VECTOR LATTICES WITH DUALS OF INTEGRAL TYPE

H. W. ELLIS*

1. Introduction. Marston Morse and William Transue **(4; 5; 6)** have developed a theory of seminormed Banach spaces of complex or real-valued functions called MT spaces. For E a locally compact space, $\Re^E$ the space of real-valued functions on E, and K the space of real-valued functions on E with compact supports, a real MT space A is a vector subspace of $\Re^E$ containing K and provided with a suitable seminorm N^A. Each element ϕ of the N^A-dual A^* of A determines a Radon measure μ_φ on E, and $\phi(f) = \int f\, d\mu_\varphi$ is valid on A where the integration is in the Bourbaki sense. Problems considered include the compatibility of different MT spaces and the relations between the MT spaces and the spaces $L_1(\phi)$ of integrals determined by the measure dual.

In this paper $\Re^E$ is replaced by an arbitrary σ-Dedekind complete space R and K by a Riesz subspace C of R. Certain order-bounded linear functionals ϕ on C, called integrals, replace the Radon measure functionals and determine integration spaces $L_1(\phi)$ by an extension method of Nakano **(7)**. If A is a Riesz subspace of R containing C and N^A is a suitable seminorm on A, every element ϕ of A^* determines an integral on C for which ϕ is the Nakano extension on A. Such a pair (A, N^A) is called an abstract Morse–Transue or AMT space. The principal MT results extend to the abstract case, the arguments now being lattice theoretic. Recent work of Luxemburg and Zaanen **(3)**, in part drawing on earlier work by Nakano, helps to clarify and extend the theory (§5 below). The theory of MT spaces and part of the theory of L^λ spaces **(2)** is included in the AMT theory for suitable choices of R and C.

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2. Integrals and their Nakano extensions. We shall use in general the definitions of **(3)**. A vector or linear lattice R is σ-Dedekind complete (sequentially continuous) if $a_n \in R^+$, $n = 1, 2, \ldots$ implies that $\wedge_1^\infty a_n \in R$; Dedekind complete (universally continuous) if for every collection $a_\lambda \in R^+$, $\lambda \in \Lambda$, $\wedge_{\lambda \in \Lambda} a_\lambda \in R$; super Dedekind complete (super-universally continuous) if it is Dedekind complete and such that there exists a countable subsequence $a_{\lambda i}$, $\lambda_i \in \Lambda$, with $\wedge_i a_{\lambda i} = \wedge_{\lambda \in \Lambda} a_\lambda$. A linear subspace of a linear lattice is called a Riesz subspace if it is closed under the lattice operations. A Riesz subspace of a σ-Dedekind complete vector lattice R is called a sequentially continuous or

σ-Dedekind complete subspace if it is a σ-Dedekind complete vector space contained in R that is closed under countable infima and suprema. A Riesz subspace S of R is an ideal if $x \in S, a \in R$ with $|a| \leqslant |x|$ implies that $a \in S$.

Let C be a Riesz subspace of a σ-Dedekind complete vector lattice R, $C^{\sim}$ the set of order-bounded linear functionals on C. If $u_n \in C^+, n = 1, 2, \ldots, \wedge_{n=1}^{\infty} u_n$ exists in R^+, the elements being considered as elements of R. If $u_n \downarrow$ and $\wedge u_n = 0$ in R, we write $u_n \downarrow 0 \ (R)$. Observe that if C is not sequentially continuous, the collection $\{u_n\}$ need not have an infimum in C or may have an infimum different from that in R.

Let $C_c^{\sim} = (\phi \in C^{\sim}: u_n \downarrow 0$ implies $\phi(u_n) \to 0$ as $n \to \infty)$ **(3**, VI**)**; $C_{\bar{c}}^{\sim} = (\phi \in C^{\sim}: u_n \downarrow 0 \ (R)$ implies that $\phi(u_n) \to 0$ as $n \to \infty)$. Always $C_{\bar{c}}^{\sim} \supset C_c^{\sim} \supset \{0\}$. It is shown in **(3**, VII, p. 674**)** that for $C = C[0, 1]$, the continuous functions on $[0, 1]$, $C_c^{\sim} = \{0\}$. For E a locally compact space, C the continuous functions on E with compact supports, $R = \Re^E$, $u_n \downarrow 0 \ (R)$ implies that $u_n \to 0$ uniformly and this implies that $C_{\bar{c}}^{\sim} = C^{\sim}$. Here again $C_c^{\sim}$ may reduce to $\{0\}$. The elements of $C_{\bar{c}}^{\sim}$ will be called C-integrals or just integrals. When C is a sequentially continuous Riesz subspace of R, $C_{\bar{c}}^{\sim} = C_c^{\sim}$. The spaces $C_c^{\sim}$ and $C_{\bar{c}}^{\sim}$ are normal subspaces of the universally continuous linear lattice $C^{\sim}$ ordered pointwise on C, the arguments for $C_{\bar{c}}^{\sim}$ being analogous to those for $C_c^{\sim}$ **(3**, VI**)**.

Using the method of Nakano **(7)** we extend each positive $\phi \in C_{\bar{c}}^{\sim}$ to upper and lower functionals on R. For each $x \in R$ we set

$\phi^*(x) = \inf (\lim_{i \to \infty} \phi(a_i)$, for all increasing sequences $(a_i\uparrow), a_i \in C$,

$$\text{with } \vee (x \wedge a_i) = x \text{ (in } R))$$

$= +\infty$ if no such sequence exists;

$\phi_*(x) = \sup (\lim_{i \to \infty} \phi(b_i)$, for all sequences $b_i\downarrow, b_i \in C$,

$$\text{with } \wedge (x \vee b_i) = x \text{ (in } R))$$

$= -\infty$, if no such sequence exists.

As shown in **(7)** the functionals ϕ^* and ϕ_* have the following properties (with values in the extended reals):

N1. $\phi^*(0) = \phi_*(0) = 0$.

N2. $\phi^*(-x) = -\phi_*(x), \phi_*(x) = -\phi^*(x)$.

N3. $\phi^*(ax) = a\phi^*(x), \phi_*(ax) = a\phi_*(x)$ if $a \geqslant 0$.

N4. $\phi^*(x) + \phi^*(y) \geqslant \phi^*(x + y) \geqslant \phi^*(x) + \phi_*(y)$,

when the sums have sense. Similarly

$$\phi_*(x) + \phi_*(y) \leqslant \phi_*(x + y) \leqslant \phi^*(x) + \phi_*(y).$$

N5. $\phi^*(x) + \phi^*(y) \geqslant \phi^*(x \vee y) + \phi^*(x \wedge y)$,

$$\phi_*(x) + \phi_*(y) \leqslant \phi_*(x \vee y) + \phi_*(x \wedge y).$$

N6. $\phi^*(x) \geqslant \phi_*(x)$.

N7. $\phi^*(x) \geqslant \phi^*(y), \phi_*(x) \geqslant \phi_*(y)$ if $x \geqslant y$.

N8. $\lim \phi^*(x_n) = \phi^*(x)$ if $x_n \uparrow x$.

N9. $\lim \phi_*(x_n) = \phi_*(x)$ if $x_n \downarrow x$.

N10. $\phi^*(a) = \phi_*(a) = \phi(a)$ for all $a \in C$.

We note the following additional properties:

2.1. If $x_n \uparrow x \ (x_n \downarrow x), x_n \in C, x \in R$, then $\phi^*(x) = \phi_*(x)$.

If $x_n \uparrow x, \phi_*(x) \geqslant \phi(x_n), n = 1, 2$, by N7 whence

$$\phi_*(x) \geqslant \lim_{n \to \infty} \phi(x_n) = \phi^*(x),$$

by N8. With N6 this gives the first part of 2.1.

2.2. From N4 and N10, if $x \in R, u \in C$,

$$\phi^*(x + u) = \phi^*(x) + \phi(u), \qquad \phi_*(x + u) = \phi_*(x) + \phi(u).$$

2.3. From 2.2 and N2, for all $u \in C$,

$$\phi^*(x) - \phi_*(x) = \phi^*(x + u) - \phi_*(x + u)$$
$$= \phi^*(x - u) + \phi^*(u - x).$$

For $\varphi \geqslant 0$ let $L_1(\phi)$ denote the elements $x \in R$ for which $\phi^*(x) = \phi_*(x)$, the common value being finite. Then $L_1(\phi)$ is a sequentially continuous Riesz subspace of R containing C. Setting $\phi = \phi^* = \phi_*$ on $L_1(\phi)$, ϕ is linear on $L_1(\phi)$.

LEMMA 2.1. *Let L be an arbitrary Riesz space and suppose that $u_n \wedge f \uparrow f$, $v_n \wedge f \uparrow f$. Then if $w_n = u_n \wedge v_n, w_n \wedge f \uparrow f$.*

Proof. Clearly $f \wedge w_n \leqslant f, n = 1, 2, \ldots$. Let h be an arbitrary upper bound of this set. Then

$$h \geqslant u_m \wedge v_m \wedge f \geqslant u_n \wedge v_m \wedge f, \qquad n \text{ fixed, } m > n,$$
$$\geqslant u_n \wedge f, \qquad\qquad\qquad n = 1, 2, \ldots.$$

Thus $h \geqslant f$ and $f = \vee (w_n \wedge f)$.

THEOREM 2.1. (i) *If $\phi \in C_{\bar{c}}^{\sim}$ and $a \geqslant 0$, then $a \phi^* = (a\phi)^*$ and $a\phi_* = (a\phi)_*$ on R.* (ii) *If $\phi_1, \phi_2 \in C_{\bar{c}}^{\sim+}$, then $(\phi_1 + \phi_2)^* = \phi_1^* + \phi_2^*$ and $(\phi_1 + \phi_2)_* = \phi_{1*} + \phi_{2*}$ on R^+.* (iii) *If $\phi_1 \geqslant \phi_2 \geqslant 0$ on C^+, $\phi_1^* \geqslant \phi_2^*$ and $\phi_{1*} \geqslant \phi_{2*}$ on R^+.*

Proof of (ii). Let $\phi = \phi_1 + \phi_2$ in $C_{\bar{c}}^{\sim+}$. Let $\{u_n\}$ be a monotone sequence in C^+. Then $0 \leqslant \phi(u_n) = \phi_1(u_n) + \phi_2(u_n) < \infty, n = 1, 2, \ldots$, and

$$\lim_{n \to \infty} \phi(u_n) = \lim_{n \to \infty} \phi_1(u_n) + \lim_{n \to \infty} \phi_2(u_n)$$

in the non-negative extended reals $\bar{\Re}^+$. If $u \in R^+$ with $\phi_1^*(u) < \infty$ and $\phi_2^*(u) < \infty$, there are sequences $u_n\uparrow$, $u_n \wedge u \uparrow u$ with the limits on the right simultaneously near $\phi_1^*(u)$ and $\phi_2^*(u)$ (using Lemma 2.1). This implies that

(*) $$\phi^*(u) \leqslant \phi_1^*(u) + \phi_2^*(u).$$

Since $\phi^*(u)$ is finite, there exist such sequences with $\lim \phi(u_n)$ arbitrarily near $\phi^*(u)$, leading to the opposite inequality and thus to equality in (*). Similar considerations show that if one side is infinite so is the other. Similar arguments using decreasing sequences give the result for the lower extensions.

THEOREM 2.2. *If* ϕ_1, $\phi_2 \in C_{\tilde{c}}^{\sim+}$ *and* $\phi = \phi_1 + \phi_2$, *then*

$$L_1(\phi) = L_1(\phi_1) \cap L_1(\phi_2).$$

Proof. On $L_1(\phi_1) \cap L_1(\phi_2)$, $\phi^{+*} = \phi^+{}_*$, $\phi^{-*} = \phi^-{}_*$ and all are finite. Thus Theorem 2.1 (ii) shows that $|\phi|^* = |\phi|_*$ on positive elements. If u is arbitrary, N4 and N2 then imply that

$$|\phi|^*(u) = |\phi|^*(u^+) - |\phi|^*(u^-) = |\phi|_*(u^+) - |\phi|_*(u^-) = |\phi|_*(u)$$

and $u \in L_1(|\phi|)$.

On $L_1(|\phi|)^+$, $|\phi|^* = |\phi|_*$ and both are finite. By Theorem 2.1, $\phi^{+*} + \phi^{-*} = |\phi|^* = |\phi|_* = \phi^+{}_* + \phi^-{}_*$. Since all are positive on positive elements and $\phi^{+*} \geqslant \phi^+{}_*$, $\phi^{-*} \geqslant \phi^-{}_*$, it follows that $\phi^{+*} = \phi^+{}_*$ and $\phi^{-*} = \phi^-{}_*$. Thus $L_1(|\phi|) \subset L_1(\phi_1) \cap L_1(\phi_2)$ and the proof is complete.

Let $\phi \in C_{\tilde{c}}^{\sim}$. Then $\phi = \phi^+ - \phi^-$ on C with ϕ^+, $\phi^- \in C_{\tilde{c}}^{\sim+}$. Writing $|\phi| = \phi^+ + \phi^-$, $|\varphi|$, φ^+, and φ^- on C have linear extensions $|\phi| = |\phi|^* = |\varphi|_*$, $\phi^+ = \phi^{+*} = \phi^+{}_*$, and $\phi^- = \phi^{-*} = \phi^-{}_*$ to $L_1(|\phi|)$. The linear extension of an arbitrary ϕ is now defined to be $\phi^+ - \phi^-$ on $L_1(|\phi|)$ and $L_1(\phi)$ is defined to coincide with $L_1(|\phi|) = L_1(\phi^+) \cap L_1(\phi^-)$. For all x in $L_1(\phi)$

$$\phi(x) = \phi^+(x) - \phi^-(x) = \phi^+(x^+) - \phi^+(x^-) - \phi^-(x^+) + \phi^-(x^-), |\phi(x)| \leqslant |\phi|(|x|)\cdot$$

A seminorm ρ on a Riesz space L is called a Riesz or monotone seminorm if $\rho(f) \leqslant \rho(g)$ when $|f| \leqslant |g|$. On $L_1(\phi) = L_1(|\phi|)$ $\rho\varphi$, defined by $\rho\varphi(u) = |\phi|(|u|)$, is a Riesz seminorm.

THEOREM 2.3. *C is dense in $L_1(\phi)$ for the $\rho\varphi$ seminorm topology.*

Proof. There is no loss of generality in assuming that $\phi > 0$ and $f > 0$. Then $\rho\varphi = \phi$ is linear on $L_1(\phi)^+$. From the definition of $\phi = \phi^*$ on $L_1(\phi)$, given $\epsilon > 0$ there exists a sequence $\{u_n\}$ in C with $u_n\uparrow$, $u_n \wedge f \uparrow f$, and

$$\lim_{n\to\infty} \phi(u_n) \geqslant \phi(f) = \phi^*(f) \geqslant \lim_{n\to\infty} \phi(u_n) - \epsilon/4, \qquad \phi(u_n \wedge f)\uparrow_{n=1}^{\infty} \phi(f).$$

Thus for n sufficiently large

$$|\phi(f) - \phi(u_n)| = |\phi(f - u_n)| < \epsilon/4,$$

$$0 \leqslant \phi(u_n) - \phi(u_n \wedge f) = \phi(u_n - u_n \wedge f) \leqslant |\phi(u_n - f)| + \phi(f - u_n \wedge f) < \epsilon/4.$$

Now

$$0 \leqslant |f - u_n| - (f - u_n) = 2(f - u_n)^- = 2(u_n - u_n \wedge f),$$

$$0 \leqslant \phi(|f - u_n|) - \phi(f - u_n) = 2\phi(u_n - u_n \wedge f) < \epsilon/2,$$

$$\phi(|f - u_n|) \leqslant |\phi(f - u_n)| + \epsilon/2 < \epsilon.$$

3. Abstract Morse–Transue (AMT) spaces. Fix R, a σ-Dedekind or Dedekind complete Riesz space, and C a Riesz subspace of R. Let A be a Riesz subspace of R containing C and suppose that A has a non-trivial Riesz seminorm N^A.

Let A^* denote the dual of A for the seminorm topology, $A_{\bar{c}}^* = A^* \cap A_{\bar{c}}^\sim$. For $\phi \in A^\sim$ let ϕ_c denote the restriction of ϕ to C. If $\phi \in A_{\bar{c}}^*$ and $u_n \downarrow 0$ (R) with the elements u_n in C and thus in A, $\phi_c(u_n) = \phi(u_n) \to 0$ as $n \to \infty$ so that $\phi_c \in C_{\bar{c}}^\sim$. Thus to each $\phi \in A_{\bar{c}}^*$ corresponds a space of integrable elements $L_1(\phi_c)$. We define

$$\Omega^A = \bigcap_{\varphi \in A_{\bar{c}}^*} L_1(\phi_c) = \bigcap_{0 \leqslant \varphi \in A_{\bar{c}}^*} L_1(\phi_c).$$

Completing the definition, A (or (A, N^A)) will be called an abstract Morse Transue (AMT) space if

  (i) $A \subset \Omega^A$.
  (ii) For every $\phi \in A_{\bar{c}}^*$, $\phi = \phi_c^* = \phi_{c*}$ on A.
  (iii) $A^* = A_{\bar{c}}^*$.

These conditions correspond respectively to the Morse–Transue conditions that: every $x \in A$ is integrable with respect to every measure in the measure dual, $\phi(x)$ can be recovered by integrating with respect to the measure corresponding to ϕ, and the measure dual is isomorphic to all of A^*. Clearly (ii) implies (i).

THEOREM 3.1. (A, N^A) is an AMT space (with respect to C and R) if and only if for every $\phi \in A^*$, $\phi_c^* = \phi_{c*} = \phi$ on A.

All that needs to be verified is that these conditions imply (iii). If $\phi \in A^*$ and $u_n \downarrow 0$ (R) with $u_n \in A$, $n = 1, 2, \dots$, $\phi_{c*}(u_n) \to 0$ by N9 so that $\phi(u_n) \to 0$ and $\phi \in A^*$.

THEOREM 3.2. If A is an AMT space, C is N^A-dense in A.

Proof. Assume the theorem to be false. Then the Hahn–Banach theorem asserts the existence of a point $x_0 \in A - C$ and $\phi' \in A^*$ with $\phi'(x) = 0$ in C, $\phi'(x_0) \neq 0$. Then $\phi'_c = 0$ on C, $\phi'_c{}^* = \phi'_{c*} = 0$ on A, contradicting (ii).

THEOREM 3.3. If (A, N^A) is an AMT space, $u_n \downarrow 0$ (R) implies that $N^A(u_n) \to 0$.

Proof. Since $A^* = A_{\bar{c}}^*$, $u_n \downarrow 0$ (R) implies that $\phi(u_n) \to 0$ for every $\phi \in A'$, i.e. $\{u_n\}$ converges weakly to zero. The argument of (**3**, VII, Lemma 22.6, p. 671) then applies to complete the proof.

COROLLARY. If (A, N^A) is an AMT space and $u_n \uparrow u$ (R) in A^+, then $N^A(u_n) \uparrow N^A(u)$.

Examples. For any Riesz seminorm N^C, (C, N^C) is an AMT space if and only if $C^* = C_{\bar{c}}^*$. Since $C^* \subset C^\sim$, this is always the case if $C^\sim = C_{\bar{c}}^\sim$.

If C is too small, the spaces in the preceding paragraph may be the only AMT spaces for C and R. For example, let X be an arbitrary set of points, $R = \mathfrak{R}^x$, where $\mathfrak{R}$ denotes the reals and R has the natural partial order. Let C

denote the constant functions on X. Then $C^{\sim} = C_{\bar{c}}^{\sim}$ is isomorphic to $\mathfrak{R}$. Let $u(x) = 1$ in X. Then for each $\phi \in C^{\sim}, f \in R$,

$$\phi^*(f) = (\sup_{x \in X} f(x))\phi(u) > (\inf_{x \in X} f(x))\phi(u) = \phi_*(f)$$

unless $f \in C$. Thus for any (A, N^4), $\Omega^4 = C$ and (i) implies that no A larger than C can be an AMT space.

For every $\phi \in C_{\bar{c}}^{\sim}$, $(L_1(\phi), \rho\varphi)$ is an AMT space.

We can assume that $\phi > 0$. Since $L_1(\phi)$ is σ-Dedekind complete, $u_n \downarrow f$ means the same in R and $L_1(\phi)$. Since $\rho\varphi$ and $\phi = \phi_*$ coincide on $L_1(\phi)^+$,

$$\rho\varphi(u_n) \downarrow 0 \qquad \text{if } u_n \downarrow 0,$$

by N9. This implies that every ϕ' in $L_1(\phi)^*$ is in $L_1(\phi^*)$ so that (iii) holds.

We next note that if $0 < \phi' \in L_1(\phi)^*$, $L_1(\phi'_c) > L_1(\phi)$. Since $\phi'_c \leqslant ||\phi'||\phi$ on C^+, $\phi'_c{}^* \leqslant (||\phi'||\phi)^* = ||\phi'||\phi^*$ on $L_1(\phi)^+ \subset R^+$ by Theorem 2.1.

If $f \in L_1(\phi)$ and $v \in C$, 2.3 implies that

$$0 \leqslant \phi'_c{}^*(f) - \phi'_{c*}(f) = \phi'_c{}^*(f - v) + \phi'_c{}^*(v - f)$$
$$\leqslant 2\phi'_c{}^*(|f - v|) \leqslant 2||\phi'||\phi(|f - v|).$$

Since C is $\rho\varphi$-dense in $L_1(\phi)$, $\phi'_c{}^*(f) = \phi'_{c*}(f)$ and $f \in L_1(\phi'_c)$.

To prove (ii) we note that for every $u \in C, f \in L_1(\varphi)$,

$$|\phi'_c(f) - \phi'(f)| \leqslant |\phi'_c(f) - \phi'_c(u)| + |\phi'(f) - \phi'(u)|$$
$$\leqslant 2||\phi'||\phi(|f - u|)$$

and the right side can be made arbitrarily small by choice of $u \in C$.

4. Extensions of AMT spaces and normed AMT spaces.

Let (A, N^4) be an arbitrary AMT space. From Banach space theory (iii) implies that for all $f \in A$,

$$N^4(f) = \sup_{\substack{\varphi \in A^* \\ ||\varphi|| \leqslant 1}} |\phi(f)|.$$

By (3, VII, Lemma 22.3), $||\phi|| = || |\phi| ||$ and, since $N^4(f) = N^4(|f|)$,

$$N^4(f) = \sup_{\substack{\phi \in A^* \\ ||\phi|| \leqslant 1}} |\phi|(|f|) = \sup_{\substack{\phi \in A^{*+} \\ ||\phi|| \leqslant 1}} \phi(|f|).$$

Using this formula, N^4 can be extended from A to Ω^4. Replacing ϕ by ϕ^* in $\varphi(|f|)$ gives an extension to all of R (valued in the non-negative extended reals). We shall use the notation N^4 also for the above extension. Then, on R, N^4 satisfies:

4.1. $0 \leqslant N^4(f) \leqslant +\infty$,

4.2. $N^4(af) = aN^4(f), a \geqslant 0$,

4.3. $N^4(f + g) \leqslant N^4(f) + N^4(g)$,

4.4. $N^4(f) \leqslant N^4(g)$ if $|f| \leqslant |g|$,

4.5. $N^4(f_n) \uparrow_{n=1}^{\infty} N^4(f)$ if $0 \leqslant f_n \uparrow f$.

From 4.4 we see that $N^4(f) = N^4(|f|)$. Property 4.5 is an easy consequence of N8 which is valid for all $\phi \in A^{*+}$.

Let $F^4 = [f \in R: N^4(f) < \infty]$. Then F^4 is an ideal of R and thus is σ-Dedekind complete. N^4 is a Riesz seminorm on F^4. F^4 is closed for the seminorm topology.

Set $\Phi^4 = [f \in R: N^4(f) = 0]$. Clearly $\Phi^4 \subset \Omega^4$, F^4.

Now $f \equiv g$ if $f - g \in \Phi^4$ defines equivalence relations on C, A, Ω^4, F^4, and R. Using boldface type for equivalence classes and spaces of equivalence classes,

$$\mathbf{f} = f + \Phi^4 \text{ in } R, F^4, \text{ and } \Omega^4; \; = f + \Phi^4 \cap A \text{ in } A;$$

$$= f + \Phi^4 \cap C \text{ in } C.$$

Then N^4 is uniquely defined on $\mathbf{R}$ and is a norm on $\mathbf{C}$, $\mathbf{A}$, and $\mathbf{F}^4$. Partially order $\mathbf{R}$ by $f \leqslant g$ if, for each $f \in \mathbf{f}$, $g \in \mathbf{g}$, $(f - g)^+ \in \Phi^4$. Then $\mathbf{R}$ is σ-Dedekind complete and $\mathbf{C}$ and $\mathbf{A}$ are Riesz subspaces of $\mathbf{R}$ with $\mathbf{C} \subset \mathbf{A} \subset \mathbf{R}$. We shall show that $(\mathbf{A}, N^4)$ is a normed AMT space with respect to $\mathbf{C}$ and $\mathbf{R}$.

Now A^* and $\mathbf{A}^*$ consist of the same elements. Suppose that $\mathbf{u}_n \downarrow \mathbf{0}$ (R). We shall show that for each $\varphi \in A^*$ there then exists a sequence $u'_n \downarrow 0$ (R) with $\{u'_n\} \subset L_1(\varphi)$, $u'_n \in \mathbf{u}_n$, $n = 1, 2, \ldots$. Since $A^* = A_{\bar{c}}^*$ by assumption, this will imply that (iii) holds for $\mathbf{A}$.

We first note that if $u \in \mathbf{u}$ and $\mathbf{u} \geqslant \mathbf{0}$, then $u^- \in \Phi^4 \cap A$ and $u^+ \in \mathbf{u}$. Let $u_1 > 0$ be an arbitrary element of $\mathbf{u}_1$, u_2 any element of $\mathbf{u}_2$. Since $\mathbf{u}_1 \geqslant \mathbf{u}_2$, $(u_2 - u_1)^+ \in \Phi^4 \cap A$. Set $u'_2 = [u_2 - (u_2 - u_1)^+]^+$. Then $u'_2 \in A$ and $u'_1 = u_1 \geqslant u_2$. The remaining elements u'_n are determined similarly by induction. Let $v = \wedge u'_n$, which exists in R since R is σ-Dedekind complete. Since $\mathbf{0} = \wedge \mathbf{u}_n$, $\mathbf{v} \leqslant \mathbf{0}$ and $v^+ \in \Phi^4$. Let $u''_n = u'_n - v^+$. Then $u''_n \downarrow 0$ in R and in $L_1(\phi)$ for every $\phi \in A^*$. It follows that $\lim \phi(\mathbf{u}_n) = \lim \phi(u''_n) = 0$ and $\phi \in \mathbf{A}_{\bar{c}}^*$.

We next show that (ii) holds for every $\phi \in \mathbf{A}^* = \mathbf{A}_{\bar{c}}^*$. Let $u \in \mathbf{u} \in A$, $\phi \in \mathbf{A}^{*+} = A^{*+}$. Then $\phi_c^*(\mathbf{u}) = \phi_c^*(u) = \phi_{c*}(u) = \phi_{c*}(\mathbf{u})$.

In the analogous MT case F^4 is always N^4-complete. Let $C = A = R$ denote the *bounded* Lebesque integrable functions on $(0, 1)$ seminormed by

$$N_1(f) = \int_0^1 |f| \, dx$$

and with identifications. Then A is an AMT space and $F^4 = R$ is not complete. That this example is typical is illustrated by the following theorem.

THEOREM 4.1. *The normed space* $\mathbf{F}^4$ *is* N^4-*complete if and only if* $\mathbf{u}_n \geqslant \mathbf{0}$, $n = 1, 2, \ldots,$

$$\sum_1^\infty N^4(\mathbf{u}_n) < \infty \quad \text{implies that} \quad \sum_1^\infty \mathbf{u}_n \in \mathbf{R}.$$

Proof. Completeness in norm implies this condition with $\sum_1^\infty \mathbf{u}_n$ in $\mathbf{F}^A$ by (**3**, VIII, Theorem 26.3). Conversely, if the condition is satisfied, $\sum_1^m \mathbf{u}_n \uparrow \sum_1^\infty \mathbf{u}_n$ so that, by 4.5 and 4.3,

$$N^A\left(\sum_1^\infty \mathbf{u}_n\right) = \lim_{m\to\infty} N^A\left(\sum_1^m \mathbf{u}_n\right) \leqslant \sum_1^\infty N^A(\mathbf{u}_n) < \infty,$$

and $\sum_1^\infty \mathbf{u}_n \in \mathbf{F}^A$. The theorem is then completed by applying Theorem 26.3 quoted above.

THEOREM 4.2. *If $f \in R$ and $N^A(f) = +\infty$, $f \notin \Omega^A$. Thus $\Omega^A \subset F^A$.* (Compare (**5**, Theorem 5.1, p. 351).)

Proof. By definition

$$N^A(f) = \sup_{\substack{\varphi \in A^{*+} \\ ||\varphi|| \leqslant 1}} \phi^*(|f|).$$

The hypothesis implies the existence in A^{*+} of a sequence $\{\phi_n\}$ with

$$\phi_n{}^*(|f|) > 2^n, \qquad ||\phi_n|| \leqslant 1, \qquad n = 1, 2, \ldots.$$

Set

$$\phi' = \sum_{n=1}^\infty \phi_n/2^n.$$

It is easily verified that $\phi' \in A^{*+}$ with $||\phi'|| \leqslant 1$.

By Theorem 2.1, for $N = 1, 2, \ldots$,

$$\phi' \geqslant \left(\sum_1^N \phi_n/2^n\right)^* = \sum_1^N \phi_n{}^*/2^n \qquad \text{on } R^+.$$

Thus

$$\phi'^*(|f|) \geqslant \sum_1^N \phi_n{}^*(|f|)/2^n > N$$

for every N. Thus $f \notin L_1(\phi')$ and therefore not in Ω^A.

5. Compatible AMT spaces. Two AMT spaces (A, N^A) and (B, N^B) (with respect to the same C and R) will be called compatible if $A^* = B^*$ and $N^A = N^B$ on C.

When A is not σ-Dedekind complete we let $A'^+ = [f \in R: f = \wedge u_n$ (in R) for some sequence $\{u_n\} \subset A^+]$ and set $A' = A'^+ - A'^+$. Then A' is the smallest σ-Dedekind complete Riesz subspace of R containing A. If $f \in A'^+$, there exist sequences $\{u_n\}, \{v_n\}$ in A^+ with $u_n \downarrow f, v_n \uparrow f$ (in R).

LEMMA 5.1. $A' \subset \Omega^A$.

Proof. Let $\phi \in A^{*+}, f \in A'^+, u_n \downarrow f, v_n \uparrow f$ (in R) with each u_n, v_n in A^+. Now

$$\phi^*(f) - \phi_*(f) = \phi^*(f) - \phi(v_n) + \phi(v_n - u_n) - \phi_*(f) - \phi(u_n).$$

Since $u_n - v_n \downarrow 0$ (R), $\phi(u_n - v_n) \to 0$. Furthermore $\phi^*(f) - \phi(v_n) \to 0$ by N8 and $\phi_*(f) - \phi(u_n) \to 0$ by N9. Thus $\phi^*(f) = \phi_*(f)$ and both are finite since f is bounded above in A. Thus $f \in L_1(\phi)$ for each positive ϕ and therefore is in Ω^A.

THEOREM 5.1. (A, N^4) and $(A,' N^{4'})$ are compatible AMT spaces.

Proof. Here $N^{4'}$ denotes the extension of N^4 from A to A'. It is thus sufficient to show that $A^* = A'^*$. If $\phi \in A^*$, $\phi^* = \phi_*$ is linear on A' since $A' \subset L_1(\phi)$. Furthermore, where $0 \leqslant u_n \uparrow f, u_n \in A, f \in A'$,

$$|\phi^*(f)| \leqslant |\phi|^*(f) = \lim_{n \to \infty} |\phi|(u_n) \leqslant ||\phi|| \lim N^4(u_n)$$
$$= ||\phi|| \, N^4(f),$$

using 4.5. Since $\geqslant$ is obvious, $||\phi^*||_{A'_*} = ||\phi||_{A_*}$. From N9 it follows that $\phi^* \in A_c'^*$.

Conversely, each ϕ in A'^{*+} determines a $\phi' \in A^* = A_{\bar{c}}^*$ by restriction to A. From linearity it follows that ϕ is monotone on A'^+. Thus if $f > 0$, $u_n \downarrow f$, $v_n \uparrow f$, $\{u_n\}, \{v_n\} \subset A$, $\phi(f - v_n), \phi(u_n - f) \leqslant \phi'(u_n - v_n)$. Since $u_n - v_n \downarrow 0 \, (R)$, $\phi'(u_n - v_n) \to 0$ and $\phi(f) = \lim \phi(u_n) = \lim \phi(v_n)$. It follows that $\phi = \phi'^*$ on A' so that the correspondence between A^* and A'^* is 1–1 and norm-preserving. This implies that $(A', N^{4'})$ is an AMT space and that this space is compatible with (A, N^4).

THEOREM 5.2. (i) $(\mathbf{A}', N^4)$ is super Dedekind complete and if $f_a, a \in \mathfrak{A}$, is a system of elements of $\mathbf{A}'^+$ filtering down to zero, $\mathbf{f}_a \downarrow_{a \in \mathfrak{a}} \mathbf{0}$, then $N^4(\mathbf{f}_a) \downarrow 0$. (ii) Every order-bounded increasing sequence in A' has a norm limit.

Proof. By Theorem 3.3 if $\mathbf{f}_n \downarrow \mathbf{0}$, $\mathbf{f}_n \in \mathbf{A}'$, $N^4(\mathbf{f}_n) \downarrow 0$. Since $\mathbf{A}'$ is σ-Dedekind complete, the conditions of $(3, \text{X}, \text{Theorem } 33.4, \text{p. } 503)$ are satisfied.

If (A, N^4) is an AMT space for C and R, (C, N^4) is compatible with (A, N^4). This follows from the definition. Obviously (C, N^4) is the smallest AMT space compatible with (A, N^4).

Let $\bar{A}$ denote the closure of A in F^4 for the N^4 seminorm topology. (Compare $(4, §13)$.)

THEOREM 5.3. $(\bar{A}, N^4)$ is an AMT space contained in Ω^4 and compatible with (A, N^4) and is maximal in the sense that every AMT space compatible with (A, N^4) is contained in $(\bar{A}, N^4)$.

Proof. We first note that $\bar{A}$ is a Riesz subspace of R. Let $u \in \bar{A}$. Since C is dense in A, there exist $u_n \in C$ with

$$\lim_{n \to \infty} N^4(u - u_n) = 0.$$

It follows easily that $\bar{A}$ is linear. Now $|u| \in R$ and each $|u_n| \in C$. Thus, since

$$||u| - |u_n|| = (|u| - |u_n|)^+ + (|u| - |u_n|)^-$$
$$\leqslant 2(|u - u_n|)^+ = 2|u - u_n|,$$
$$\lim_{n \to \infty} N^4(|u| - |u_n|) = 0,$$

and $|u| \in \bar{A}$. This implies that u^+ and $u^- \in \bar{A}$ and $\bar{A}$ is closed under the lattice operations.

On $\bar{A}$

$$N^4(u) = \sup_{\substack{\phi \in \bar{A}^{*+} \\ ||\phi|| \leqslant 1}} \phi^*(|u|).$$

With u, u_n as in the preceding paragraph, and each $\phi \in A^{*+}$, using 2.3,

$$\phi^*(u) - \phi_*(u) = \phi^*(u - u_n) + \phi^*(u_n - u) \leqslant 2||\phi|| \, N^A(u - u_n),$$

$n = 1, 2, \ldots$. It follows that $u \in \Omega^A$, $\bar{A} \subset \Omega^A$. Thus if $\phi \in A^*$, $\phi = \phi^*$ is linear on A. By continuity, ϕ is bounded on $\bar{A}$ with the same norm as on A. Using N9, each $\phi \in A^*$ extends to $\phi \in \bar{A}_c^*$.

Let $\phi \in \bar{A}^*$, $\phi' = \phi|A$. Then ϕ' extends to $\bar\phi'$ on $\bar{A}$. Let $u \in \bar{A}$, $N^A(u - u_n) \to 0$, $u_n \in C$. Then

$$|\phi(u) - \bar\phi'(u)| = |\phi(u - u_n) + \bar\phi'(u_n - u)| \leqslant (||\phi|| - ||\phi'||) N^A(u - u_n),$$

$n = 1, 2, \ldots$, and $\phi = \bar\phi'$.

COROLLARY 1. $\bar{A}$ *is complete if* F^A *is complete.*

COROLLARY 2. *If* $A = L_1(\phi)$, $A = \bar{A}$.

In this case $A = \Omega^A$ and is therefore maximal.

COROLLARY 3. $\bar{\mathbf{A}} \supset \mathbf{A}'$.

THEOREM 5.4. *Every* $\mathbf{f} \in \Omega^A$ *that is majorized above and below by elements of* $\mathbf{A}'$ *(equivalently of* $\mathbf{C}$*) is in* $\mathbf{A}'$, *i.e.* $\mathbf{A}'$ *is an ideal of* Ω^A.

Proof. Let $\mathbf{f} \in \Omega^{A+}$ and suppose that there exists $\mathbf{v} \in \mathbf{C}$ with $\mathbf{f} \leqslant \mathbf{v}$. Choose $0 \leqslant v \in \mathbf{v} \cap \mathbf{C}, f \in \mathbf{f} \cap \mathbf{A}'$ with $0 \leqslant f \leqslant v$, and let $\phi \in A^{*+}$. Then $\phi^*(f) < \infty$ and there exist sequences $u_{ni} \uparrow_{i=1}^{\infty}$ with $u_{ni} \wedge f \uparrow_i f$ and

$$\lim_{i \to \infty} \phi(u_{ni}) \geqslant \phi^*(f) \geqslant \lim_{i \to \infty} \phi(u_{ni}) - 1/n, \qquad n = 1, \ldots.$$

The sequences can be assumed to be bounded above by v and thus $f_n = \vee_i u_{ni}$ exists in A' since A' is σ-Dedekind complete. Using Lemma 2.1 and induction, it can be assumed that $f \leqslant f_n \downarrow_n$. Thus

$$\phi^*(f) \leqslant \phi(f_n) \leqslant \phi^*(f) - 1/n, \qquad n = 1, 2, \ldots,$$

and $\lim_{n \to \infty} \phi(f_n) = \phi^*(f)$.

Let $f_\varphi = \wedge f_n$. Then $\phi^*(f_\varphi) = \phi^*(f)$. Since $\mathbf{A}'$ is Dedekind complete by Theorem 5.2, $\mathbf{f}' = \wedge_{\varphi \in A^{*+}} \mathbf{f}_\varphi$ exists in $\mathbf{A}'$ with $\mathbf{f}' \geqslant \mathbf{f}$. Since $\Omega^A \subset \mathbf{L}_1(\phi)$, each ϕ is linear on Ω^A. Thus

$$\phi^*(\mathbf{f}' - \mathbf{f}) = \phi^*(\mathbf{f}') - \phi^*(\mathbf{f}) = 0,$$

for each $\phi \in A^{*+}$, $N^A(\mathbf{f}' - \mathbf{f}) = 0$ and $\mathbf{f}' = \mathbf{f} \in \mathbf{A}'$.

THEOREM 5.5. $\bar{\mathbf{A}}$ *is an ideal of* Ω^A. *If every increasing sequence of elements of* $\mathbf{A}'^+$ *that is bounded in norm is a Cauchy sequence, then* $\bar{\mathbf{A}} = \Omega^A$. *Theorem 5.2 extends to* $\bar{A}$.

Proof. By Theorem 3.3, if $\mathbf{u}_n \downarrow \mathbf{0}$, $\mathbf{u}_n \in \bar{\mathbf{A}}$, then $N^A(\mathbf{u}_n) \downarrow 0$ as $n \to \infty$. If $\mathbf{f} \in \bar{\mathbf{A}}^+$, $N^A(\mathbf{f}) < \infty$ and there exists $\mathbf{u}_n \wedge \mathbf{f} \uparrow \mathbf{f}$, with $\mathbf{u}_n \in \mathbf{C}$. By the preceding theorem $\mathbf{u}_n \wedge \mathbf{f} \in \mathbf{A}' \subset \bar{\mathbf{A}}$. Then $\mathbf{f} - \mathbf{u}_n \wedge \mathbf{f} \downarrow \mathbf{0}$ so that $N^A(\mathbf{f} - \mathbf{u}_n \wedge \mathbf{f}) \to 0$ as $n \to \infty$. If $0 < \mathbf{g} \leqslant \mathbf{f}$,

$$\mathbf{g} - \mathbf{u}_n \wedge \mathbf{g} = (\mathbf{g} - \mathbf{u}_n)^+ \leqslant (\mathbf{f} - \mathbf{u}_n)^+ = \mathbf{f} - \mathbf{u}_n \wedge \mathbf{f}$$

and $N^A(g - \mathbf{u}_n \wedge \mathfrak{g}) \to 0$ as $n \to \infty$. Thus $\mathfrak{g} \in \bar{\mathbf{A}}' = \bar{\mathbf{A}}$. It follows that $\bar{\mathbf{A}}$ is an ideal of Ω^A.

If $\mathbf{f} \in \Omega^A$, $N^A(\mathbf{f}) < \infty$ by Theorem 4.2. There then exists a sequence $\{\mathbf{u}_n \wedge \mathbf{f}\}$ in $\mathbf{A}'$ with $\mathbf{u}_n \wedge \mathbf{f} \uparrow \mathbf{f}$. By 4.5

$$N^A(\mathbf{u}_n \wedge \mathbf{f}) \uparrow N^A(\mathbf{f}).$$

Since $\mathbf{u}_n \wedge \mathbf{f}$ is a Cauchy sequence by hypothesis, $N^A(\mathbf{f} - \mathbf{u}_n \wedge \mathbf{f}) \to 0$ as $n \to \infty$ and $\mathbf{f} \in \bar{\mathbf{A}}' = \bar{\mathbf{A}}$. (Compare (**5**, Theorem 7.2, p. 359).)

THEOREM 5.6. (Ω^A, N^A) *is an* AMT *space if and only if* $\Omega^A = \bar{\mathbf{A}}$.

Proof. If $\Omega^A = \bar{\mathbf{A}}$, Theorem 5.3 applies. Now Ω^A is a Riesz subspace of $\mathbf{R}$ normed by N^A. Since $\bar{\mathbf{A}} \subset \Omega^A$ and compatibility is an equivalence relation, (Ω^A, N^A) cannot be compatible with $(\mathbf{A}, N^A)$ if $\Omega^A \neq \bar{\mathbf{A}}$. For each ϕ in A^* the Nakano extension gives an extension $\phi^* = \phi_*$ on Ω^A that is N^A-bounded. If $\phi \in \Omega^{A*+}$, ϕ determines $\phi' \in A^*$ by restriction to $\mathbf{A}$ and ϕ' extends to Ω^A. Since Ω^A is linear, non-compatibility implies the existence of some $\bar{\phi} = \phi - \phi'$ in Ω^{A*} with $\bar{\phi} = 0$ on $\mathbf{A}'$. We can assume that $\bar{\phi} > 0$ and that there exists $\mathbf{x}$ with $\bar{\phi}(\mathbf{x}) > 0$. As in the previous theorem, there exists $\mathbf{u}_n \uparrow \mathbf{x}$ with each $\mathbf{u}_n \in \mathbf{A}'$. Thus $\mathbf{x} - \mathbf{u}_n \downarrow \mathbf{0}$ but $\bar{\phi}(\mathbf{x} - \mathbf{u}_n) = \bar{\phi}(\mathbf{x})$ for all n. Thus $\bar{\phi} \in \Omega^{A*}$ and not in $\Omega_{\bar{c}}^{A*}$, condition (iii) is violated, and Ω^A is not an AMT space.

The following example illustrates the preceding result. Compare (**3**, XI, p. 509). Let R denote the space of sequences $x = (x_1, x_2, \ldots)$; $C = c_0$, the space of sequences convergent to zero; $A = C$, N^A the uniform norm on A. Then $A = A' = \bar{A}$ is norm complete so that $C^\sim = C^*$ (**3**, VIII, Theorem 26.4). It is easy to show that C^* is equivalent to l_1 by the correspondence $y(x) = \sum x_i y_i$. If $u_n \downarrow 0$, $N^A(u_n) \downarrow 0$ as $n \to \infty$. This implies that $C^\sim = C_{\bar{c}}^\sim$ and, noting that C is σ-Dedekind complete (actually super Dedekind complete),

$$C^\sim = C_c^\sim = C_{\bar{c}}^\sim = C^* = l_1$$

Thus (C, N^A) is a normed AMT space.

Let $0 < y \in C^*$; $0 < x \in R$, $x^n = (x_1{}^n, x_2{}^n, \ldots)$, $x_i{}^n = x_i$, $i = 1, 2, \ldots, n$; $= 0, i > n$. Then $y^*(x) = y_*(x) \leqslant \infty$ on R^+ by 2.1. The extended N^A is given by

$$N^A(x) = \sup_i |x_i|,$$

and $\Omega^A = F^A = l\infty$. The Hahn–Banach theorem implies the existence of Banach generalized limits $\phi(x) = (\mathrm{H}-\mathrm{B})\lim x_n$ with

$$\overline{\lim}_{n \to \infty} x_n \geqslant \phi(x) \geqslant \underline{\lim}_{n \to \infty} x_n,$$

$0 \neq \phi \in (l\infty)^*$. For every $x \in C = \bar{A}$, $\phi(x) = 0$.

6. Real MT spaces as AMT spaces. In this section we let E be an arbitrary locally compact space, C the functions continuous on E with compact supports,

and $R = \mathfrak{R}^E$. Then $C^\sim$ corresponds to the space of all Radon measures on E and $C^\sim = C_c^\sim$.

For $\phi \in C^{\sim+}$ we consider $L_1(\phi) = L_1 (\phi^*)$ under the norm $N^A = \rho\varphi$ extended by $N^A(u) = \varphi^*(|u|)$ to R. It is well known that $L_1(\phi^*)$ is complete under $\rho\varphi$, but we include a proof adapted to the present context. We shall show that the hypotheses of Theorem 4.1 are satisfied so that $\mathbf{F}^A$ is complete and thus $L_1(\phi)$ is complete since it is norm closed in $\mathbf{F}^A$.

Let $\mathbf{u}_n > \mathbf{0}$, $n = 1, 2, \ldots$, $\sum_1^\infty \phi^*(\mathbf{u}_n) < \infty$. There exist $u_n \in \mathbf{u}_n$, $u_n > 0$, $u_n \uparrow_n$ and thus there exists $u(t)$ valued in $\bar{\mathfrak{R}}^+$ with $u_n(t) \uparrow u(t)$ for all $t \in E$. Let $E_0 = [t : u(t) < \infty]$, $u'(t) = u(t)\chi_{E_0}$. Then $u' \in R^+$, $\sum_1^n u_i\chi_{E_0} \uparrow u'$ and, from 4.5 and 4.3, $\phi^*(u') \leqslant \sum_1^\infty \phi^*(u_n) < \infty$ and $\mathbf{u}' \in \mathbf{F}^A$. To show that $\mathbf{u}' = \sum_1^\infty \mathbf{u}_n$ we show that for each n, $u_n - u_n\chi_{E_0} \in \Phi^A$. Suppose that there exists n_0 with

$$u' = u_{n_0}\chi_{E_0}, \qquad \phi^*(u') = a > 0.$$

Since $(\sum_1^n u_i) \wedge (Ku') \uparrow (Ku')$ for every K, $0 < K < \infty$, by 4.5

$$\lim_{n\to\infty} \phi^*\left(\sum_1^n u_i \right) \geqslant \lim_{n\to\infty} \phi^*\left[\left(\sum_1^n u_i \right) \wedge (Ku') \right] = \phi^*(Ku') = Ka.$$

By hypothesis,

$$\phi^*\left(\sum_1^n u_i \right) \leqslant \sum_1^\infty \phi^*(u_i) < \infty \qquad \text{for every } n.$$

Thus we have a contradiction for n, K sufficiently large.

It follows that $\mathbf{L}_1(\phi)$ is a completion of $\mathbf{C}$ for $\rho\varphi$. Since this is also the case for the space of equivalence classes of functions integrable with respect to the Radon measure ϕ for the Bourbaki theory and the two theories coincide on C, both theories give the same spaces of integrable functions. It follows that if $\mathbf{A}$ is a Riesz substance of $\mathbf{R}$ and $\mathbf{C} \subset \mathbf{A} \subset \mathbf{R}$ and N^A is a Riesz seminorm on $\mathbf{A}$, then the MT and AMT spaces $\mathbf{\Omega}^A$ coincide and the MT and AMT extensions of N^A coincide on $\mathbf{\Omega}^A$ and $(\mathbf{A}, N^A)$ is an AMT space if and only if it is an MT space.

We next compare the extensions of N^A to R^+. In the Bourbaki theory (1), $\phi \in C^\sim$ is first extended to lower semi-continuous functions valued in $\bar{\mathfrak{R}}^+$ and then to arbitrary functions valued in $\bar{\mathfrak{R}}^+$. We have defined ϕ^* and ϕ_* above only for functions valued in $\mathfrak{R}$. The two theories are comparable for equivalence classes of functions if we consider, in the Bourbaki case, only those equivalence classes containing finite functions. We shall use the notation $\int^*|\mathbf{f}| \, d\phi$ for the Bourbaki extensions of ϕ, $\mathbf{F}^A(B)$ for the $\mathbf{f} \in \mathbf{R}$ with $\int^*|\mathbf{f}| \, d\phi < \infty$. Note that if $\mathbf{f} \in \mathbf{F}^A(B)$, $\mathbf{f}$ contains finite elements. We shall show that $\mathbf{F}^A = \mathbf{F}^A(B)$.

We note first that if $\phi^*(f) < \infty$, $\epsilon > 0$, there exist $u_n \in C$ with $u_n \uparrow$, $u_n \wedge f \uparrow f$ and $\phi^*(f) > \lim_{n\to\infty} \phi(u_n) - \epsilon$.

Now $u = \vee_n u_n$ exists in $\bar{\mathfrak{R}}^E$ and is l.s.c. with

$$\int^* u \, d\phi = \lim \int u_n \, d\phi = \lim \phi(u_n) < \phi^*(f) + \epsilon,$$

$$\int^*_{\substack{u \geqslant f \\ u \text{ l.s.c.}}} f \, d\phi = \inf \int^* u \, d\phi < \phi^*(f) + \epsilon.$$

Since $\epsilon > 0$ is arbitrary, $\int^* \mathbf{f} \, d\phi \leqslant \phi^*(\mathbf{f})$ and $\mathbf{f} \in \mathbf{F}^A$.

By **(1**, Prop. 5, p. 150**)** if $f \geqslant 0$ is l.s.c. and $\int^* f \, d\phi < \infty$, then $\mathbf{f} \in \mathbf{L_1}(\phi)$. This implies that $\phi^*(\mathbf{f}) = \phi_*(\mathbf{f}) = \int^* \mathbf{f} \, d\phi$,

$$\int^* \mathbf{f} \, d\phi = \inf_{\substack{g \geqslant f \\ g \text{ l.s.c.}}} \int^* \mathbf{g} \, d\phi = \inf_{\substack{g \geqslant f \\ g \text{ l.s.c.}}} \phi^*(\mathbf{g}) \geqslant \phi^*(\mathbf{f}).$$

Thus $\mathbf{F}^A = \mathbf{F}^A(B)$ and $\int^*$ and ϕ^* coincide on $\mathbf{F}^A = \mathbf{F}^A(B)$.

We have seen that the theory of AMT spaces (for the C and R in this section) and the theory of MT spaces are equivalent. Thus Theorem 5.6 answers affirmatively a question on p. 116 of **(6)**.

7. L^λ-spaces as AMT spaces. Let (X, S, μ) denote an arbitrary σ-finite measure space, R the measurable functions in $\mathfrak{R}^x$ with the natural order, C the Riesz subspace of R of simple functions (i.e. of functions each assuming only a finite number of real values different from zero with each value assumed in a set in S of finite measure). Let A be a Riesz subspace of R containing C.

We want to determine when $\mathbf{A}$ is both an AMT space and an $\mathbf{L}^\lambda$-space **(2)**. We assume that N^A is a non-trivial Riesz seminorm on $\mathbf{A}$. If N^A is to extend to a length function, it must satisfy **(2**, p. 576**)**,

$$(\text{L1})' \quad N^A(\chi_e) = 0 \text{ if } \mu(e) = 0, \, e \in S.$$

By Theorem 3.3 we also need

6.1. $N^A(u_n) \downarrow 0$ if $u_n \downarrow 0 \, (R), \, u_n \in A, \, n = 1, 2, \ldots$.

We consider (A, N^A) where N^A is a Riesz seminorm satisfying $(\text{L1})'$ and 6.1.

Let $\phi \in A^{*+}$. Then $\phi_c \in C_{\bar{c}}^{\sim}$. If $f \in R^+$, there exists a sequence $\{u_n\} \subset C^+$ with $u_n \uparrow f$. This implies by 2.1 that $\phi^* = \phi_*$ on R^+. If $f \in A$,

$$|\phi^*(f) - \phi(f)| \leqslant 2||\phi||N^A(f - u_n), \qquad n = 1, 2, \ldots,$$

so that $\phi = \phi^* = \phi_*$ on A and (A, N^A) is an AMT space by Theorem 3.1.

On R^+ the extended N^A satisfies 4.1–4.5 of §4 and $(\text{L1})'$ and these conditions imply that (L1) in **(2)** is also satisfied. N^A can be extended to all measurable functions valued in $\mathfrak{R}^x$ by (L5) and is then a length function **(2)**. The space L^λ, $\lambda = N^A$, of equivalence classes of functions modulo λ-negligible functions is an L^λ-space and is therefore complete **(2**, Theorem 3.1, p. 579**)**. In the present case

$$\mathbf{L}^\lambda = \mathbf{\Omega}^A = \mathbf{F}^A \supset \bar{\mathbf{A}}.$$

From Theorem 5.6, L^λ is an AMT space if and only if $\mathbf{\Omega}^A = \bar{\mathbf{A}}$.

Let $\phi \in A^{*+}$. Then ϕ defines a set function on S by $\phi(e) = \phi(\chi_e)$. From 6.1 it follows that ϕ is countably additive and from $(\text{L1})'$ that it is absolutely continuous with respect to μ. The Radon–Nikodym theorem then asserts the existence of a positive measurable function g with

$$\phi(f) = \int fg \, d\mu \leqslant \infty,$$

for all $f \in R^+$. Both sides are finite on $\mathbf{L}^\lambda$ and, where λ^* is the length function conjugate to λ, it follows easily that $||\phi|| = \lambda^*(\mathbf{g})$ and $\mathbf{g} \in \mathbf{L}^{\lambda^*}$. This implies that $(L^\lambda)^* = L^{\lambda^*}$.

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California Institute of Technology,
Pasadena, California, and
Queen's University,
Kingston, Ontario