and the *Triebel–Lizorkin–Morrey space* $\dot{\mathcal{E}}^\alpha_{u,p,q}(\mathbb{R}^n)$ are, respectively, defined to be the set of

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all $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$ such that

$$\|f\|_{\dot{\mathcal{N}}^\alpha_{u,p,q}(\mathbb{R}^n)} := \Big\{ \sum_{k\in\mathbb{Z}} \sup_{Q\in\mathcal{Q}} \frac{1}{|Q|^{q/p-q/u}} \Big\| 2^{k\alpha} \varphi_k * f \Big\|^q_{L^p(Q)} \Big\}^{1/q} < \infty,$$

$$\|f\|_{\dot{\mathcal{E}}^\alpha_{u,p,q}(\mathbb{R}^n)} := \sup_{Q\in\mathcal{Q}} \frac{1}{|Q|^{1/p-1/u}} \Big\| \Big\{ \sum_{k\in\mathbb{Z}} 2^{k\alpha q} |\varphi_k * f|^q \Big\}^{1/q} \Big\|_{L^p(Q)} < \infty$$

with the usual modification made when $q = \infty$.

Remark 1.2 (i) We point out that the spaces

$$\dot{B}^{s,\tau}_{p,q}(\mathbb{R}^n), \quad \dot{F}^{s,\tau}_{p,q}(\mathbb{R}^n), \quad \dot{\mathcal{N}}^\alpha_{u,p,q}(\mathbb{R}^n), \quad \text{and} \quad \dot{\mathcal{E}}^\alpha_{u,p,q}(\mathbb{R}^n)$$

are independent of the choice of functions φ satisfying (1.2) (see [23, 31, 32]).

(ii) Let φ be as in (1.2). Then it is well known that there exists a function $\psi \in \mathcal{S}(\mathbb{R}^n)$ satisfying (1.2) such that $\sum_{k\in\mathbb{Z}} \widehat{\varphi}(2^k\xi)\widehat{\psi}(2^k\xi) = 1$ for all $\xi \in \mathbb{R}^n \setminus \{0\}$; see [18, (6.9)].

(iii) If $\tau \in [0, 1/p)$, then the sums $\sum_{k=-\log_2 l(Q)}^{\infty}$ in Definition 1.1(i) can be equivalently replaced by $\sum_{k\in\mathbb{Z}}$ (see [24, Theorem 1(ii) and Proposition 3.3]).

(iv) Obviously,

$$\dot{B}^{s,0}_{p,q}(\mathbb{R}^n) = \dot{B}^s_{p,q}(\mathbb{R}^n) \quad \text{and} \quad \dot{F}^{s,0}_{p,q}(\mathbb{R}^n) = \dot{F}^s_{p,q}(\mathbb{R}^n),$$

where $\dot{B}^s_{p,q}(\mathbb{R}^n)$ and $\dot{F}^s_{p,q}(\mathbb{R}^n)$ are, respectively, the *homogeneous Besov and Triebel–Lizorkin spaces* (see [17, 28, 36]).

The main results of this article are stated as follows.

Theorem 1.3 *Let $\ell \in \mathbb{N}$ and $\alpha \in (0, 2\ell)$. Let $B_{\ell,2^{-k}}$ for $k \in \mathbb{Z}$ be as in (1.1) with t replaced by 2^{-k}.*

(i) *Let $p \in (1, \infty)$, $\tau \in [0, 1/p)$, and $q \in (0, \infty]$.*

If $f \in \dot{B}^{\alpha,\tau}_{p,q}(\mathbb{R}^n)$, then there exists some $g \in L^p_{\mathrm{loc}}(\mathbb{R}^n) \cap \mathcal{S}'_\infty(\mathbb{R}^n)$ such that $g = f$ in $\mathcal{S}'_\infty(\mathbb{R}^n)$ and

$$\|\|g\|\|_{\dot{B}^{\alpha,\tau}_{p,q}(\mathbb{R}^n)} \le C \|f\|_{\dot{B}^{\alpha,\tau}_{p,q}(\mathbb{R}^n)} \tag{1.3}$$

with C being a positive constant independent of f, where

$$\|\|g\|\|_{\dot{B}^{\alpha,\tau}_{p,q}(\mathbb{R}^n)} := \sup_{Q\in\mathcal{Q}} \frac{1}{|Q|^\tau} \Big\{ \sum_{k\in\mathbb{Z}} 2^{k\alpha q} \Big\| g - B_{\ell,2^{-k}} g \Big\|^q_{L^p(Q)} \Big\}^{1/q}.$$

Conversely, if $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) \cap \mathcal{S}'_\infty(\mathbb{R}^n)$ and $\|\|f\|\|_{\dot{B}^{\alpha,\tau}_{p,q}(\mathbb{R}^n)} < \infty$, then

$$f \in \dot{B}^{\alpha,\tau}_{p,q}(\mathbb{R}^n) \quad \textit{and} \quad \|f\|_{\dot{B}^{\alpha,\tau}_{p,q}(\mathbb{R}^n)} \le C\, \|\| f\|\|_{\dot{B}^{\alpha,\tau}_{p,q}(\mathbb{R}^n)}$$

with C being a positive constant independent of f.

(ii) *Let $p \in (0, \infty)$, $\tau \in (1/p, \infty)$ and $q \in (0, \infty]$, or $p \in (0, \infty)$, $\tau = 1/p$ and $q = \infty$, or $p = \infty$, $\tau \in [0, \infty)$ and $q \in (0, \infty]$.*

If $f \in \dot{B}^{\alpha,\tau}_{p,q}(\mathbb{R}^n)$, then there exists some $g \in L^1_{\mathrm{loc}}(\mathbb{R}^n) \cap \mathcal{S}'_\infty(\mathbb{R}^n)$ such that $f = g$ in $\mathcal{S}'_\infty(\mathbb{R}^n)$ and $\|\|g\|\|_{\dot{B}^{\alpha,\tau}_{p,q}(\mathbb{R}^n)} \le C\|f\|_{\dot{B}^{\alpha,\tau}_{p,q}(\mathbb{R}^n)}$ with C being a positive constant independent of f, where

$$\|\|g\|\|_{\dot{B}^{\alpha,\tau}_{p,q}(\mathbb{R}^n)} := \sup_{k\in\mathbb{Z}} \sup_{y\in\mathbb{R}^n} 2^{k[\alpha+n(\tau-1/p)]} |g(y) - B_{\ell,2^{-k}} g(y)|.$$

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Conversely, if

$$f \in L^1_{\rm loc}(\mathbb{R}^n) \cap \mathcal{S}'_\infty(\mathbb{R}^n) \quad \textit{and} \quad |||\, f|||_{\dot{B}^{\alpha,\tau}_{p,q}(\mathbb{R}^n)} < \infty,$$

then

$$f \in \dot{B}^{\alpha,\tau}_{p,q}(\mathbb{R}^n) \quad \textit{and} \quad \|f\|_{\dot{B}^{\alpha,\tau}_{p,q}(\mathbb{R}^n)} \le C\, |||\, f|||_{\dot{B}^{\alpha,\tau}_{p,q}(\mathbb{R}^n)}$$

with C being a positive constant independent of f.

Remark 1.4 In Theorem 1.3, we formally gave two different definitions of

$$|||\cdot|||_{\dot{B}^{\alpha,\tau}_{p,q}(\mathbb{R}^n)}\,.$$

Please notice that in part (i) we have the restriction $\tau \in [0, 1/p)$, whereas in part (ii) we have, beside others, the restriction $\tau \in [1/p, \infty)$.

Theorem 1.5 *Let $\ell \in \mathbb{N}$ and $\alpha \in (0, 2\ell)$.*

(i) *Let $p \in (1, \infty)$, $\tau \in [0, 1/p)$ and $q \in (1, \infty]$. If $f \in \dot{F}^{\alpha,\tau}_{p,q}(\mathbb{R}^n)$, then there exists $g \in L^p_{\rm loc}(\mathbb{R}^n) \cap \mathcal{S}'_\infty(\mathbb{R}^n)$ such that $g = f$ in $\mathcal{S}'_\infty(\mathbb{R}^n)$ and $|||g|||_{\dot{F}^{\alpha,\tau}_{p,q}(\mathbb{R}^n)} \le C\|f\|_{\dot{F}^{\alpha,\tau}_{p,q}(\mathbb{R}^n)}$ with C being a positive constant independent of f, where*

$$|||g|||_{\dot{F}^{\alpha,\tau}_{p,q}(\mathbb{R}^n)} := \sup_{Q\in\mathcal{Q}} \frac{1}{|Q|^\tau} \Big\| \Big\{ \sum_{k\in\mathbb{Z}} 2^{k\alpha q} |g - B_{\ell,2^{-k}} g|^q \Big\}^{1/q} \Big\|_{L^p(Q)}.$$

Conversely, if $f \in L^1_{\rm loc}(\mathbb{R}^n) \cap \mathcal{S}'_\infty(\mathbb{R}^n)$ and $|||f|||_{\dot{F}^{\alpha,\tau}_{p,q}(\mathbb{R}^n)} < \infty$, then

$$f \in \dot{F}^{\alpha,\tau}_{p,q}(\mathbb{R}^n) \quad \textit{and} \quad \|f\|_{\dot{F}^{\alpha,\tau}_{p,q}(\mathbb{R}^n)} \le C\, |||\, f|||_{\dot{F}^{\alpha,\tau}_{p,q}(\mathbb{R}^n)}$$

with C being a positive constant independent of f.

(ii) *Let $p \in (0, \infty)$, $\tau \in [\frac{1}{p}, \infty)$ and $q \in (1, \infty]$. If $f \in \dot{F}^{\alpha,\tau}_{p,q}(\mathbb{R}^n)$, then there exists $g \in L^1_{\rm loc}(\mathbb{R}^n) \cap \mathcal{S}'_\infty(\mathbb{R}^n)$ such that $f = g$ in $\mathcal{S}'_\infty(\mathbb{R}^n)$ and $|||g|||_{\dot{F}^{\alpha,\tau}_{p,q}(\mathbb{R}^n)} \le C\|f\|_{\dot{F}^{\alpha,\tau}_{p,q}(\mathbb{R}^n)}$ with C being a positive constant independent of f, where, when $\tau \in (1/p, \infty)$ and $q \in (0, \infty]$, or $\tau = 1/p$ and $q = \infty$,*

$$|||g|||_{\dot{F}^{\alpha,\tau}_{p,q}(\mathbb{R}^n)} := \sup_{k\in\mathbb{Z}} \sup_{y\in\mathbb{R}^n} 2^{k[\alpha+n(\tau-1/p)]} |g(y) - B_{\ell,2^{-k}} g(y)|$$

and, when $\tau = 1/p$ and $q \in (0, \infty)$,

$$|||g|||_{\dot{F}^{\alpha,\tau}_{p,q}(\mathbb{R}^n)} := \sup_{Q\in\mathcal{Q}} \Big\{ \frac{1}{|Q|} \int_Q \sum_{k=\log_2 l(Q)}^{\infty} 2^{k[\alpha+n(\tau-1/p)]q} |g(y) - B_{\ell,2^{-k}} g(y)|^q \, dy \Big\}^{1/q}.$$

Conversely, if $f \in L^1_{\rm loc}(\mathbb{R}^n) \cap \mathcal{S}'_\infty(\mathbb{R}^n)$ and $|||f|||_{\dot{F}^{\alpha,\tau}_{p,q}(\mathbb{R}^n)} < \infty$, then $f \in \dot{F}^{\alpha,\tau}_{p,q}(\mathbb{R}^n)$ and $\|f\|_{\dot{F}^{\alpha,\tau}_{p,q}(\mathbb{R}^n)} \le C\, |||\, f|||_{\dot{F}^{\alpha,\tau}_{p,q}(\mathbb{R}^n)}$ with C being a positive constant independent of f.

Remark 1.6 (i) Since $\dot{B}^{\alpha,0}_{p,q}(\mathbb{R}^n) = \dot{B}^{\alpha}_{p,q}(\mathbb{R}^n)$ and $\dot{F}^{\alpha,0}_{p,q}(\mathbb{R}^n) = \dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)$, we know that Theorems 1.3 and 1.5 when $\tau = 0$ generalize [8, Theorem 1.3].

(ii) Recall that, for any $\alpha \in (0, 1)$,

$$\dot{F}^{\alpha,\frac{1}{2}-\frac{\alpha}{n}}_{2,2}(\mathbb{R}^n) = Q_\alpha(\mathbb{R}^n)$$

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(see [31, Corollary 3.1]), where the *space* $Q_\alpha(\mathbb{R}^n)$ is defined to be the set of all measurable functions $f \in L^2_{\rm loc}(\mathbb{R}^n)$ such that

$$\|f\|_{Q_\alpha(\mathbb{R}^n)} := \sup_R \Big\{ \frac{1}{|R|^{1-2\alpha/n}} \int_R \int_R \frac{|f(x)-f(y)|^2}{|x-y|^{n+2\alpha}}\,dx\,dy \Big\}^{1/2} < \infty,$$

with the supremum being taken over all cubes R in $\mathbb{R}^n$ (see [5, 6, 14]). By this and Theorem 1.5(i), we obtain the following equivalent characterization of $Q_\alpha(\mathbb{R}^n)$ with $\alpha \in (0,1)$: $f \in Q_\alpha(\mathbb{R}^n)$ if and only if $f \in L^2_{\rm loc}(\mathbb{R}^n) \cap \mathcal{S}'_\infty(\mathbb{R}^n)$ and $|||f|||_{Q_\alpha(\mathbb{R}^n)} < \infty$; moreover, $|||f|||_{Q_\alpha(\mathbb{R}^n)} \sim \|f\|_{Q_\alpha(\mathbb{R}^n)}$, with the implicit equivalent positive constants independent of f, where

$$|||f|||_{Q_\alpha(\mathbb{R}^n)} := \sup_{R\in\mathcal{Q}} \frac{1}{|R|^{1/2-\alpha/n}} \Big\{ \int_R \sum_{k\in\mathbb{Z}} 2^{2k\alpha} |f(x) - B_{2^{-k}} f(x)|^2\,dx \Big\}^{1/2}.$$

(iii) Since it was proved in [24, Theorem 1(ii)] that $\dot{\mathcal{E}}^\alpha_{u,p,q}(\mathbb{R}^n) = \dot{F}^{\alpha,\frac{1}{p}-\frac{1}{u}}_{p,q}(\mathbb{R}^n)$ with equivalent quasi-norms, it follows that Theorem 1.5(i) also gives a new characterization of the homogeneous Triebel–Lizorkin–Morrey space $\dot{\mathcal{E}}^\alpha_{u,p,q}(\mathbb{R}^n)$.

Similarly, we have the following equivalent characterization of the Besov–Morrey space via the differences $\{f - B_{\ell,2^{-k}} f\}_{k\in\mathbb{Z}}$.

Theorem 1.7 *Let $\ell \in \mathbb{N}$ and $\alpha \in (0, 2\ell)$. Let $q \in (0,\infty]$ and $1 < p \le u < \infty$.*
If $f \in \dot{\mathcal{N}}^\alpha_{u,p,q}(\mathbb{R}^n)$, then there exists $g \in L^p_{\rm loc}(\mathbb{R}^n) \cap \mathcal{S}'_\infty(\mathbb{R}^n)$ such that $g = f$ in $\mathcal{S}'_\infty(\mathbb{R}^n)$ and

$$|||g|||_{\dot{\mathcal{N}}^\alpha_{u,p,q}(\mathbb{R}^n)} \le C\|f\|_{\dot{\mathcal{N}}^\alpha_{u,p,q}(\mathbb{R}^n)} \tag{1.4}$$

with C being a positive constant independent of f, where

$$|||g|||_{\dot{\mathcal{N}}^\alpha_{u,p,q}(\mathbb{R}^n)} := \Big\{ \sum_{k\in\mathbb{Z}} 2^{k\alpha q} \sup_{Q\in\mathcal{Q}} \frac{1}{|Q|^{(1/p-1/u)q}} \Big\| g - B_{\ell,2^{-k}} g \Big\|^q_{L^p(Q)} \Big\}^{1/q}.$$

Conversely, if $f \in L^1_{\rm loc}(\mathbb{R}^n) \cap \mathcal{S}'_\infty(\mathbb{R}^n)$ and $|||f|||_{\dot{\mathcal{N}}^\alpha_{u,p,q}(\mathbb{R}^n)} < \infty$, then

$$f \in \dot{\mathcal{N}}^\alpha_{u,p,q}(\mathbb{R}^n) \quad \textit{and} \quad \|f\|_{\dot{\mathcal{N}}^\alpha_{u,p,q}(\mathbb{R}^n)} \le C\, |||f|||_{\dot{\mathcal{N}}^\alpha_{u,p,q}(\mathbb{R}^n)}$$

with C being a positive constant independent of f.

This article is organized as follows. We prove Theorems 1.3, 1.5, and 1.7 in Section 2 by borrowing some ideas from the proof of [8, Theorem 1.3]. The main step in the proofs of Theorems 1.3(i) and 1.5(i) consists in writing $f - B_{\ell,2^{-k}} f$ as a sum of convolutions by using the representation of f obtained in [36, Proposition 8.2] via the Calderón reproducing formula. Then the pointwise estimates of the related convolution kernels established in [8] are used to control the difference $f - B_{\ell,2^{-k}} f$ by certain maximal functions. Finally, by establishing a Fefferman–Stein vector-valued inequality of the Hardy–Littlewood maximal operator (see Proposition 2.3 below) on Morrey spaces and the fact that, when $\tau \in [0, 1/p)$, the sum $\sum_{k=-\log_2 l(Q)}^\infty$ in definitions of $\dot{B}^{\alpha,\tau}_{p,q}(\mathbb{R}^n)$ and $\dot{F}^{\alpha,\tau}_{p,q}(\mathbb{R}^n)$ can be equivalently replaced by $\sum_{j\in\mathbb{Z}}$ (see Remark 1.2(iii)), we complete the proofs of Theorems 1.3(i) and 1.5(i). Observing that, when p, q, τ,

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and α being as in Theorem 1.3(ii) or Theorem 1.5(ii), the Besov-type or the Triebel–Lizorkin-type spaces simply become the Besov or the Triebel–Lizorkin spaces (see Lemma 2.5). Thus, by this and [8, Theorem 1.3], the proofs of Theorems 1.3(ii) and 1.5(ii) become trivial.

In Section 3, we further obtain the inhomogeneous variants of Theorems 1.3 and 1.5 (see Theorems 3.3 and 3.4 below).

As mentioned in Remark 1.6(i), when $\tau = 0$, Theorems 1.3 and 1.5 go back to the characterizations of $\dot{B}^{\alpha}_{p,q}(\mathbb{R}^n)$ and $\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)$ via averages on balls for all $\alpha \in (0,\infty)$ obtained in [8, Theorem 1.3], which was also proved in [35, Theorem 1.1] whenever $\alpha \in (0,2)$. It should be pointed out that, for the higher smoothness order $\alpha \in (2N, 2N+2)$ with $N \in \mathbb{N}$, [35, Theorem 1.2] presents another equivalent characterization of $\dot{B}^{\alpha}_{p,q}(\mathbb{R}^n)$ and $\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)$ via Taylor expansions, which is different from Theorems 1.3 and 1.5. We also point out that, differently from the Littlewood–Paley type characterizations in Theorems 1.3 and 1.5, the last two authors in [34] characterized $\dot{B}^{\alpha}_{p,q}(\mathbb{R}^n)$ and $\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)$ with $\alpha \in (0,\infty)$, $p \in (1,\infty]$, and $q \in (0,\infty]$ ($q \in (1,\infty]$ for $\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)$) via some pointwise inequalities involving ball averages, which are in spirit more close to the pointwise characterizations of these spaces via Hajłasz gradient sequences obtained in [21].

Finally, we make some conventions on notation in this article. Let $\mathbb{N} := \{1, 2, \dots\}$ and $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$. We denote by C a *positive constant* that is independent of the main parameters, but may depend on n, α, τ, p, q, or u and vary from line to line. The *symbol* $f \lesssim g$ means $f \le Cg$. If $f \lesssim g$ and $g \lesssim f$, then we write $f \sim g$. For any $s \in \mathbb{R}$, we use the *symbol* $\lfloor s \rfloor$ to denote the maximal integer that is less or equal to s.

2 Proofs of Theorems 1.3, 1.5, and 1.7

In this section, we give the proofs of Theorems 1.3, 1.5, and 1.7. We begin with the following notation. For all $x \in \mathbb{R}^n$, let $J(x) := \frac{1}{|B(0,1)|}\chi_{B(0,1)}(x)$ and, for all $t \in (0,\infty)$, $J_t(x) := t^{-n}J(x/t)$. Then, for all $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) \cap \mathcal{S}'(\mathbb{R}^n)$, $t \in (0,\infty)$ and $x \in \mathbb{R}^n$,

$$(B_{\ell,t}f)(x) = -\frac{2}{\binom{2\ell}{\ell}} \sum_{j=1}^{\ell} (-1)^j \binom{2\ell}{\ell - j} (f * J_{jt})(x)$$

and hence, for all $\xi \in \mathbb{R}^n$,

$$(B_{\ell,t}f)^\wedge(\xi) = m_\ell(t\xi)\widehat{f}(\xi), \tag{2.1}$$

where, for all $x \in \mathbb{R}^n$, $m_\ell(x) := -\frac{2}{\binom{2\ell}{\ell}} \sum_{j=1}^{\ell}(-1)^j\binom{2\ell}{\ell-j}\widehat{J}(jx)$.

The following lemma is just [8, Lemma 2.1].

Lemma 2.1 *For all $\ell \in \mathbb{N}$ and $s \in \mathbb{R}$, let*

$$A_l(s) := \gamma_n \frac{4^\ell}{\binom{2\ell}{\ell}} \int_0^1 (1-u^2)^{\frac{n-1}{2}} \Big(\sin\frac{us}{2}\Big)^{2\ell}\, du,$$

where $\gamma_n := [\int_0^1 (1-u^2)^{\frac{n-1}{2}}\, du]^{-1}$. Then, for all $x \in \mathbb{R}^n$, $m_\ell(x) = 1 - A_\ell(|x|)$. Furthermore, $(\cdot)^{-2\ell}A_\ell(\cdot)$ is a smooth function on $\mathbb{R}$ and there exist positive constants c_1 and

c_2 such that, for all $s \in (0,4]$, $c_1 \le s^{-2\ell} A_\ell(s) \le c_2$ and, for all $i \in \mathbb{N}$,

$$\sup_{s\in\mathbb{R}} \Big| \Big(\frac{d}{ds}\Big)^i \Big(\frac{A_\ell(s)}{s^{2\ell}}\Big) \Big| < \infty.$$

Recall that the *Hardy–Littlewood maximal operator* $\mathcal{M}$ is defined by setting, for all $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, $\mathcal{M}(f)(x) := \sup_{B\ni x} \frac{1}{|B|}\int_B |f(y)|\,dy$, where the supremum is taken over all balls B in $\mathbb{R}^n$ containing x. Notice that, for all $f \in \mathcal{S}'(\mathbb{R}^n)$ and $m \in \mathcal{S}(\mathbb{R}^n)$, $m * f$ is a well-defined smooth function (see [20, Theorem 2.3.20]). Then we have the following conclusion, which is a slight variant of [8, Lemma 2.2], the details being omitted.

Lemma 2.2 *Let $\{T_t\}_{t\in(0,\infty)}$ be a family of operators given by*

$$T_t f(x) := \big([m(t\cdot)]^\vee\big) * f(x), \quad x \in \mathbb{R}^n,\ t \in (0,\infty),\ f \in \mathcal{S}'(\mathbb{R}^n) \cap L^1_{\mathrm{loc}}(\mathbb{R}^n),$$

for some $m \in \mathcal{S}(\mathbb{R}^n)$. Then there exists a positive constant C such that for all $f \in \mathcal{S}'(\mathbb{R}^n) \cap L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and all $x \in \mathbb{R}^n$,

$$\sup_{t\in(0,\infty)} \big|T_t f(x)\big| \le C\Big[\,\|\nabla^{n+1} m\|_{L^1(\mathbb{R}^n)} + \|m\|_{L^1(\mathbb{R}^n)}\Big]\mathcal{M}(f)(x).$$

We also need the following Fefferman–Stein vector-valued inequality with respect to the Hardy–Littlewood maximal function. The scalar case can be found in [4] or [1].

Proposition 2.3 *Let $p \in (1,\infty)$ and $\tau \in [0, 1/p)$.*

(i) *If $q \in (1,\infty)$, then there exists a positive constant C such that for all sequences $\{f_j\}_{j\in\mathbb{Z}}$ of locally integrable functions on $\mathbb{R}^n$,*

$$\sup_{Q\in\mathcal{Q}} \frac{1}{|Q|^\tau} \Big\| \Big\{ \sum_{j\in\mathbb{Z}} [\mathcal{M}(f_j)]^q \Big\}^{\frac{1}{q}} \Big\|_{L^p(Q)} \le C \sup_{Q\in\mathcal{Q}} \frac{1}{|Q|^\tau} \Big\| \Big\{ \sum_{j\in\mathbb{Z}} |f_j|^q \Big\}^{\frac{1}{q}} \Big\|_{L^p(Q)}.$$

(ii) *If $q \in (0,\infty)$, then there exists a positive constant C such that for all sequences $\{f_j\}_{j\in\mathbb{Z}}$ of locally integrable functions on $\mathbb{R}^n$,*

$$\sup_{Q\in\mathcal{Q}} \frac{1}{|Q|^\tau} \Big[\sum_{j\in\mathbb{Z}} \|\mathcal{M}(f_j)\|^q_{L^p(Q)} \Big]^{\frac{1}{q}} \le C \sup_{Q\in\mathcal{Q}} \frac{1}{|Q|^\tau} \Big[\sum_{j\in\mathbb{Z}} \|f_j\|^q_{L^p(Q)} \Big]^{\frac{1}{q}}. \tag{2.2}$$

We observe that the conclusion of Proposition 2.3(i) is a slight variant of [27, Lemma 2.5], in which the supremum is taken over all balls, the details of its proof being omitted. Moreover, the proof of Proposition 2.3(ii) is similar to that of [27, Lemma 2.5]; for completeness, we give some details.

Proof of Proposition 2.3(ii) For any given cube $Q \in \mathcal{Q}$ and $j \in \mathbb{Z}$, let $f_j^0 := f_j\chi_{2Q}$ and, for each $i \in \mathbb{N}$, $f_j^{(i)} := f_j\chi_{(2^{i+1}Q)\setminus(2^iQ)}$. Then for all $x \in \mathbb{R}^n$,

$$f_j(x) = \sum_{i=0}^\infty f_j^{(i)}(x).$$

When $q \in (1, \infty)$, by the Fatou Lemma and the Minkowski inequality, we have

$$\Big[\sum_{j\in\mathbb{Z}}\big\|\mathcal{M}(f_j)\big\|^q_{L^p(Q)}\Big]^{\frac{1}{q}} \le \sum_{i=0}^{\infty}\Big\{\sum_{j\in\mathbb{Z}}\big\|\mathcal{M}\big(f_j^{(i)}\big)\big\|^q_{L^p(Q)}\Big\}^{\frac{1}{q}} =: \sum_{i=0}^{\infty}\mathrm{H}(i, Q). \tag{2.3}$$

For $i = 0$, notice that there exist 4^n dyadic cubes $\{R_\eta\}_{\eta=1}^{4^n}$ of $\mathbb{R}^n$ such that $l(R_\eta) = l(Q)$ and $2Q \subset \bigcup_{\eta=1}^{4^n} R_\eta$. By this and the boundedness of $\mathcal{M}$ on $L^p(\mathbb{R}^n)$ for $p \in (1,\infty)$ (see [15] or [26]), we know that

$$\begin{aligned}\mathrm{H}(0, Q) &\lesssim \Big\{\sum_{j\in\mathbb{Z}}\big\|f_j\big\|^q_{L^p(2Q)}\Big\}^{\frac{1}{q}} \lesssim \sum_{\eta=1}^{4^n}\Big\{\sum_{j\in\mathbb{Z}}\big\|f_j\big\|^q_{L^p(R_\eta)}\Big\}^{\frac{1}{q}} \\ &\lesssim |Q|^\tau \sup_{\widetilde{Q}\in\mathcal{Q}}\frac{1}{|\widetilde{Q}|^\tau}\Big[\sum_{j\in\mathbb{Z}}\big\|f_j\big\|^q_{L^p(\widetilde{Q})}\Big]^{\frac{1}{q}}.\end{aligned} \tag{2.4}$$

For $i \in \mathbb{N}$, by the definition of $\mathcal{M}$, the fact that $\operatorname{supp} f_j^{(i)} \subset (2^{i+1}Q)\backslash(2^iQ)$, and the Hölder inequality, we find that, for any $x \in Q$,

$$\mathcal{M}\big(f_j^{(i)}\big)(x) \lesssim [2^i l(Q)]^{-n}\int_{2^{i+1}Q}|f_j(y)|\,dy \lesssim [2^i l(Q)]^{-\frac{n}{p}}\|f_j\|_{L^p(2^{i+1}Q)}. \tag{2.5}$$

Since there exist 4^n dyadic cubes $\{\widetilde{R}_\eta\}_{\eta=1}^{4^n}$ satisfying that

$$2^{i+1}Q \subset \bigcup_{\eta=1}^{4^n}\widetilde{R}_\eta \quad \text{and} \quad l(\widetilde{R}_\eta) = 2^i l(Q),$$

from this and (2.5), we deduce that

$$\begin{aligned}\mathrm{H}(i, Q) &\lesssim 2^{-\frac{n}{p}i}\Big\{\sum_{j\in\mathbb{Z}}\|f_j\|^q_{L^p(2^{i+1}Q)}\Big\}^{\frac{1}{q}} \lesssim 2^{-\frac{n}{p}i}\sum_{\eta=1}^{4^n}\Big\{\sum_{j\in\mathbb{Z}}\|f_j\|^q_{L^p(\widetilde{R}_\eta)}\Big\}^{\frac{1}{q}} \\ &\lesssim 2^{(\tau-\frac{1}{p})ni}|Q|^\tau \sup_{\widetilde{Q}\in\mathcal{Q}}\frac{1}{|\widetilde{Q}|^\tau}\Big[\sum_{j\in\mathbb{Z}}\big\|f_j\big\|^q_{L^p(\widetilde{Q})}\Big]^{\frac{1}{q}}.\end{aligned} \tag{2.6}$$

Combining (2.3), (2.4), (2.6), and the fact that $\tau \in [0, 1/p)$, we conclude that, for any given dyadic cube Q,

$$\Big[\sum_{j\in\mathbb{Z}}^{\infty}\big\|\mathcal{M}(f_j)\big\|^q_{L^p(Q)}\Big]^{\frac{1}{q}} \lesssim |Q|^\tau \sup_{\widetilde{Q}\in\mathcal{Q}}\frac{1}{|\widetilde{Q}|^\tau}\Big[\sum_{j\in\mathbb{Z}}\big\|f_j\big\|^q_{L^p(\widetilde{Q})}\Big]^{\frac{1}{q}},$$

which implies that (2.2) holds true. This finishes the proof of Proposition 2.3(ii). ∎

From the proofs of [36, Proposition 8.2] and [16, (2.6)], we easily deduce the following conclusion, the details being omitted.

Lemma 2.4 *Let $p \in (1, \infty]$ ($p \in (1,\infty)$ for $\dot{F}^{\alpha,\tau}_{p,q}(\mathbb{R}^n)$), $q \in (0, \infty]$, $\tau \in [0, \infty)$, and $\alpha \in (0, \infty)$. Then $\dot{F}^{\alpha,\tau}_{p,q}(\mathbb{R}^n)$ (or $\dot{B}^{\alpha,\tau}_{p,q}(\mathbb{R}^n)$) $\subset L^p_{\mathrm{loc}}(\mathbb{R}^n)$ in the sense of $\mathcal{S}'_\infty(\mathbb{R}^n)$; precisely, for any $f \in \dot{F}^{\alpha,\tau}_{p,q}(\mathbb{R}^n)$ or $\dot{B}^{\alpha,\tau}_{p,q}(\mathbb{R}^n)$, there exists a sequence $\{P_j\}_{j\in\mathbb{Z}}$ of polynomials of degree not more than $\lfloor \alpha + n\tau - n/p \rfloor$ such that $\sum_{j\in\mathbb{Z}}(\psi_j * \varphi_j * f + P_j)$ converges in $L^p_{\mathrm{loc}}(\mathbb{R}^n)$ to a function $g \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ and $f = g$ in $\mathcal{S}'_\infty(\mathbb{R}^n)$, where ψ and φ are as in Remark 1.2(ii).*

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To show Theorems 1.3 and 1.5, we also need the following Lemma 2.5. The conclusions (i) and (iii) of Lemma 2.5 were obtained in [33, Theorem 1], and (ii) was established in [17, Corollary 5.7]. Recall that, for any $s \in \mathbb{R}$ and $q \in (0, \infty]$, the space $\dot{F}^s_{\infty,q}(\mathbb{R}^n)$ is defined to be the set of all $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$ such that

$$\|f\|_{\dot{F}^s_{\infty,q}(\mathbb{R}^n)} := \sup_{Q \in \mathcal{Q}} \Big\{ \frac{1}{|Q|} \int_Q \sum_{k=-\log_2 l(Q)}^{\infty} 2^{ksq} |\varphi_k * f(y)|^q \, dy \Big\}^{1/q} < \infty$$

with the usual modification made when $q = \infty$ (see [17]), and the space $\dot{B}^s_{\infty,\infty}(\mathbb{R}^n)$ is defined to be the sets of all $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$ such that

$$\|f\|_{\dot{B}^s_{\infty,\infty}(\mathbb{R}^n)} := \sup_{k \in \mathbb{Z}} \sup_{x \in \mathbb{R}^n} 2^{ks} |f * \varphi_k(x)| < \infty.$$

Lemma 2.5 *Let $s \in \mathbb{R}$.*

(i) *For all $p \in (0, \infty)$, $q \in (0, \infty)$, and $\tau \in (1/p, \infty)$, or $q = \infty$ and $\tau \in [1/p, \infty)$, it holds true that $\dot{F}^{s,\tau}_{p,q}(\mathbb{R}^n) = \dot{F}^{s+n(\tau-1/p)}_{\infty,\infty}(\mathbb{R}^n)$ with equivalent quasi-norms.*

(ii) *For all $p \in (0, \infty)$, $q \in (0, \infty)$, it holds true that $\dot{F}^{s,1/p}_{p,q}(\mathbb{R}^n) = \dot{F}^s_{\infty,q}(\mathbb{R}^n)$ with equivalent quasi-norms.*

(iii) *For all $p \in (0, \infty]$, $q \in (0, \infty)$, and $\tau \in (1/p, \infty)$, or $q = \infty$ and $\tau \in [1/p, \infty)$, it holds true that $\dot{B}^{s,\tau}_{p,q}(\mathbb{R}^n) = \dot{B}^{s+n(\tau-1/p)}_{\infty,\infty}(\mathbb{R}^n)$ with equivalent quasi-norms.*

Now we prove Theorem 1.3.

Proof of Theorem 1.3 To prove this theorem, we let φ and ψ be as in Remark 1.2(ii).

We first show (i). Let $f \in \dot{B}^{\alpha,\tau}_{p,q}(\mathbb{R}^n)$ and

$$g := \sum_{j \in \mathbb{Z}} (\psi_j * \varphi_j * f + P_j) \tag{2.7}$$

be as in Lemma 2.4. Then by Lemma 2.4, we know that $g \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ and $f = g$ in $\mathcal{S}'_\infty(\mathbb{R}^n)$. Thus, we identify f with g in the proof below.

Now we prove (1.3). To this end, let Q be a given dyadic cube. Notice that the degree of each P_j is not more than $\lfloor \alpha + n\tau - \frac{n}{p} \rfloor < 2\ell$ and, for any $k \in \mathbb{Z} \cap [-\log_2 l(Q), \infty)$ and any polynomial P of degree less than 2ℓ, $P - B_{\ell,2^{-k}} P = 0$. Then it follows from (2.7) that for all $k \in \mathbb{Z}$,

$$g - B_{\ell,2^{-k}} g = \Big(\sum_{j \geq k} + \sum_{j < k} \Big) (I - B_{\ell,2^{-k}})(\psi_j * \varphi_j * f), \tag{2.8}$$

where I denotes the identity operator.

For the sum $\sum_{j \geq k}$, by (1.1) and the fact that, for all $j \in \mathbb{Z}$ and $h \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, $|\psi_j * h| \lesssim \mathcal{M}(h)$, we know that for all $x \in \mathbb{R}^n$,

$$\Big| (I - B_{\ell,2^{-k}})(\psi_j * \varphi_j * f)(x) \Big| \lesssim \mathcal{M}\Big(\mathcal{M}(\varphi_j * f) \Big)(x). \tag{2.9}$$

Using this, the Minkowski inequality, and the Hardy-type inequality (see [10, Lemma 2.3.4] or [8, Lemma 2.4]), we find that

$$\sum_{k\in\mathbb{Z}} 2^{k\alpha q}\Big\|\sum_{j\geq k}(I-B_{\ell,2^{-k}})(\psi_j*\varphi_j*f)\Big\|^q_{L^p(Q)} \lesssim \sum_{k\in\mathbb{Z}} 2^{k\alpha q}\Big[\sum_{j\geq k}\Big\|\mathcal{M}\Big(\mathcal{M}(\varphi_j*f)\Big)\Big\|_{L^p(Q)}\Big]^q \lesssim \sum_{k\in\mathbb{Z}} 2^{k\alpha q}\Big\|\mathcal{M}\Big(\mathcal{M}(\varphi_k*f)\Big)\Big\|^q_{L^p(Q)}. \tag{2.10}$$

For the sum $\sum_{j<k}$, from (2.1) and Lemma 2.1, we deduce that, for all $\xi\in\mathbb{R}^n$,

$$\Big[(I-B_{\ell,2^{-k}})(\psi_j*\varphi_j*f)\Big]^\wedge(\xi) = \widehat{\psi}(2^{-j}\xi)[1-m_\ell(2^{-k}\xi)](\varphi_j*f)^\wedge(\xi) = \widehat{\psi}(2^{-j}\xi)A_\ell(2^{-k}\xi)(\varphi_j*f)^\wedge(\xi),$$

where $A_\ell(\xi):=1-m_\ell(\xi)$ for all $\xi\in\mathbb{R}^n$. Thus, we obtain

$$(I-B_{\ell,2^{-k}})(\psi_j*\varphi_j*f)(x) = \Big([m^\ell_{k,j}(2^{-j}\cdot)]^\vee\Big)*\varphi_j*f(x), \quad \text{for all } x\in\mathbb{R}^n, \tag{2.11}$$

where $m^\ell_{k,j}(\xi):=\widehat{\psi}(\xi)A_\ell(2^{j-k}|\xi|)$ for all $\xi\in\mathbb{R}^n$.

Since $\widehat{\psi}\in\mathcal{S}(\mathbb{R}^n)$ has compact support, it follows from Lemma 2.1 that, for any $k,\ j\in\mathbb{Z}$, $m^\ell_{k,j}\in\mathcal{S}(\mathbb{R}^n)$. Moreover, observing that $\psi\in\mathcal{S}(\mathbb{R}^n)$ and $\operatorname{supp}\widehat{\psi}\subset\{\xi\in\mathbb{R}^n: \frac12\leq|\xi|\leq2\}$, by Lemma 2.1 again, we then know that, for all $j<k$, $\beta\in\mathbb{Z}^n_+$ and $\xi\in\mathbb{R}^n$,

$$|\partial^\beta m^\ell_{k,j}(\xi)|\lesssim 2^{2\ell(j-k)}\chi_{\{\xi\in\mathbb{R}^n:\ \frac12\leq|\xi|\leq2\}}(\xi)$$

and hence $\|m^\ell_{k,j}\|_{L^1(\mathbb{R}^n)}+\|\nabla^{n+1}m^\ell_{k,j}\|_{L^1(\mathbb{R}^n)}\lesssim 2^{2\ell(j-k)}$. Using this, (2.11), and Lemma 2.2, we find that, for all $j\in\mathbb{Z}$ with $j<k$ and $x\in\mathbb{R}^n$,

$$\Big|(I-B_{\ell,2^{-k}})(\psi_j*\varphi_j*f)(x)\Big|\lesssim 2^{2\ell(j-k)}\mathcal{M}(\varphi_j*f)(x). \tag{2.12}$$

Therefore, from (2.12), the Hardy-type inequality (see [10, Lemma 2.3.4] or [8, Lemma 2.4]), and the fact that $\alpha\in(0,2\ell)$, we conclude that

$$\sum_{k\in\mathbb{Z}} 2^{k\alpha q}\Big\|\sum_{j<k}(I-B_{\ell,2^{-k}})(\psi_j*\varphi_j*f)\Big\|^q_{L^p(Q)} \lesssim \sum_{k\in\mathbb{Z}} 2^{k(\alpha-2\ell)q}\Big[\sum_{j<k}2^{2\ell j}\|\mathcal{M}(\varphi_j*f)\|_{L^p(Q)}\Big]^q \lesssim \sum_{k\in\mathbb{Z}} 2^{k\alpha q}\|\mathcal{M}(\varphi_k*f)\|^q_{L^p(Q)}. \tag{2.13}$$

Combining (2.8), (2.10), (2.13), Proposition 2.3(ii), and Remark 1.2(iii), we conclude that

$$|||g|||_{\dot{B}^{\alpha,\tau}_{p,q}(\mathbb{R}^n)} \lesssim \sup_{Q\in\mathcal{Q}}\frac{1}{|Q|^\tau}\Big\{\sum_{k\in\mathbb{Z}}2^{k\alpha q}\Big\|\mathcal{M}\Big(\mathcal{M}(\varphi_k*f)\Big)\Big\|^q_{L^p(Q)}\Big\}^{1/q} \lesssim \sup_{Q\in\mathcal{Q}}\frac{1}{|Q|^\tau}\Big\{\sum_{k\in\mathbb{Z}}2^{k\alpha q}\|\varphi_k*f\|^q_{L^p(Q)}\Big\}^{1/q}\sim\|f\|_{\dot{B}^{\alpha,\tau}_{p,q}(\mathbb{R}^n)},$$

which implies (1.3).

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Conversely, let $f \in L^1_{\rm loc}(\mathbb{R}^n) \cap \mathcal{S}'_\infty(\mathbb{R}^n)$ and $|||f|||_{\dot{B}^{\alpha,\tau}_{p,q}(\mathbb{R}^n)} < \infty$. Then by [8, (2.16)], we find that, for all $k \in \mathbb{Z}$ and $x \in \mathbb{R}^n$,

$$|f * \varphi_k(x)| \lesssim \mathcal{M}(f - B_{\ell,2^{-k}} f)(x). \tag{2.14}$$

Thus, from this and Proposition 2.3(ii), we deduce that

$$\begin{aligned}\|f\|_{\dot{B}^{\alpha,\tau}_{p,q}(\mathbb{R}^n)} &\lesssim \sup_{Q\in\mathcal{Q}} \frac{1}{|Q|^\tau} \Big\{ \sum_{k\in\mathbb{Z}} 2^{k\alpha q} \big\| \mathcal{M}(f - B_{\ell,2^{-k}} f) \big\|^q_{L^p(Q)} \Big\}^{1/q} \\ &\lesssim \sup_{Q\in\mathcal{Q}} \frac{1}{|Q|^\tau} \Big\{ \sum_{k\in\mathbb{Z}} 2^{k\alpha q} \big\| f - B_{\ell,2^{-k}} f \big\|^q_{L^p(Q)} \Big\}^{1/q} \sim |||f|||_{\dot{B}^{\alpha,\tau}_{p,q}(\mathbb{R}^n)},\end{aligned}$$

namely, $f \in \dot{B}^{\alpha,\tau}_{p,q}(\mathbb{R}^n)$ and $\|f\|_{\dot{B}^{\alpha,\tau}_{p,q}(\mathbb{R}^n)} \lesssim |||f|||_{\dot{B}^{\alpha,\tau}_{p,q}(\mathbb{R}^n)}$. This finishes the proof of Theorem 1.3(i).

Next we prove (ii) by considering two cases.

Case I: $p \in (0,\infty)$, $\tau \in (1/p,\infty)$, and $q \in (0,\infty]$ or $p \in (0,\infty)$, $\tau = \frac{1}{p}$, and $q = \infty$ or $p = \infty$, $\tau \in (0,\infty)$, and $q \in (0,\infty]$. In this case, by Lemma 2.5(iii), it holds true that $\dot{B}^{\alpha,\tau}_{p,q}(\mathbb{R}^n) = \dot{B}^{\alpha+n(\tau-1/p)}_{\infty,\infty}(\mathbb{R}^n)$. From this and [8, Theorem 1.3(i)], we further deduce that, in this case, the conclusion (ii) of Theorem 1.3 holds true.

Case II: $p = \infty$, $\tau = 0$ and $q \in (0,\infty]$. In this case, $\dot{B}^{\alpha,\tau}_{p,q}(\mathbb{R}^n) = \dot{B}^{\alpha}_{\infty,q}(\mathbb{R}^n)$. Then the desired conclusion also follows from [8, Theorem 1.3(i)]. This finishes the proof of Theorem 1.3(ii) and hence the proof of Theorem 1.3. ■

Proof of Theorem 1.5 The proof of (i) is similar to that of Theorem 1.3(i), the details being omitted.

To prove (ii), if $\tau \in (1/p,\infty)$ and $q \in (1,\infty]$, then, by Lemma 2.5(i), we know that

$$\dot{F}^{\alpha,\tau}_{p,q}(\mathbb{R}^n) = \dot{F}^{\alpha+n(\tau-1/p)}_{\infty,\infty}(\mathbb{R}^n). \tag{2.15}$$

Moreover, by Lemma 2.5(ii), we find that, for all $q \in (0,\infty]$, $\dot{F}^{\alpha,1/p}_{p,q}(\mathbb{R}^n) = \dot{F}^{\alpha}_{\infty,q}(\mathbb{R}^n)$. Therefore, by this, (2.15) and [8, Theorem 1.3(ii)], we conclude that the conclusion (ii) of Theorem 1.5 holds true. This finishes the proof of Theorem 1.5. ■

Proof of Theorem 1.7 Let $f \in \dot{\mathcal{N}}^{\alpha}_{u,p,q}(\mathbb{R}^n)$. Then, by (i) and (iii) of Definition 1.1, we find that $f \in \dot{B}^{\alpha,\frac{1}{p}-\frac{1}{u}}_{p,q}(\mathbb{R}^n)$. Thus, by Theorem 1.3, we know that there exists $g \in L^p_{\rm loc}(\mathbb{R}^n)$ such that $g = f$ in $\mathcal{S}'_\infty(\mathbb{R}^n)$. Moreover, (2.8) holds true.

Next we show (1.4). By (2.9), the Minkowski inequality, the Hardy-type inequality (see [10, Lemma 2.3.4] or [8, Lemma 2.4]) and the boundedness of $\mathcal{M}$ on the Morrey space (see Proposition 2.3(ii)), we find that

$$\begin{aligned}&\sum_{k\in\mathbb{Z}} 2^{k\alpha q} \sup_{Q\in\mathcal{Q}} \frac{1}{|Q|^{(1/p-1/u)q}} \Big\| \sum_{j\ge k} (I - B_{\ell,2^{-k}})(\psi_j * \varphi_j * f) \Big\|^q_{L^p(Q)} \\ &\quad\lesssim \sum_{k\in\mathbb{Z}} 2^{k\alpha q} \Big[\sum_{j\ge k} \sup_{Q\in\mathcal{Q}} \frac{1}{|Q|^{1/p-1/u}} \Big\| \mathcal{M}\Big(\mathcal{M}(\varphi_j * f)\Big) \Big\|_{L^p(Q)} \Big]^q \\ &\quad\lesssim \sum_{k\in\mathbb{Z}} 2^{k\alpha q} \sup_{Q\in\mathcal{Q}} \frac{1}{|Q|^{(1/p-1/u)q}} \Big\| \varphi_k * f \Big\|^q_{L^p(Q)}.\end{aligned} \tag{2.16}$$

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On the other hand, by (2.12), the Minkowski inequality and the Hardy-type inequality again (see [10, Lemma 2.3.4] or [8, Lemma 2.4]), we know that

$$
\begin{aligned}
(2.17)\qquad &\sum_{k\in\mathbb{Z}} 2^{k\alpha q} \sup_{Q\in\mathcal{Q}} \frac{1}{|Q|^{(1/p-1/u)q}} \Big\| \sum_{j<k} (I-B_{\ell,2^{-k}})(\psi_j * \varphi_j * f)\Big\|_{L^p(Q)}^q \\
&\lesssim \sum_{k\in\mathbb{Z}} 2^{k(\alpha-2\ell)q} \Big[\sum_{j<k} 2^{2\ell j} \sup_{Q\in\mathcal{Q}} \frac{1}{|Q|^{1/p-1/u}} \Big\| \mathcal{M}\Big(\varphi_j * f\Big)\Big\|_{L^p(Q)}\Big]^q \\
&\lesssim \sum_{k\in\mathbb{Z}} 2^{k\alpha q} \sup_{Q\in\mathcal{Q}} \frac{1}{|Q|^{(1/p-1/u)q}} \|\varphi_k * f\|_{L^p(Q)}^q.
\end{aligned}
$$

Combining (2.8), (2.16) and (2.17), we conclude that (1.4) holds true.

Conversely, let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)\cap \mathcal{S}'_\infty(\mathbb{R}^n)$ and $|||f|||_{\dot{\mathcal{N}}^{\alpha}_{u,p,q}(\mathbb{R}^n)} < \infty$. Then, from (2.14) and the boundedness of $\mathcal{M}$ on the Morrey space (see Proposition 2.3(ii)), we deduce that

$$
\begin{aligned}
\|f\|_{\dot{\mathcal{N}}^{\alpha}_{u,p,q}(\mathbb{R}^n)} &\lesssim \Big\{\sum_{k\in\mathbb{Z}} 2^{k\alpha q} \sup_{Q\in\mathcal{Q}} \frac{1}{|Q|^{(1/p-1/u)q}} \Big\|\mathcal{M}(f-B_{\ell,2^{-k}}f)\Big\|_{L^p(Q)}^q\Big\}^{1/q} \\
&\lesssim \Big\{\sum_{k\in\mathbb{Z}} 2^{k\alpha q} \sup_{Q\in\mathcal{Q}} \frac{1}{|Q|^{(1/p-1/u)q}} \Big\|f-B_{\ell,2^{-k}}f\Big\|_{L^p(Q)}^q\Big\}^{1/q} \sim |||f|||_{\dot{\mathcal{N}}^{\alpha}_{u,p,q}(\mathbb{R}^n)},
\end{aligned}
$$

which implies that $f \in \dot{\mathcal{N}}^{\alpha}_{u,p,q}(\mathbb{R}^n)$. This finishes the proof of Theorem 1.7. ∎

3 Inhomogeneous Spaces

In this section, we aim to present the inhomogeneous versions of Theorems 1.3 and 1.5. For the corresponding conclusions in case of Besov and Triebel–Lizorkin spaces we refer the reader to [8, Theorem 3.1].

A pair of functions (φ,Φ) is said to be *admissible* if $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfies (1.2), $\Phi \in \mathcal{S}(\mathbb{R}^n)$ satisfies $\operatorname{supp}\widehat{\Phi} \subset \{\xi\in\mathbb{R}^n : |\xi|\le 2\}$, and $|\widehat{\Phi}(\xi)| \ge \text{constant} > 0$ when $|\xi| \le \frac{5}{3}$.

Definition 3.1 Let $\tau \in [0,\infty)$, $\alpha\in\mathbb{R}$, $p,\ q\in(0,\infty]$, and (φ,Φ) be a pair of admissible functions. Then the *inhomogeneous Besov-type space* $B^{s,\tau}_{p,q}(\mathbb{R}^n)$ and the *inhomogeneous Triebel–Lizorkin-type space* $F^{s,\tau}_{p,q}(\mathbb{R}^n)$ $(p\in(0,\infty))$ are, respectively, defined to be the sets of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$
\|f\|_{B^{s,\tau}_{p,q}(\mathbb{R}^n)} := \sup_{Q\in\mathcal{Q}} \frac{1}{|Q|^\tau}\Big\{\sum_{k=\max\{0,-\log_2 l(Q)\}}^{\infty} 2^{k\alpha q}\|\varphi_k * f\|_{L^p(Q)}^q\Big\}^{1/q} < \infty,
$$

$$
\|f\|_{F^{s,\tau}_{p,q}(\mathbb{R}^n)} := \sup_{Q\in\mathcal{Q}} \frac{1}{|Q|^\tau}\Big\|\Big\{\sum_{k=\max\{0,-\log_2 l(Q)\}}^{\infty} 2^{k\alpha q}|\varphi_k * f|^q\Big\}^{1/q}\Big\|_{L^p(Q)} < \infty,
$$

where φ_0 is replaced by Φ.

In what follows, denote by $\mathcal{C}(\mathbb{R}^n)$ the set of all complex-valued uniformly continuous functions on $\mathbb{R}^n$ equipped with the sup-norm and by $\mathcal{C}^\infty(\mathbb{R}^n)$ the set of all smooth functions on $\mathbb{R}^n$. Let $\Psi\in\mathcal{C}^\infty(\mathbb{R}^n)$ be a radial function with compact support such that, when $|x|\le 1$, $\Psi(x) = 1$ and, when $|x|\ge 3/2$, $\Psi(x)=0$. If we let $\Psi^0 := \Psi$ and

$\Psi^{(j)}(\cdot) := \Psi(2^{-j}\cdot) - \Psi(2^{-j+1}\cdot)$ for all $j \in \mathbb{N}$, then we obtain a smooth decomposition of unity, namely, for all $x \in \mathbb{R}^n$, $\sum_{j=0}^{\infty} \Psi^{(j)}(x) = 1$. For all $x \in \mathbb{R}^n$, let

$$\varphi_0(x) := \widehat{\Psi}(-x) \quad \text{and} \quad \varphi := \widehat{\Psi(2\cdot)}(-x). \tag{3.1}$$

Then, for all $j \in \mathbb{N}$ and $x \in \mathbb{R}^n$, $\varphi_j(x) := 2^{jn}\varphi(2^j x) = \widehat{\Psi^{(j)}}(-x)$.

For any $1 < p \le u < \infty$, let $\mathcal{M}_p^u(\mathbb{R}^n)$ denote the well-known *Morrey space* that is defined to be the set of all p-locally integrable functions f such that

$$\|f\|_{\mathcal{M}_p^u(\mathbb{R}^n)} := \sup_{Q \in \mathcal{Q}} \frac{1}{|Q|^{1/p-1/u}} \left\{ \int_Q |f(x)|^p \, dx \right\}^{1/p} < \infty.$$

We first present the following technical lemma.

Lemma 3.2 *Let $q \in (0,\infty]$, $s \in (0,\infty)$, $p \in [1,\infty]$, and $\tau \in [0, 1/p)$.*

(i) *For all $f \in B_{p,q}^{s,\tau}(\mathbb{R}^n)$, it holds true that*

$$\|f\|_{B_{p,q}^{s,\tau}(\mathbb{R}^n)} \sim \|f\|_{\mathcal{M}_p^{p/(1-p\tau)}(\mathbb{R}^n)} + \widetilde{\|f\|}_{B_{p,q}^{s,\tau}(\mathbb{R}^n)}$$

with the implicit equivalent positive constants independent of f, where

$$\widetilde{\|f\|}_{B_{p,q}^{s,\tau}(\mathbb{R}^n)} := \sup_{Q \in \mathcal{Q}} \frac{1}{|Q|^\tau} \left\{ \sum_{k=1}^{\infty} 2^{k\alpha q} \|\varphi_k * f\|_{L^p(Q)}^q \right\}^{1/q}.$$

(ii) *For all $f \in F_{p,q}^{s,\tau}(\mathbb{R}^n)$, it holds true that*

$$\|f\|_{F_{p,q}^{s,\tau}(\mathbb{R}^n)} \sim \|f\|_{\mathcal{M}_p^{p/(1-p\tau)}(\mathbb{R}^n)} + \widetilde{\|f\|}_{F_{p,q}^{s,\tau}(\mathbb{R}^n)}$$

with the implicit equivalent positive constants independent of f, where

$$\widetilde{\|f\|}_{F_{p,q}^{s,\tau}(\mathbb{R}^n)} := \sup_{Q \in \mathcal{Q}} \frac{1}{|Q|^\tau} \left\| \left\{ \sum_{k=1}^{\infty} 2^{k\alpha q} |\varphi_k * f|^q \right\}^{1/q} \right\|_{L^p(Q)}.$$

Proof By similarity, we only prove (ii). From [36, Proposition 2.1(ii)], we deduce that, for all $\tau \in [0,\infty)$, $s \in (0,\infty)$, $p \in [1,\infty]$ and $q \in (0,\infty]$, $F_{p,q}^{s,\tau}(\mathbb{R}^n) \hookrightarrow F_{p,1}^{0,\tau}(\mathbb{R}^n)$, which, together with the fact that

$$F_{p,1}^{0,\tau}(\mathbb{R}^n) \hookrightarrow \mathcal{M}_p^{p/(1-p\tau)}(\mathbb{R}^n)$$

(see [36, Proposition 2.7(i)]), implies that

$$F_{p,q}^{s,\tau}(\mathbb{R}^n) \hookrightarrow \mathcal{M}_p^{p/(1-p\tau)}(\mathbb{R}^n).$$

More precisely, let φ_0 and φ be as in (3.1). Then $f = \sum_{j=0}^{\infty} \varphi_j * f$ in $\mathcal{S}'(\mathbb{R}^n)$, and it was proved in [36, Proposition 2.7(i)] that $\sum_{j=0}^{\infty} \varphi_j * f$ converges in $\mathcal{M}_p^{p/(1-p\tau)}(\mathbb{R}^n)$. Moreover, for any $f \in F_{p,q}^{s,\tau}(\mathbb{R}^n)$, we have

$$\|f\|_{\mathcal{M}_p^{p/(1-p\tau)}(\mathbb{R}^n)} \lesssim \|f\|_{F_{p,q}^{s,\tau}(\mathbb{R}^n)}. \tag{3.2}$$

On the other hand, due to $\tau \in [0, 1/p)$, by [36, Corollary 3.3(i)], we find that, for any $f \in F_{p,q}^{s,\tau}(\mathbb{R}^n)$,

$$\|f\|_{F_{p,q}^{s,\tau}(\mathbb{R}^n)} \sim \sup_{Q \in \mathcal{Q}} \frac{1}{|Q|^\tau} \left\| \left\{ \sum_{k=0}^{\infty} 2^{k\alpha q} |\varphi_k * f|^q \right\}^{1/q} \right\|_{L^p(Q)}.$$

Thus, if $f \in F^{s,\tau}_{p,q}(\mathbb{R}^n)$, then, by the boundedness of $\mathcal{M}$ on the Morrey space (see, for example, [1, Theorem 12]) and the fact that $|\Phi * f| \lesssim \mathcal{M}(f)$, we conclude that

$$\|f\|_{F^{s,\tau}_{p,q}(\mathbb{R}^n)} \lesssim \|\Phi * f\|_{\mathcal{M}^{p/(1-p\tau)}_{p}(\mathbb{R}^n)} + \|\widetilde{f}\|_{F^{s,\tau}_{p,q}(\mathbb{R}^n)} \lesssim \|f\|_{\mathcal{M}^{p/(1-p\tau)}_{p}(\mathbb{R}^n)} + \|\widetilde{f}\|_{F^{s,\tau}_{p,q}(\mathbb{R}^n)}.$$

Conversely, from (3.2), we easily deduce that for any $f \in F^{s,\tau}_{p,q}(\mathbb{R}^n)$,

$$\|f\|_{\mathcal{M}^{p/(1-p\tau)}_{p}(\mathbb{R}^n)} + \|\widetilde{f}\|_{F^{s,\tau}_{p,q}(\mathbb{R}^n)} \lesssim \|f\|_{F^{s,\tau}_{p,q}(\mathbb{R}^n)}.$$

This finishes the proof of Lemma 3.2. ∎

By Lemma 3.2, [33, Theorem 2], [36, Proposition 2.4(iii)], [8, Theorem 3.1], and some arguments similar to those used in the proof of Theorem 1.3, we obtain the following results, the details being omitted.

Theorem 3.3 *Let $\ell \in \mathbb{N}$ and $\alpha \in (0, 2\ell)$.*

(i) *Let $p \in (1,\infty)$, $\tau \in [0, 1/p)$, and $q \in (0,\infty]$. Then $f \in B^{\alpha,\tau}_{p,q}(\mathbb{R}^n)$ if and only if $f \in \mathcal{S}'(\mathbb{R}^n) \cap \mathcal{M}^{p/(1-p\tau)}_{p}(\mathbb{R}^n)$ and*

$$|||f|||_{B^{\alpha,\tau}_{p,q}(\mathbb{R}^n)} := \|f\|_{\mathcal{M}^{p/(1-p\tau)}_{p}(\mathbb{R}^n)} + \sup_{Q\in\mathcal{Q}} \frac{1}{|Q|^{\tau}} \Big\{ \sum_{k=1}^{\infty} 2^{k\alpha q} \|f - B_{\ell,2^{-k}} f\|^{q}_{L^p(Q)} \Big\}^{1/q}$$

is finite. Moreover, $|||\cdot|||_{B^{\alpha,\tau}_{p,q}(\mathbb{R}^n)} \sim \|f\|_{B^{\alpha,\tau}_{p,q}(\mathbb{R}^n)}$ with the implicit equivalent positive constants independent of f.

(ii) *Let $p \in (0,\infty)$, $\tau \in (1/p, \infty)$, and $q \in (0,\infty]$ or $p \in (0,\infty)$, $\tau = \frac{1}{p}$, and $q = \infty$ or $p = \infty$, $\tau \in [0,\infty)$ and $q \in (0,\infty]$. Then $f \in B^{\alpha,\tau}_{p,q}(\mathbb{R}^n)$ if and only if $f \in \mathcal{C}(\mathbb{R}^n)$ and*

$$|||f|||_{B^{\alpha,\tau}_{p,q}(\mathbb{R}^n)} := \|f\|_{L^\infty(\mathbb{R}^n)} + \sup_{k\in\mathbb{N}} 2^{k[\alpha+n(\tau-1/p)]} \|f - B_{\ell,2^{-k}} f\|_{L^\infty(\mathbb{R}^n)} < \infty.$$

Theorem 3.4 *Let $\ell \in \mathbb{N}$ and $\alpha \in (0, 2\ell)$.*

(i) *Let $p \in (1,\infty)$, $\tau \in [0, 1/p)$, and $q \in (1,\infty]$. Then $f \in F^{\alpha,\tau}_{p,q}(\mathbb{R}^n)$ if and only if $f \in \mathcal{S}'(\mathbb{R}^n) \cap \mathcal{M}^{p/(1-p\tau)}_{p}(\mathbb{R}^n)$ and*

$$|||f|||_{F^{\alpha,\tau}_{p,q}(\mathbb{R}^n)} := \|f\|_{\mathcal{M}^{p/(1-p\tau)}_{p}(\mathbb{R}^n)} + \sup_{Q\in\mathcal{Q}} \frac{1}{|Q|^{\tau}} \Big\| \Big\{ \sum_{k=1}^{\infty} 2^{k\alpha q} |f - B_{\ell,2^{-k}} f|^{q} \Big\}^{1/q} \Big\|_{L^p(Q)}$$

is finite. Moreover, $|||\cdot|||_{F^{\alpha,\tau}_{p,q}(\mathbb{R}^n)} \sim \|f\|_{F^{\alpha,\tau}_{p,q}(\mathbb{R}^n)}$ with the implicit equivalent positive constants independent of f.

(ii) *Let $p \in (0,\infty)$, $\tau \in [\frac{1}{p}, \infty)$ and $q \in (1,\infty]$. Then $f \in F^{\alpha,\tau}_{p,q}(\mathbb{R}^n)$ if and only if $f \in \mathcal{M}^{p/(1-p\tau)}_{p}(\mathbb{R}^n) \cap \mathcal{S}'_\infty(\mathbb{R}^n)$ and $|||f|||_{F^{\alpha,\tau}_{p,q}(\mathbb{R}^n)}$ is finite, where, when $\tau \in (1/p, \infty)$ and $q \in (0,\infty]$, or $\tau = 1/p$ and $q = \infty$,*

$$|||f|||_{F^{\alpha,\tau}_{p,q}(\mathbb{R}^n)} := \|f\|_{\mathcal{M}^{p/(1-p\tau)}_{p}(\mathbb{R}^n)} + \sup_{k\in\mathbb{N}} \sup_{y\in\mathbb{R}^n} 2^{k[\alpha+n(\tau-1/p)]} |g(y) - B_{\ell,2^{-k}} g(y)|,$$

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and when $\tau = 1/p$ and $q \in (0,\infty)$,

$$\begin{aligned}|||f|||_{F^{\alpha,\tau}_{p,q}(\mathbb{R}^n)} &:= \|f\|_{L^\infty(\mathbb{R}^n)} \\ &\quad + \sup_{\substack{Q\in\mathcal{Q}\\ \ell(Q)\le 1}} \left\{ \frac{1}{|Q|}\int_Q \sum_{k=-\log_2 l(Q)}^{\infty} 2^{k[\alpha+n(\tau-1/p)]}|f(y)-B_{\ell,2^{-k}}f(y)|^q\,dy\right\}^{1/q}.\end{aligned}$$

Moreover, $\|f\|_{F^{\alpha,\tau}_{p,q}(\mathbb{R}^n)} \sim |||f|||_{F^{\alpha,\tau}_{p,q}(\mathbb{R}^n)}$ with the implicit equivalent positive constants independent of f.

We end this section by the following interesting remark.

Remark 3.5 (i) Let $m \in \mathbb{N}$ and $1 < p \le u < \infty$. Then by [25, Theorem 3.1], we know that

$$F^{m,\frac{1}{p}-\frac{1}{u}}_{p,2}(\mathbb{R}^n) = W^m\mathcal{M}^u_p(\mathbb{R}^n),$$

where $W^m\mathcal{M}^u_p(\mathbb{R}^n)$ denotes the *Sobolev–Morrey space*, which is defined to be the set of all $f \in \mathcal{M}^u_p(\mathbb{R}^n)$ such that all distributional derivatives $D^\alpha f$ of order $\alpha \in \mathbb{Z}^n_+$ with $|\alpha| \le m$ belong to $\mathcal{M}^u_p(\mathbb{R}^n)$, equipped with the norm

$$\|f\|_{W^m\mathcal{M}^u_p(\mathbb{R}^n)} := \sum_{|\alpha|\le m}\|D^\alpha f\|_{\mathcal{M}^u_p(\mathbb{R}^n)}.$$

From this and Theorem 3.4, we easily deduce that, for all $\ell \in \mathbb{N}$, $m \in \mathbb{N}\cap(0,2\ell)$ and $1 < p \le u < \infty$, $f \in W^m\mathcal{M}^u_p(\mathbb{R}^n)$ if and only if $f \in \mathcal{S}'(\mathbb{R}^n)\cap\mathcal{M}^u_p(\mathbb{R}^n)$ and

$$|||f|||_{W^m\mathcal{M}^u_p(\mathbb{R}^n)} := \|f\|_{\mathcal{M}^u_p(\mathbb{R}^n)} + \left\|\left\{\sum_{k=1}^{\infty}2^{2k\alpha}|f-B_{\ell,2^{-k}}f|^2\right\}^{1/2}\right\|_{\mathcal{M}^u_p(\mathbb{R}^n)} < \infty;$$

moreover, $|||f|||_{W^m\mathcal{M}^u_p(\mathbb{R}^n)}$ is equivalent to $\|f\|_{W^m\mathcal{M}^u_p(\mathbb{R}^n)}$ with the implicit equivalent positive constants independent of f.

(ii) Similar result for the inhomogeneous Besov–Morrey space $\mathcal{N}^\alpha_{u,p,q}(\mathbb{R}^n)$ (see [22,27,36]) corresponding to Theorem 3.3(i) also holds true, the details being omitted here.

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