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REVERSES OF THE CAUCHY–BUNYAKOVSKY–SCHWARZ INEQUALITY FOR n-TUPLES OF COMPLEX NUMBERS

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Some new reverses of the Cauchy–Bunyakovsky–Schwarz inequality for n-tuples of real and complex numbers related to Cassels and Shisha–Mond results are given.

1. INTRODUCTION

Let $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{b} = (b_1, \dots, b_n)$ be two positive n-tuples with the property that there exists the positive numbers m_i, M_i $(i = 1, 2)$ such that

$$0 < m_1 \leqslant a_i \leqslant M_1 < \infty \quad \text{and} \quad 0 < m_2 \leqslant b_i \leqslant M_2 < \infty, \tag{1.1}$$

for each $i \in \{1, \dots, n\}$.

The following reverses of the Cauchy–Bunyakovsky–Schwarz inequality are well known in the literature:

1. PÓLYA–SZEGÖ'S INEQUALITY [8]

$$\frac{\sum_{k=1}^n a_k^2 \sum_{k=1}^n b_k^2}{\left(\sum_{k=1}^n a_k b_k\right)^2} \leqslant \frac{1}{4}\left(\sqrt{\frac{M_1 M_2}{m_1 m_2}} + \sqrt{\frac{m_1 m_2}{M_1 M_2}}\right)^2; \tag{1.2}$$

2. SHISHA–MOND'S INEQUALITY [9]

$$\frac{\sum_{k=1}^n a_k^2}{\sum_{k=1}^n a_k b_k} - \frac{\sum_{k=1}^n a_k b_k}{\sum_{k=1}^n b_k^2} \leqslant \left(\sqrt{\frac{M_1}{m_2}} - \sqrt{\frac{m_1}{M_2}}\right)^2; \tag{1.3}$$

3. OZEKI'S INEQUALITY [7]

$$\sum_{k=1}^n a_k^2 \sum_{k=1}^n b_k^2 - \left(\sum_{k=1}^n a_k b_k\right)^2 \leqslant \frac{1}{4} n^2 (M_1 M_2 - m_1 m_2)^2; \tag{1.4}$$

4. DIAZ–METCALF'S INEQUALITY [1]

$$\sum_{k=1}^n b_k^2 + \frac{m_2 M_2}{m_1 M_1} \sum_{k=1}^n a_k^2 \leqslant \left(\frac{M_2}{m_1} + \frac{m_2}{M_1}\right) \sum_{k=1}^n a_k b_k. \tag{1.5}$$

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If the weight $\mathbf{w} = (w_1, \dots, w_n)$ is a positive n-tuple, then we have the following inequalities, which are also well known.

5. Cassels' inequality [**10**]

If the positive n-tuples $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{b} = (b_1, \dots, b_n)$ satisfy the condition

$$0 < m \leqslant \frac{a_k}{b_k} \leqslant M < \infty \quad \text{for each} \quad k \in \{1, \dots, n\}, \tag{1.6}$$

where m, M are given, then

$$\frac{\sum_{k=1}^{n} w_k a_k^2 \sum_{k=1}^{n} w_k b_k^2}{\left(\sum_{k=1}^{n} w_k a_k b_k\right)^2} \leqslant \frac{(M+m)^2}{4mM}. \tag{1.7}$$

6. Greub–Reinboldt's inequality [**4**]

If $\mathbf{a}$ and $\mathbf{b}$ satisfy the condition (1.1), then

$$\frac{\sum_{k=1}^{n} w_k a_k^2 \sum_{k=1}^{n} w_k b_k^2}{\left(\sum_{k=1}^{n} w_k a_k b_k\right)^2} \leqslant \frac{(M_1 M_2 + m_1 m_2)^2}{4 m_1 m_2 M_1 M_2}. \tag{1.8}$$

7. Generalised Diaz–Metcalf inequality [**1**] (see also [**6**, p. 123])

If $u, v \in [0,1]$ and $v \leqslant u$, $u + v = 1$ and (1.6) holds, then one has the inequality

$$u \sum_{k=1}^{n} w_k b_k^2 + vmM \sum_{k=1}^{n} w_k a_k^2 \leqslant (vm + uM) \sum_{k=1}^{n} w_k a_k b_k. \tag{1.9}$$

8. Klamkin–McLenaghan's inequality [**5**]

If $\mathbf{a}$ and $\mathbf{b}$ satisfy (1.6), then we have the inequality

$$\sum_{k=1}^{n} w_k a_k^2 \sum_{k=1}^{n} w_k b_k^2 - \left(\sum_{k=1}^{n} w_k a_k b_k\right)^2 \leqslant \left(\sqrt{M} - \sqrt{m}\right)^2 \sum_{k=1}^{n} w_k a_k b_k \sum_{k=1}^{n} w_k a_k^2. \tag{1.10}$$

For other reverse results of the Cauchy–Bunyakovsky–Schwarz inequality, see the recent survey online [**3**].

The main aim of this paper is to point out some new reverse inequalities of the classical Cauchy–Bunyakovsky–Schwarz result for both real and complex n-tuples.

2. Some Reverses of the Cauchy-Bunyakovsky–Schwarz Inequality

The following result holds.

Theorem 1. *Let $\mathbf{a} = (a_1, \dots, a_n)$, $\mathbf{b} = (b_1, \dots, b_n) \in \mathbb{K}^n$, where $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and $\mathbf{p} = (p_1, \dots, p_n) \in \mathbb{R}_+^n$ with $\sum_{i=1}^{n} p_i = 1$. If $b_i \neq 0$, $i \in \{1, \dots, n\}$ and there exists the constant $\alpha \in \mathbb{K}$ and $r > 0$ such that for any $k \in \{1, \dots, n\}$*

$$\frac{a_k}{b_k} \in \overline{D}(\alpha, r) := \left\{ z \in \mathbb{K} \mid |z - \alpha| \leqslant r \right\}, \tag{2.1}$$

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then we have the inequality

$$(2.2)\qquad \sum_{k=1}^{n} p_k |a_k|^2 + \left(|\alpha|^2 - r^2\right) \sum_{k=1}^{n} p_k |b_k|^2 \leqslant 2\operatorname{Re}\left[\overline{\alpha}\Big(\sum_{k=1}^{n} p_k a_k b_k\Big)\right] \leqslant 2|\alpha| \cdot \left|\sum_{k=1}^{n} p_k a_k b_k\right|.$$

The constant $c = 2$ is best possible in the sense that it cannot be replaced by a smaller constant.

PROOF: From (2.1) we have $\left|a_k - \alpha\overline{b}_k\right|^2 \leqslant r|b_k|^2$ for each $k \in \{1, \dots, n\}$, which is clearly equivalent to

$$(2.3)\qquad |a_k|^2 + \left(|\alpha|^2 - r^2\right)|b_k|^2 \leqslant 2\operatorname{Re}\left[\overline{\alpha}(a_k b_k)\right]$$

for each $k \in \{1, \dots, n\}$.

Multiplying (2.3) with $p_k \geqslant 0$ and summing over k from 1 to n, we deduce the first inequality in (1.2). The second inequality is obvious.

To prove the sharpness of the constant 2, assume that under the hypothesis of the theorem there exists a constant $c > 0$ such that

$$(2.4)\qquad \sum_{k=1}^{n} p_k |a_k|^2 + \left(|\alpha|^2 - r^2\right) \sum_{k=1}^{n} p_k |b_k|^2 \leqslant c\operatorname{Re}\left[\overline{\alpha}\Big(\sum_{k=1}^{n} p_k a_k b_k\Big)\right],$$

provided $a_k/\overline{b_k} \in \overline{D}(\alpha, r)$, $k \in \{1, \dots, n\}$.

Assume that $n = 2$, $p_1 = p_2 = 1/2$, $b_1 = b_2 = 1$, $\alpha = r > 0$ and $a_2 = 2r$, $a_1 = 0$. Then $|a_2/b_2 - \alpha| = r$, $|a_1/b_1 - \alpha| = r$ showing that the condition (2.1) holds. For these choices, the inequality (2.4) becomes $2r^2 \leqslant cr^2$, giving $c \geqslant 2$. ∎

The case where the disk $\overline{D}(\alpha, r)$ does not contain the origin, that is, $|\alpha| > r$, provides the following interesting reverse of the Cauchy-Bunyakovsky-Schwarz inequality.

THEOREM 2. *Let* **a**, **b**, **p** *as in Theorem 1 and assume that $|\alpha| > r > 0$. Then we have the inequality*

$$(2.5)\qquad \sum_{k=1}^{n} p_k|a_k|^2 \sum_{k=1}^{n} p_k|b_k|^2 \leqslant \frac{1}{|\alpha|^2 - r^2}\left\{\operatorname{Re}\left[\overline{\alpha}\Big(\sum_{k=1}^{n} p_k a_k b_k\Big)\right]\right\}^2 \leqslant \frac{|\alpha|^2}{|\alpha|^2 - r^2}\left|\sum_{k=1}^{n} p_k a_k b_k\right|^2.$$

The constant $c = 1$ in the first and second inequality is best possible in the sense that it cannot be replaced by a smaller constant.

PROOF: Since $|\alpha| > r$, we may divide (2.2) by $\sqrt{|\alpha|^2 - r^2} > 0$ to obtain

$$(2.6)\qquad \frac{1}{\sqrt{|\alpha|^2 - r^2}} \sum_{k=1}^{n} p_k|a_k|^2 + \sqrt{|\alpha|^2 - r^2} \sum_{k=1}^{n} p_k|b_k|^2 \leqslant \frac{2}{\sqrt{|\alpha|^2 - r^2}} \operatorname{Re}\left[\overline{\alpha}\Big(\sum_{k=1}^{n} p_k a_k b_k\Big)\right].$$

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On the other hand, by the use of the following elementary inequality

$$\frac{1}{\beta}p + \beta q \geqslant 2\sqrt{pq} \quad \text{for } \beta > 0 \text{ and } p, q \geqslant 0, \tag{2.7}$$

we may state that

$$2\Big(\sum_{k=1}^{n} p_k |a_k|^2\Big)^{1/2} \cdot \Big(\sum_{k=1}^{n} p_k |b_k|^2\Big)^{1/2} \leqslant \frac{1}{\sqrt{|\alpha|^2 - r^2}} \sum_{k=1}^{n} p_k |a_k|^2 + \sqrt{|\alpha|^2 - r^2} \sum_{k=1}^{n} p_k |b_k|^2. \tag{2.8}$$

Utilising (2.6) and (2.8), we deduce

$$\Big(\sum_{k=1}^{n} p_k |a_k|^2\Big)^{1/2} \cdot \Big(\sum_{k=1}^{n} p_k |b_k|^2\Big)^{1/2} \leqslant \frac{1}{\sqrt{|\alpha|^2 - r^2}} \operatorname{Re}\Big[\overline{\alpha}\Big(\sum_{k=1}^{n} p_k a_k b_k\Big)\Big],$$

which is clearly equivalent to the first inequality in (2.6).

The second inequality is obvious.

To prove the sharpness of the constant, assume that (2.5) holds with a constant $c > 0$, that is,

$$\sum_{k=1}^{n} p_k |a_k|^2 \sum_{k=1}^{n} p_k |b_k|^2 \leqslant \frac{c}{|\alpha|^2 - r^2} \Big\{\operatorname{Re}\Big[\overline{\alpha}\Big(\sum_{k=1}^{n} p_k a_k b_k\Big)\Big]\Big\}^2 \tag{2.9}$$

provided $a_k/\overline{b_k} \in \overline{D}(\alpha, r)$ and $|\alpha| > r$.

For $n = 2$, $b_2 = b_1 = 1$, $p_1 = p_2 = 1/2$, $a_2, a_1 \in \mathbb{R}$, $\alpha, r > 0$ and $\alpha > r$, we get from (2.9) that

$$\frac{a_1^2 + a_2^2}{2} \leqslant \frac{c\alpha^2}{\alpha^2 - r^2}\Big(\frac{a_1 + a_2}{2}\Big)^2. \tag{2.10}$$

If we choose $a_2 = \alpha + r$, $a_1 = \alpha - r$, then $|a_i - \alpha| \leqslant r$, $i = 1, 2$ and by (2.10) we deduce

$$\alpha^2 + r^2 \leqslant \frac{c\alpha^4}{\alpha^2 - r^2},$$

which is clearly equivalent to

$$(c-1)\alpha^4 + r^4 \geqslant 0 \quad \text{for } \alpha > r > 0.$$

If in this inequality we choose $\alpha = 1$, $r = \varepsilon \in (0, 1)$ and let $\varepsilon \to 0+$, then we deduce $c \geqslant 1$. ∎

The following corollary is a natural consequence of the above theorem.

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Corollary 1. *Under the assumptions of Theorem 2, we have the following additive reverse of the Cauchy–Bunyakovsky–Schwarz inequality*

$$0 \leqslant \sum_{k=1}^{n} p_k |a_k|^2 \sum_{k=1}^{n} p_k |b_k|^2 - \left|\sum_{k=1}^{n} p_k a_k b_k\right|^2 \leqslant \frac{r^2}{|\alpha|^2 - r^2}\left|\sum_{k=1}^{n} p_k a_k b_k\right|^2. \tag{2.11}$$

The constant $c = 1$ is best possible in the sense mentioned above.

Remark 1. If in Theorem 1, we assume that $|\alpha| = r$, then we obtain the inequality:

$$\sum_{k=1}^{n} p_k |a_k|^2 \leqslant 2\operatorname{Re}\left[\overline{\alpha}\Big(\sum_{k=1}^{n} p_k a_k b_k\Big)\right] \leqslant 2|\alpha|\left|\sum_{k=1}^{n} p_k a_k b_k\right|. \tag{2.12}$$

The constant 2 is sharp in both inequalities.

We also remark that, if $r > |\alpha|$, then (2.2) may be written as

$$\sum_{k=1}^{n} p_k |a_k|^2 \leqslant \left(r^2 - |\alpha|^2\right)\sum_{k=1}^{n} p_k |b_k|^2 + 2\operatorname{Re}\left[\overline{\alpha}\Big(\sum_{k=1}^{n} p_k a_k b_k\Big)\right] \leqslant \left(r^2 - |\alpha|^2\right)\sum_{k=1}^{n} p_k |b_k|^2 + 2|\alpha|\left|\sum_{k=1}^{n} p_k a_k b_k\right|. \tag{2.13}$$

The following reverse of the Cauchy–Bunyakovsky–Schwarz inequality also holds.

Theorem 3. *Let* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{p}$ *be as in Theorem 1 and assume that* $\alpha \in \mathbb{K}$, $\alpha \neq 0$ *and* $r > 0$. *Then we have the inequalities*

$$\begin{aligned} 0 &\leqslant \left(\sum_{k=1}^{n} p_k|a_k|^2\right)^{1/2} \cdot \left(\sum_{k=1}^{n} p_k|b_k|^2\right)^{1/2} - \left|\sum_{k=1}^{n} p_k a_k b_k\right| \\ &\leqslant \left(\sum_{k=1}^{n} p_k|a_k|^2\right)^{1/2} \cdot \left(\sum_{k=1}^{n} p_k|b_k|^2\right)^{1/2} - \operatorname{Re}\left[\frac{\overline{\alpha}}{|\alpha|}\Big(\sum_{k=1}^{n} p_k a_k b_k\Big)\right] \\ &\leqslant \frac{1}{2}\cdot\frac{r^2}{|\alpha|}\sum_{k=1}^{n} p_k|b_k|^2. \end{aligned} \tag{2.14}$$

The constant 1/2 is best possible in the sense mentioned above.

Proof: From Theorem 1, we have

$$\sum_{k=1}^{n} p_k |a_k|^2 + |\alpha|^2 \sum_{k=1}^{n} p_k |b_k|^2 \leqslant 2\operatorname{Re}\left[\overline{\alpha}\Big(\sum_{k=1}^{n} p_k a_k b_k\Big)\right] + r^2 \sum_{k=1}^{n} p_k |b_k|^2. \tag{2.15}$$

Since $\alpha \neq 0$, we can divide (2.15) by $|\alpha|$, getting

$$(2.16)\qquad \frac{1}{|\alpha|}\sum_{k=1}^{n} p_k\,|a_k|^2 + |\alpha|\sum_{k=1}^{n} p_k\,|b_k|^2 \leqslant 2\,\mathrm{Re}\left[\frac{\overline{\alpha}}{|\alpha|}\Bigl(\sum_{k=1}^{n} p_k a_k b_k\Bigr)\right] + \frac{r^2}{|\alpha|}\sum_{k=1}^{n} p_k|b_k|^2.$$

Utilising the inequality (2.7), we may state that

$$(2.17)\qquad 2\Bigl(\sum_{k=1}^{n} p_k\,|a_k|^2\Bigr)^{1/2}\cdot\Bigl(\sum_{k=1}^{n} p_k\,|b_k|^2\Bigr)^{1/2} \leqslant \frac{1}{|\alpha|}\sum_{k=1}^{n} p_k\,|a_k|^2 + |\alpha|\sum_{k=1}^{n} p_k\,|b_k|^2\,.$$

Making use of (2.16) and (2.17), we deduce the second inequality in (2.14).

The first inequality in (2.14) is obvious.

To prove the sharpness of the constant 1/2, assume that there exists a $c > 0$ such that

$$(2.18)\quad \Bigl(\sum_{k=1}^{n} p_k|a_k|^2\Bigr)^{1/2}\cdot\Bigl(\sum_{k=1}^{n} p_k|b_k|^2\Bigr)^{1/2} - \mathrm{Re}\left[\frac{\overline{\alpha}}{|\alpha|}\Bigl(\sum_{k=1}^{n} p_k a_k b_k\Bigr)\right] \leqslant c\cdot\frac{r^2}{|\alpha|}\sum_{k=1}^{n} p_k|b_k|^2,$$

provided $|a_k/\overline{b_k} - \alpha| \leqslant r$, $\alpha \neq 0$, $r > 0$.

If we choose $n = 2$, $\alpha > 0$, $b_1 = b_2 = 1$, $a_1 = \alpha + r$, $a_2 = \alpha - r$, then from (2.18) we deduce

$$(2.19)\qquad \sqrt{r^2+\alpha^2} - \alpha \leqslant c\frac{r^2}{\alpha}.$$

If we multiply (2.19) with $\sqrt{r^2+\alpha^2} + \alpha > 0$ and then divide it by $r > 0$, we deduce

$$(2.20)\qquad 1 \leqslant \frac{\sqrt{r^2+\alpha^2}+\alpha}{\alpha}\cdot c$$

for any $r > 0$, $\alpha > 0$.

If in (2.20) we let $r \to 0+$, then we get $c \geqslant 1/2$, and the sharpness of the constant is proved. ∎

3. A Cassels Type Inequality for Complex Numbers

The following result holds.

Theorem 4. *Let* $\mathbf{a} = (a_1, \dots, a_n)$, $\mathbf{b} = (b_1, \dots, b_n) \in \mathbb{K}^n$, *where* $\mathbb{K} = \mathbb{R}, \mathbb{C}$ *and* $\mathbf{p} = (p_1, \dots, p_n) \in \mathbb{R}_+^n$ *with* $\sum_{i=1}^{n} p_i = 1$. *If* $b_i \neq 0$, $i \in \{1, \dots, n\}$ *and there exist the constants* $\gamma, \Gamma \in \mathbb{K}$ *with* $\mathrm{Re}(\Gamma\overline{\gamma}) > 0$ *and* $\Gamma \neq \gamma$, *so that either*

$$(3.1)\qquad \left|\frac{a_k}{\overline{b_k}} - \frac{\gamma+\Gamma}{2}\right| \leqslant \frac{1}{2}|\Gamma - \gamma| \ \text{ for each } \ k \in \{1, \dots, n\},$$

or, equivalently,

$$\text{(3.2)} \qquad \operatorname{Re}\left[\left(\Gamma - \frac{a_k}{b_k}\right)\left(\frac{\overline{a_k}}{\overline{b_k}} - \overline{\gamma}\right)\right] \geqslant 0 \quad \text{for each } k \in \{1, \dots, n\}$$

holds, then we have the inequalities

$$\text{(3.3)} \qquad \sum_{k=1}^{n} p_k |a_k|^2 \sum_{k=1}^{n} p_k |b_k|^2 \leqslant \frac{1}{2\operatorname{Re}(\Gamma\overline{\gamma})} \left\{ \operatorname{Re}\left[(\overline{\gamma} + \overline{\Gamma}) \sum_{k=1}^{n} p_k a_k b_k \right] \right\}^2 \leqslant \frac{|\Gamma + \gamma|^2}{4\operatorname{Re}(\Gamma\overline{\gamma})} \left| \sum_{k=1}^{n} p_k a_k b_k \right|^2.$$

The constants $1/2$ *and* $1/4$ *are best possible in* (3.3).

PROOF: The fact that the relations (3.1) and (3.2) are equivalent follows by the simple fact that for $z, u, U \in \mathbb{C}$, the following inequalities are equivalent

$$\left| z - \frac{u+U}{2} \right| \leqslant \frac{1}{2} |U - u|$$

and

$$\operatorname{Re}\left[(u - z)(\overline{z} - \overline{u}) \right] \geqslant 0.$$

Define $\alpha = (\gamma + \Gamma)/2$ and $r = |\Gamma - \gamma|/2$. Then

$$|\alpha|^2 - r^2 = \frac{|\Gamma + \gamma|^2}{4} - \frac{|\Gamma - \gamma|^2}{4} = \operatorname{Re}(\Gamma\overline{\gamma}) > 0.$$

Consequently, we may apply Theorem 2, and the inequalities (3.3) are proved.

The sharpness of the constants may be proven in a similar way to that in the proof of Theorem 2, and we omit the details. ∎

The following additive version also holds.

COROLLARY 2. *With the assumptions in Theorem 4, we have*

$$\text{(3.4)} \qquad \sum_{k=1}^{n} p_k |a_k|^2 \sum_{k=1}^{n} p_k |b_k|^2 - \left| \sum_{k=1}^{n} p_k a_k b_k \right|^2 \leqslant \frac{|\Gamma - \gamma|^2}{4\operatorname{Re}(\Gamma\overline{\gamma})} \left| \sum_{k=1}^{n} p_k a_k b_k \right|^2.$$

The constant $1/4$ *is also best possible.*

REMARK 2. With the above assumptions and if $\operatorname{Re}(\Gamma\overline{\gamma}) = 0$, then by the use of Remark 1, we may deduce the inequality

$$\text{(3.5)} \qquad \sum_{k=1}^{n} p_k |a_k|^2 \leqslant \operatorname{Re}\left[(\overline{\gamma} + \overline{\Gamma}) \sum_{k=1}^{n} p_k a_k b_k \right] \leqslant |\Gamma + \gamma| \left| \sum_{k=1}^{n} p_k a_k b_k \right|.$$

If $\operatorname{Re}(\Gamma\overline{\gamma}) < 0$, then, by Remark 1, we also have

$$\text{(3.6)} \qquad \sum_{k=1}^{n} p_k |a_k|^2 \leqslant -\operatorname{Re}(\Gamma\overline{\gamma}) \sum_{k=1}^{n} p_k |b_k|^2 + \operatorname{Re}\left[(\overline{\Gamma} + \overline{\gamma}) \sum_{k=1}^{n} p_k a_k b_k \right] \leqslant -\operatorname{Re}(\Gamma\overline{\gamma}) \sum_{k=1}^{n} p_k |b_k|^2 + |\Gamma + \gamma| \left| \sum_{k=1}^{n} p_k a_k b_k \right|.$$

REMARK 3. If $a_k, b_k > 0$ and there exist the constants $m, M > 0$ $(M > m)$ with

$$m \leqslant \frac{a_k}{b_k} \leqslant M \quad \text{for each} \quad k \in \{1, \dots, n\}, \tag{3.7}$$

then, obviously (3.1) holds with $\gamma = m$, $\Gamma = M$, also $\Gamma\overline{\gamma} = Mm > 0$ and by (3.3) we deduce

$$\sum_{k=1}^{n} p_k a_k^2 \sum_{k=1}^{n} p_k b_k^2 \leqslant \frac{(M+m)^2}{4mM} \left(\sum_{k=1}^{n} p_k a_k b_k \right)^2, \tag{3.8}$$

that is, Cassels' inequality.

4. A Shisha–Mond Type Inequality for Complex Numbers

The following result holds.

THEOREM 5. *Let* $\mathbf{a} = (a_1, \dots, a_n)$, $\mathbf{b} = (b_1, \dots, b_n) \in \mathbb{K}^n$, *where* $\mathbb{K} = \mathbb{R}, \mathbb{C}$ *and* $\mathbf{p} = (p_1, \dots, p_n) \in \mathbb{R}_+^n$ *with* $\sum_{i=1}^{n} p_i = 1$. *If* $b_i \neq 0$, $i \in \{1, \dots, n\}$ *and there exist the constants* $\gamma, \Gamma \in \mathbb{K}$ *such that* $\Gamma \neq \gamma, -\gamma$ *and either*

$$\left| \frac{a_k}{b_k} - \frac{\gamma + \Gamma}{2} \right| \leqslant \frac{1}{2} |\Gamma - \gamma| \quad \text{for each} \quad k \in \{1, \dots, n\}, \tag{4.1}$$

or, equivalently,

$$\operatorname{Re}\left[\left(\Gamma - \frac{a_k}{b_k} \right) \left(\frac{\overline{a_k}}{\overline{b_k}} - \overline{\gamma} \right) \right] \geqslant 0 \quad \text{for each} \quad k \in \{1, \dots, n\}, \tag{4.2}$$

holds, then we have the inequalities

$$\begin{aligned} 0 &\leqslant \left(\sum_{k=1}^{n} p_k |a_k|^2 \right)^{1/2} \cdot \left(\sum_{k=1}^{n} p_k |b_k|^2 \right)^{1/2} - \left| \sum_{k=1}^{n} p_k a_k b_k \right| \\ &\leqslant \left(\sum_{k=1}^{n} p_k |a_k|^2 \right)^{1/2} \cdot \left(\sum_{k=1}^{n} p_k |b_k|^2 \right)^{1/2} - \operatorname{Re}\left[\frac{\overline{\Gamma} + \overline{\gamma}}{|\Gamma + \gamma|} \sum_{k=1}^{n} p_k a_k b_k \right] \\ &\leqslant \frac{1}{4} \cdot \frac{|\Gamma - \gamma|^2}{|\Gamma + \gamma|} \sum_{k=1}^{n} p_k |b_k|^2. \end{aligned} \tag{4.3}$$

The constant $1/4$ *is best possible in the sense that it cannot be replaced by a smaller constant.*

PROOF: Follows by Theorem 3 on choosing $\alpha = (\gamma + \Gamma)/2 \neq 0$ and $r = |\Gamma - \gamma|/2 > 0$.

The proof for the best constant follows in a similar way to that in the proof of Theorem 3 and we omit the details. ∎

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REMARK 4. If $a_k, b_k > 0$ and there exists the constants $m, M > 0$ $(M > m)$ with

$$m \leqslant \frac{a_k}{b_k} \leqslant M \quad \text{for each} \quad k \in \{1, \dots, n\}, \tag{4.4}$$

then we have the inequality

$$\begin{aligned} 0 &\leqslant \left(\sum_{k=1}^{n} p_k a_k^2\right)^{1/2} \cdot \left(\sum_{k=1}^{n} p_k b_k^2\right)^{1/2} - \sum_{k=1}^{n} p_k a_k b_k \\ &\leqslant \frac{1}{4} \cdot \frac{(M-m)^2}{(M+m)} \sum_{k=1}^{n} p_k b_k^2. \end{aligned} \tag{4.5}$$

The constant $1/4$ is best possible. For $p_k = 1/n$, $k \in \{1, \dots, n\}$, we recapture the result from [**3**, Theorem 5.21] that has been obtained from a reverse inequality due to Shisha and Mond [**8**].

5. Further Reverses of the Cauchy–Bunyakovsky–Schwarz Inequality

The following result holds.

THEOREM 6. *Let* $\mathbf{a} = (a_1, \dots, a_n)$, $\mathbf{b} = (b_1, \dots, b_n) \in \mathbb{K}^n$ *and* $r > 0$ *such that for* $p_i \geqslant 0$ *with* $\sum_{i=1}^{n} p_i = 1$

$$\sum_{i=1}^{n} p_i |b_i - \overline{a_i}|^2 \leqslant r^2 < \sum_{i=1}^{n} p_i |a_i|^2. \tag{5.1}$$

Then we have the inequality

$$\begin{aligned} 0 &\leqslant \sum_{i=1}^{n} p_i |a_i|^2 \sum_{i=1}^{n} p_i |b_i|^2 - \left|\sum_{i=1}^{n} p_i a_i b_i\right|^2 \\ &\leqslant \sum_{i=1}^{n} p_i |a_i|^2 \sum_{i=1}^{n} p_i |b_i|^2 - \left[\operatorname{Re}\left(\sum_{i=1}^{n} p_i a_i b_i\right)\right]^2 \\ &\leqslant r^2 \sum_{i=1}^{n} p_i |b_i|^2. \end{aligned} \tag{5.2}$$

The constant $c = 1$ in front of r^2 is best possible in the sense that it cannot be replaced by a smaller constant.

PROOF: From the first condition in (5.1), we have

$$\sum_{i=1}^{n} p_i \left[|b_i|^2 - 2\operatorname{Re}(b_i a_i) + |a_i|^2\right] \leqslant r^2,$$

giving

$$\sum_{i=1}^{n} p_i |b_i|^2 + \sum_{i=1}^{n} p_i |a_i|^2 - r^2 \leqslant 2\operatorname{Re}\left(\sum_{i=1}^{n} p_i a_i b_i\right). \tag{5.3}$$

Since, by the second condition in (5.1) we have

$$\sum_{i=1}^{n} p_i |a_i|^2 - r^2 > 0,$$

we may divide (5.3) by $\sqrt{\sum_{i=1}^{n} p_i |a_i|^2 - r^2} > 0$, getting

$$\frac{\sum_{i=1}^{n} p_i|b_i|^2}{\sqrt{\sum_{i=1}^{n} p_i|a_i|^2 - r^2}} + \sqrt{\sum_{i=1}^{n} p_i|a_i|^2 - r^2} \leqslant \frac{2\operatorname{Re}(\sum_{i=1}^{n} p_i a_i b_i)}{\sqrt{\sum_{i=1}^{n} p_i|a_i|^2 - r^2}}. \tag{5.4}$$

Utilising the elementary inequality

$$\frac{p}{\alpha} + q\alpha \geqslant 2\sqrt{pq} \quad \text{for } p, q \geqslant 0 \text{ and } \alpha > 0, \tag{5.5}$$

we may write that

$$2\sqrt{\sum_{i=1}^{n} p_i|b_i|^2} \leqslant \frac{\sum_{i=1}^{n} p_i|b_i|^2}{\sqrt{\sum_{i=1}^{n} p_i|a_i|^2 - r^2}} + \sqrt{\sum_{i=1}^{n} p_i|a_i|^2 - r^2}. \tag{5.6}$$

Combining (5.5) with (5.6) we deduce

$$\sqrt{\sum_{i=1}^{n} p_i |b_i|^2} \leqslant \frac{\operatorname{Re}(\sum_{i=1}^{n} p_i a_i b_i)}{\sqrt{\sum_{i=1}^{n} p_i|a_i|^2 - r^2}}. \tag{5.7}$$

Taking the square in (5.7), we obtain

$$\sum_{i=1}^{n} p_i|b_i|^2 \Big(\sum_{i=1}^{n} p_i|a_i|^2 - r^2\Big) \leqslant \Big[\operatorname{Re}\Big(\sum_{i=1}^{n} p_i a_i b_i\Big)\Big]^2,$$

giving the third inequality in (5.2).

The other inequalities are obvious.

To prove the sharpness of the constant, assume, under the hypothesis of the theorem, that there exists a constant $c > 0$ such that

$$\sum_{i=1}^{n} p_i |a_i|^2 \sum_{i=1}^{n} p_i |b_i|^2 - \Big[\operatorname{Re}\Big(\sum_{i=1}^{n} p_i a_i b_i\Big)\Big]^2 \leqslant cr^2 \sum_{i=1}^{n} p_i |b_i|^2, \tag{5.8}$$

provided

$$\sum_{i=1}^{n} p_i |b_i - \overline{a_i}|^2 \leqslant r^2 < \sum_{i=1}^{n} p_i |a_i|^2.$$

Let $r = \sqrt{\varepsilon}$, $\varepsilon \in (0,1)$, $a_i, e_i \in \mathbb{C}$, $i \in \{1, \dots, n\}$ with $\sum_{i=1}^{n} p_i |a_i|^2 = \sum_{i=1}^{n} p_i |e_i|^2 = 1$ and $\sum_{i=1}^{n} p_i a_i e_i = 0$. Put $b_i = \overline{a_i} + \sqrt{\varepsilon} e_i$. Then, obviously

$$\sum_{i=1}^{n} p_i |b_i - \overline{a_i}|^2 = r^2, \quad \sum_{i=1}^{n} p_i |a_i|^2 = 1 > r$$

and

$$\sum_{i=1}^{n} p_i |b_i|^2 = \sum_{i=1}^{n} p_i |a_i|^2 + \varepsilon \sum_{i=1}^{n} p_i |e_i|^2 = 1 + \varepsilon,$$

$$\operatorname{Re}\left(\sum_{i=1}^{n} p_i a_i b_i\right) = \sum_{i=1}^{n} p_i |a_i|^2 = 1$$

and thus

$$\sum_{i=1}^{n} p_i |a_i|^2 \sum_{i=1}^{n} p_i |b_i|^2 - \left[\operatorname{Re}\left(\sum_{i=1}^{n} p_i a_i b_i\right)\right]^2 = \varepsilon.$$

Using (5.8), we may write

$$\varepsilon \leqslant c\varepsilon(1+\varepsilon) \quad \text{for} \quad \varepsilon \in (0,1),$$

giving $1 \leqslant c(1+\varepsilon)$ for $\varepsilon \in (0,1)$. Making $\varepsilon \to 0+$, we deduce $c \geqslant 1$. ∎

The following result also holds.

Theorem 7. *Let* $\mathbf{x} = (x_1, \dots, x_n)$, $\mathbf{y} = (y_1, \dots, y_n) \in \mathbb{K}^n$, $\mathbf{p} = (p_1, \dots, p_n) \in \mathbb{R}_+^n$ *with* $\sum_{i=1}^{n} p_i = 1$ *and* $\gamma, \Gamma \in \mathbb{K}$ *such that* $\operatorname{Re}(\gamma\overline{\Gamma}) > 0$ *and either*

$$\sum_{i=1}^{n} p_i \operatorname{Re}\Big[\big(\Gamma\overline{y_i} - x_i\big)\big(\overline{x_i} - \overline{\gamma} y_i\big)\Big] \geqslant 0, \tag{5.9}$$

or, equivalently,

$$\sum_{i=1}^{n} p_i \left| x_i - \frac{\gamma+\Gamma}{2} \cdot \overline{y_i} \right|^2 \leqslant \frac{1}{4} |\Gamma - \gamma|^2 \sum_{i=1}^{n} p_i |y_i|^2. \tag{5.10}$$

Then we have the inequalities

$$\begin{aligned} \sum_{i=1}^{n} p_i |x_i|^2 \sum_{i=1}^{n} p_i |y_i|^2 &\leqslant \frac{1}{4} \cdot \frac{\{\operatorname{Re}[(\overline{\Gamma}+\overline{\gamma}) \sum_{i=1}^{n} p_i x_i y_i]\}^2}{\operatorname{Re}(\Gamma\overline{\gamma})} \\ &\leqslant \frac{1}{4} \cdot \frac{|\Gamma+\gamma|^2}{\operatorname{Re}(\Gamma\overline{\gamma})} \left| \sum_{i=1}^{n} p_i x_i y_i \right|^2. \end{aligned} \tag{5.11}$$

The constant $1/4$ *is best possible in both inequalities.*

Proof: Define $b_i = x_i$ and $a_i = (\overline{\Gamma} + \overline{\gamma})/2 \cdot y_i$ and $r = |\Gamma - \gamma|/2 \left(\sum_{i=1}^{n} p_i |y_i|^2\right)^{1/2}$. Then, by (5.10)

$$\begin{aligned} \sum_{i=1}^{n} p_i |b_i - \overline{a_i}|^2 &= \sum_{i=1}^{n} p_i \left| x_i - \frac{\gamma+\Gamma}{2} \cdot \overline{y_i} \right|^2 \\ &\leqslant \frac{1}{4} |\Gamma - \gamma|^2 \sum_{i=1}^{n} p_i |y_i|^2 = r^2, \end{aligned}$$

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showing that the first condition in (5.1) is satisfied.

We also have

$$\sum_{i=1}^{n} p_i|a_i|^2 - r^2 = \sum_{i=1}^{n} p_i\left|\frac{\Gamma+\gamma}{2}\right|^2|y_i|^2 - \frac{1}{4}|\Gamma-\gamma|^2\sum_{i=1}^{n} p_i|y_i|^2$$
$$= \operatorname{Re}(\Gamma\overline{\gamma})\sum_{i=1}^{n} p_i|y_i|^2 > 0$$

since $\operatorname{Re}(\gamma\overline{\Gamma}) > 0$, and thus the condition in (5.1) is also satisfied.

Using the second inequality in (5.2), one may write

$$\sum_{i=1}^{n} p_i\left|\frac{\Gamma+\gamma}{2}\right|^2|y_i|^2\sum_{i=1}^{n} p_i|x_i|^2 - \left[\operatorname{Re}\sum_{i=1}^{n} p_i\left(\frac{\overline{\Gamma}+\overline{\gamma}}{2}\right)y_i x_i\right]^2$$
$$\leqslant \frac{1}{4}|\Gamma-\gamma|^2\sum_{i=1}^{n} p_i|y_i|^2\sum_{i=1}^{n} p_i|x_i|^2,$$

giving

$$\frac{|\Gamma+\gamma|^2-|\Gamma-\gamma|^2}{4}\sum_{i=1}^{n} p_i|y_i|^2\sum_{i=1}^{n} p_i|x_i|^2 \leqslant \frac{1}{4}\operatorname{Re}\left[(\overline{\Gamma}+\overline{\gamma})\sum_{i=1}^{n} p_i x_i y_i\right]^2,$$

which is clearly equivalent to the first inequality in (5.11).

The second inequality in (5.11) is obvious.

To prove the sharpness of the constant 1/4, assume that the first inequality in (5.11) holds with a constant $C > 0$, that is,

$$\sum_{i=1}^{n} p_i|x_i|^2\sum_{i=1}^{n} p_i|y_i|^2 \leqslant C\cdot\frac{\left\{\operatorname{Re}\left[(\overline{\Gamma}+\overline{\gamma})\sum_{i=1}^{n} p_i x_i y_i\right]\right\}^2}{\operatorname{Re}(\Gamma\overline{\gamma})}, \tag{5.12}$$

provided $\operatorname{Re}(\gamma\overline{\Gamma}) > 0$ and either (5.9) or (5.10) holds.

Assume that $\Gamma, \gamma > 0$ and let $x_i = \gamma\overline{y}_i$. Then (5.9) holds true and by (5.12) we deduce

$$\gamma^2\left(\sum_{i=1}^{n} p_i|y_i|^2\right)^2 \leqslant C\frac{(\Gamma+\gamma)^2\gamma^2\left(\sum_{i=1}^{n} p_i|y_i|^2\right)^2}{\Gamma\gamma},$$

giving

$$\Gamma\gamma \leqslant C(\Gamma+\gamma)^2 \quad \text{for any} \quad \Gamma,\gamma > 0. \tag{5.13}$$

Let $\varepsilon \in (0,1)$ and choose in (5.13) $\Gamma = 1+\varepsilon$, $\gamma = 1-\varepsilon > 0$ to get $1-\varepsilon^2 \leqslant 4C$ for any $\varepsilon \in (0,1)$. Letting $\varepsilon \to 0+$, we deduce $C \geqslant 1/4$ and the sharpness of the constant is proved.

Finally, we note that the conditions (5.9) and (5.10) are equivalent since in an inner product space $(H, \langle\cdot,\cdot\rangle)$ for any vectors $x, z, Z \in H$ one has $\operatorname{Re}\langle Z-x, x-z\rangle \geqslant 0$ if and only if $\left\|x-(z+Z)/2\right\| \leqslant \|Z-z\|/2$ [1]. We omit the details. ∎

6. More Reverses of the Cauchy-Bunyakovsky-Schwarz Inequality

The following result holds.

Theorem 8. *Let* $\mathbf{a} = (a_1, \dots, a_n)$, $\mathbf{b} = (b_1, \dots, b_n) \in \mathbb{K}^n$ *and* $\mathbf{p} = (p_1, \dots, p_n) \in \mathbb{R}_+^n$ *with* $\sum_{i=1}^{n} p_i = 1$. *If* $r > 0$ *and the following condition is satisfied*

$$\sum_{i=1}^{n} p_i \left|b_i - \overline{a_i}\right|^2 \leqslant r^2, \tag{6.1}$$

then we have the inequalities

$$\begin{aligned} 0 &\leqslant \left(\sum_{i=1}^{n} p_i \left|b_i\right|^2 \sum_{i=1}^{n} p_i \left|a_i\right|^2\right)^{1/2} - \left|\sum_{i=1}^{n} p_i a_i b_i\right| \\ &\leqslant \left(\sum_{i=1}^{n} p_i |b_i|^2 \sum_{i=1}^{n} p_i |a_i|^2\right)^{1/2} - \left|\sum_{i=1}^{n} p_i \operatorname{Re}(a_i b_i)\right| \\ &\leqslant \left(\sum_{i=1}^{n} p_i |b_i|^2 \sum_{i=1}^{n} p_i |a_i|^2\right)^{1/2} - \sum_{i=1}^{n} p_i \operatorname{Re}(a_i b_i) \\ &\leqslant \frac{1}{2} r^2. \end{aligned} \tag{6.2}$$

The constant $1/2$ *is best possible in* (6.2) *in the sense that it cannot be replaced by a smaller constant.*

Proof: The condition (6.1) is clearly equivalent to

$$\sum_{i=1}^{n} p_i \left|b_i\right|^2 + \sum_{i=1}^{n} p_i \left|a_i\right|^2 \leqslant 2 \sum_{i=1}^{n} p_i \operatorname{Re}(b_i a_i) + r^2. \tag{6.3}$$

Using the elementary inequality

$$2\left(\sum_{i=1}^{n} p_i \left|b_i\right|^2 \sum_{i=1}^{n} p_i \left|a_i\right|^2\right)^{1/2} \leqslant \sum_{i=1}^{n} p_i \left|b_i\right|^2 + \sum_{i=1}^{n} p_i \left|a_i\right|^2 \tag{6.4}$$

and (6.3), we deduce

$$2\left(\sum_{i=1}^{n} p_i \left|b_i\right|^2 \sum_{i=1}^{n} p_i \left|a_i\right|^2\right)^{1/2} \leqslant 2 \sum_{i=1}^{n} p_i \operatorname{Re}(b_i a_i) + r^2, \tag{6.5}$$

giving the last inequality in (6.2). The other inequalities are obvious.

To prove the sharpness of the constant $1/2$, assume that

$$0 \leqslant \left(\sum_{i=1}^{n} p_i \left|b_i\right|^2 \sum_{i=1}^{n} p_i \left|a_i\right|^2\right)^{1/2} - \sum_{i=1}^{n} p_i \operatorname{Re}(b_i a_i) \leqslant c r^2 \tag{6.6}$$

for any $\mathbf{a}, \mathbf{b} \in \mathbb{K}^n$ and $r > 0$ satisfying (6.1).

Assume that $\mathbf{a}$, $\mathbf{e} \in H$, $\mathbf{e} = (e_1, \dots, e_n)$ with $\sum_{i=1}^{n} p_i |a_i|^2 = \sum_{i=1}^{n} p_i |e_i|^2 = 1$ and $\sum_{i=1}^{n} p_i a_i e_i = 0$. If $r = \sqrt{\varepsilon}$, $\varepsilon > 0$, and if we define $\mathbf{b} = \overline{\mathbf{a}} + \sqrt{\varepsilon}\mathbf{e}$ where $\overline{\mathbf{a}} = (\overline{a_1}, \dots, \overline{a_n}) \in \mathbb{K}^n$, then $\sum_{i=1}^{n} p_i |b_i - \overline{a_i}|^2 = \varepsilon = r^2$, showing that the condition (6.1) is satisfied.

On the other hand,

$$\begin{aligned}
\left(\sum_{i=1}^{n} p_i |b_i|^2 \sum_{i=1}^{n} p_i |a_i|^2\right)^{1/2} - \sum_{i=1}^{n} p_i \operatorname{Re}(b_i a_i)
&= \left(\sum_{i=1}^{n} p_i \left|\overline{a_i} + \sqrt{\varepsilon} e_i\right|^2\right)^{1/2} - \sum_{i=1}^{n} p_i \operatorname{Re}\left[(\overline{a_i} + \sqrt{\varepsilon} e_i) a_i\right] \\
&= \left(\sum_{i=1}^{n} p_i |a_i|^2 + \varepsilon \sum_{i=1}^{n} |e_i|^2\right)^{1/2} - \sum_{i=1}^{n} p_i |a_i|^2 \\
&= \sqrt{1+\varepsilon} - 1.
\end{aligned}$$

Utilising (6.6), we conclude that

$$\sqrt{1+\varepsilon} - 1 \leqslant c\varepsilon \quad \text{for any} \quad \varepsilon > 0. \tag{6.7}$$

Multiplying (6.7) by $\sqrt{1+\varepsilon} + 1 > 0$ and thus dividing by $\varepsilon > 0$, we get

$$(\sqrt{1+\varepsilon} - 1)c \geqslant 1 \quad \text{for any} \quad \varepsilon > 0. \tag{6.8}$$

Letting $\varepsilon \to 0+$ in (6.8), we deduce $c \geqslant 1/2$, and the theorem is proved. ∎

Finally, the following result also holds.

Theorem 9. *Let $\mathbf{x} = (x_1, \dots, x_n)$, $\mathbf{y} = (y_1, \dots, y_n) \in \mathbb{K}^n$, $\mathbf{p} = (p_1, \dots, p_n) \in \mathbb{R}_+^n$ with $\sum_{i=1}^{n} p_i = 1$, and $\gamma, \Gamma \in \mathbb{K}$ with $\Gamma \neq \gamma, -\gamma$, so that either*

$$\sum_{i=1}^{n} p_i \operatorname{Re}\left[(\Gamma \overline{y_i} - x_i)(\overline{x_i} - \overline{\gamma} y_i)\right] \geqslant 0, \tag{6.9}$$

or, equivalently,

$$\sum_{i=1}^{n} p_i \left| x_i - \frac{\gamma + \Gamma}{2} \cdot \overline{y_i} \right|^2 \leqslant \frac{1}{4} |\Gamma - \gamma|^2 \sum_{i=1}^{n} p_i |y_i|^2 \tag{6.10}$$

holds. Then we have the inequalities

$$\begin{aligned}
0 &\leqslant \left(\sum_{i=1}^{n} p_i |x_i|^2 \sum_{i=1}^{n} p_i |y_i|^2\right)^{1/2} - \left|\sum_{i=1}^{n} p_i x_i y_i\right| \\
&\leqslant \left(\sum_{i=1}^{n} p_i |x_i|^2 \sum_{i=1}^{n} p_i |y_i|^2\right)^{1/2} - \left|\sum_{i=1}^{n} p_i \operatorname{Re}\left[\frac{\overline{\Gamma} + \overline{\gamma}}{|\Gamma + \gamma|} x_i y_i\right]\right|
\end{aligned} \tag{6.11}$$

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$$\begin{aligned}&\leqslant \left(\sum_{i=1}^{n} p_i|x_i|^2 \sum_{i=1}^{n} p_i\,|y_i|^2\right)^{1/2} - \sum_{i=1}^{n} p_i \operatorname{Re}\Big[\frac{\overline{\Gamma}+\overline{\gamma}}{|\Gamma+\gamma|} x_i y_i\Big] \\ &\leqslant \frac{1}{4}\cdot\frac{|\Gamma-\gamma|^2}{|\Gamma+\gamma|}\sum_{i=1}^{n} p_i|y_i|^2.\end{aligned}$$

The constant $1/4$ *in the last inequality is best possible.*

PROOF: Consider $b_i = x_i$, $a_i = (\overline{\Gamma}+\overline{\gamma})/2 \cdot y_i$, $i \in \{1, \dots, n\}$ and

$$r := \frac{1}{2}|\Gamma-\gamma|\left(\sum_{i=1}^{n} p_i|y_i|^2\right)^{1/2}.$$

Then, by (6.10), we have

$$\sum_{i=1}^{n} p_i|b_i - \overline{a_i}|^2 = \sum_{i=1}^{n} p_i\left|x_i - \frac{\gamma+\Gamma}{2}\cdot y_i\right|^2 \leqslant \frac{1}{4}|\Gamma-\gamma|^2\sum_{i=1}^{n} p_i|y_i|^2 = r^2$$

showing that (6.1) is valid.

By the use of the last inequality in (6.2), we have

$$\begin{aligned}0 &\leqslant \left(\sum_{i=1}^{n} p_i|x_i|^2 \sum_{i=1}^{n} p_i\left|\frac{\Gamma+\gamma}{2}\right|^2|y_i|^2\right)^{1/2} - \sum_{i=1}^{n} p_i \operatorname{Re}\Big[\frac{\overline{\Gamma}+\overline{\gamma}}{2} x_i y_i\Big] \\ &\leqslant \frac{1}{8}|\Gamma-\gamma|^2\sum_{i=1}^{n} p_i|y_i|^2.\end{aligned}$$

Dividing by $|\Gamma+\gamma|/2 > 0$, we deduce

$$\begin{aligned}0 &\leqslant \left(\sum_{i=1}^{n} p_i\,|x_i|^2 \sum_{i=1}^{n} p_i\,|y_i|^2\right)^{1/2} - \sum_{i=1}^{n} p_i \operatorname{Re}\Big[\frac{\overline{\Gamma}+\overline{\gamma}}{|\Gamma+\gamma|} x_i y_i\Big] \\ &\leqslant \frac{1}{4}\cdot\frac{|\Gamma-\gamma|^2}{|\Gamma+\gamma|}\sum_{i=1}^{n} p_i|y_i|^2,\end{aligned}$$

which is the last inequality in (6.11).

The other inequalities are obvious.

To prove the sharpness of the constant $1/4$, assume that there exists a constant $c > 0$, such that

$$\left(\sum_{i=1}^{n} p_i|x_i|^2 \sum_{i=1}^{n} p_i|y_i|^2\right)^{1/2} - \sum_{i=1}^{n} p_i \operatorname{Re}\Big[\frac{\overline{\Gamma}+\overline{\gamma}}{|\Gamma+\gamma|} x_i y_i\Big] \leqslant c\cdot\frac{|\Gamma-\gamma|^2}{|\Gamma+\gamma|}\sum_{i=1}^{n} p_i|y_i|^2, \tag{6.12}$$

provided either (6.9) or (6.10) holds.

Let $n = 2$, $\mathbf{y} = (1, 1)$, $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$, $\mathbf{p} = (1/2, 1/2)$ and $\Gamma, \gamma > 0$ with $\Gamma > \gamma$. Then by (6.12) we deduce

$$\sqrt{2}\sqrt{x_1^2 + x_2^2} - (x_1 + x_2) \leqslant 2c\frac{(\Gamma-\gamma)^2}{\Gamma+\gamma}. \tag{6.13}$$

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If $x_1 = \Gamma$, $x_2 = \gamma$, then $(\Gamma - x_1)(x_1 - \gamma) + (\Gamma - x_2)(x_2 - \gamma) = 0$, showing that the condition (6.9) is valid for $n = 2$ and $\mathbf{p}$, $\mathbf{x}$, $\mathbf{y}$ as above. Replacing x_1 and x_2 in (6.13), we deduce

$$\sqrt{2}\sqrt{\Gamma^2 + \gamma^2} - (\Gamma + \gamma) \leqslant 2c\frac{(\Gamma - \gamma)^2}{\Gamma + \gamma}. \tag{6.14}$$

If in (6.14) we choose $\Gamma = 1 + \varepsilon$, $\gamma = 1 - \varepsilon$ with $\varepsilon \in (0,1)$, we deduce

$$\sqrt{1 + \varepsilon^2} - 1 \leqslant 2c\varepsilon^2. \tag{6.15}$$

Finally, multiplying (6.15) with $\sqrt{1 + \varepsilon^2} + 1 > 0$ and then dividing by ε^2, we deduce

$$1 \leqslant 2c\left(\sqrt{1 + \varepsilon^2} + 1\right) \text{ for any } \varepsilon > 0. \tag{6.16}$$

Letting $\varepsilon \to 0+$ in (6.16), we get $c \geqslant 1/4$, and the sharpness of the constant is proved. □

REMARK 5. The integral version may be stated in a canonical way. The corresponding inequalities for integrals will be considered in another work devoted to positive linear functionals with complex values that is in preparation.

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