If instead of taking the projections of OU on the given chords, the projections on another set drawn through O at right angles to the given set are taken, a similar result is obtained for the sum of the sines of such a series of angles.

If the common difference of the angles is a multiple of $\frac{2\pi}{n}$, but not of 2π, the same results are obtained.

ALEX. D. RUSSELL.

Direct Proofs of Theorems in Elementary Geometry.

(1) If the straight line joining two points subtends equal angles at two other points on the same side of it, the four points are concyclic.

(2) If a pair of opposite angles of a quadrilateral are supplementary, its vertices are concyclic.

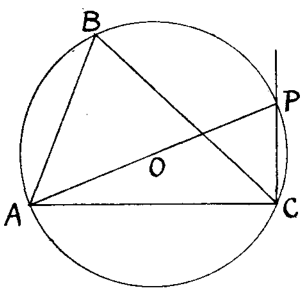

Fig. 1.

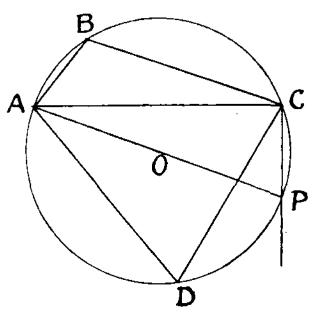

Fig. 2.

(1) Let A, C be the two points and B one of the other points. Let $\angle ABC$ be acute (Fig. 1).

Let O be the circumcentre of $\triangle ABC$; join AO and produce it to meet the perpendicular to AC through C in P.

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Then $OA = OC$ and $\angle ACP = 90°$. $\therefore OA = OP$.

$\therefore$ the circumscribing circle of $\triangle ABC$ passes through P. But $\angle P = \frac{1}{2} \angle AOC = \angle B =$ constant.

$\therefore$ B lies on the fixed circle which circumscribes the fixed right-angled triangle ACP in which $\angle P =$ given $\angle B$.

If $\angle B$ is obtuse (Fig. 2), B lies on the circumscribing circle of the fixed right-angled triangle ACP in which $\angle P = 180° - \angle B$

(2) If in the quadrilateral $ABCD$ the angles B and D are supplementary, D being acute (Fig. 2), then by the previous theorem B and D both lie on the fixed circle which circumscribes the fixed right-angled triangle ACP in which $\angle ACP = \angle D$.

R. F. BLADES.

An Elementary Proof of Feuerbach's Theorem.

Let O be the centre of the circumscribing circle of $\triangle ABC$, A_1 the middle point of BC, and EA_1OF the diameter at right angles to BC. Draw AX perpendicular to BC and produce it to meet the circle in K. Let H be the orthocentre of $\triangle ABC$; join OH and bisect it in N, the centre of the nine-point circle.

Draw OY perpendicular to and bisecting AK.

Join EA, which bisects $\angle BAC$ and contains the incentre I; draw ID, NM perpendicular to BC. Join AF and draw AG perpendicular to EF; also draw PIQ parallel to BC and meeting EF in P and AX in Q.

Then we have $AH = 2OA_1$, $HK = 2HX$, $AI \,.\, IE = 2Rr$.

Also from similar triangles $\dfrac{PI}{IE} = \dfrac{FG}{AF}$ and $\dfrac{IQ}{AI} = \dfrac{AF}{FE}$.

Thus $\dfrac{PI \,.\, IQ}{AI \,.\, IE} = \dfrac{FG}{FE}$, so that $\dfrac{PI \,.\, IQ}{2R \,.\, r} = \dfrac{FG}{2R}$, and $PI \,.\, IQ = r \,.\, FG$.

Now the projection of IN on $FE = ID - NM = r - \frac{1}{2}(OA_1 + HX)$
$= r - \frac{1}{4}(AH + HK) = r - \frac{1}{2}AY$.