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EXTENSIONS OF FILTERS AND FIELDS OF SETS (I).

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Abstract

We investigate the problem of the existence of filters with some properties. This leads to a solution of two problems of Ulam concerning σ-fields on the real line.

DEFINITION. Let $\mathscr{F}$ be a uniform filter on κ. A family $\mathscr{A} \subseteq \mathscr{F}$ is a *basis for* $\mathscr{F}$ if $|\mathscr{A}| \leqq \kappa$ and for every $F \in \mathscr{F}$ there is some $A \in \mathscr{A}$ such that $A \subseteq F$. A family $\mathscr{A} \subseteq [\kappa]^\kappa$ is a *pseudobasis for* $\mathscr{F}$ if $|\mathscr{A}| \leqq \kappa$ and for every $F \in \mathscr{F}$ there is some $A \in \mathscr{A}$ such that $A \subseteq F$. A subset $L \subseteq \kappa$ of the cardinality κ is a *Lusin set for* $\mathscr{F}$ if $|L - F| < \kappa$ for all $F \in \mathscr{F}$. Let $\mathscr{F}$ be a filter on κ and let $\delta \geqq 2$, we say that $\mathscr{F}$ has the *property* $\boldsymbol{U}(\delta)$ if there exists a family $\mathscr{U} \subseteq [\kappa]^{\geq \delta}$ of pairwise disjoint sets such that every selector of $\mathscr{U}$ is $\mathscr{F}$-stationary.

It is well known that the filter on the real line which is dual to the ideal of the sets of Lebesgue measure zero has a basis. The same holds for the filter of comeager sets. It follows from the Continuum Hypothesis **CH** (or Martin's Axiom **A**) that these filters have Lusin sets. For comeager sets see Sierpiński (1934), pages 36 and 81, and for the Lebesgue measurability see Sierpiński (1934), pages 80 and 82.

If we assume **CH** then a Lusin set in our sense for the filter of comeager sets is a set with the property L in the terminology of Sierpiński (1934), p. 81.

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A Lusin set in our sense for the filter of sets whose complements have Lebesgue measure zero is a set with the property S in the terminology of Sierpiński (1934), p. 81.

Filters without any Lusin set were considered by Prikry (1974), where they appear under the name of filters "dense modulo sets of power $< \kappa$". The property $\mathbf{U}(\delta)$ is connected with some problems of Ulam studied in §.4. of this paper.

The following theorem presents the relations between the notions introduced above.

THEOREM 0. *Let $\mathcal{F}$ be a uniform filter on κ. The following are true:*

(a) *If $\mathcal{F}$ has a basis, then $\mathcal{F}$ has a pseudobasis and $\mathcal{F}$ has a pseudobasis if and only if $\mathcal{F}$ has the property* $\mathbf{U}(\kappa)$.

(b) *If $\mathcal{F}$ has a Lusin set, then $\mathcal{F}$ has a pseudobasis.*

(c) *For κ-complete filters, if $\mathcal{F}$ has a basis, then $\mathcal{F}$ has a Lusin set.*

(d) *If $\mathcal{F}$ has the property* $\mathbf{U}(\delta_2)$, *then $\mathcal{F}$ has the property* $\mathbf{U}(\delta_1)$ *for $\delta_1 < \delta_2$.*

(e) *For every uncountable regular κ, no κ-complete normal filter on κ has the property* $\mathbf{U}(2)$.

(f) *For every regular $\lambda \leqq \kappa$* there is a *λ-complete uniform filter on κ without the property* $\mathbf{U}(2)$.

(g) *every regular $\lambda \leqq \kappa$ there is a λ-complete uniform filter on κ with a Lusin set and without any basis.*

(h) *For every regular $\lambda \leqq \kappa$ there is a λ-complete uniform filter on κ with a pseudobasis and without any Lusin set.*

(i) *For every regular $\lambda < \kappa$ there is a λ-complete uniform filter on κ with a basis and without any Lusin set.*

(j) *For every cardinal δ (finite or infinite) such that $2 \leqq \delta < \kappa$ and every regular $\lambda \leqq \kappa$ there is a λ-complete uniform filter on κ with the property* $\mathbf{U}(\delta)$ *and without the property* $\mathbf{U}(\delta^+)$.

The proof of Theorem. 0, is given in §1. In §2, we investigate the possibility of extending a λ-complete filter on κ to a λ-complete filter without the property $\mathbf{U}(2)$. Then, in §3, we apply the results of §2, to the problem of the extension of fields of sets to proper fields of sets which contain selectors of every family of disjoint sets of power $\geqq 2$. In §4, we give a negative solution to two similar problems of Ulam concerning σ-fields on the real line.

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0. Terminology and notation

$|X|$ denotes the cardinality of X. Small greek letters denote ordinals. If α is an ordinal then $\alpha = \{\xi : \xi < \alpha\}$. The symbols, κ, λ, σ (with indices if necessary) denote infinite cardinals and δ any (finite or infinite) cardinal. The cofinality of κ is denoted by $\mathrm{cf}(\kappa)$ and the cardinal successor of κ by κ^+. The set of natural numbers is denoted by ω and $\omega_1 = \omega^+$. We shall identify 2^ω with the set of reals. A cardinal κ is a *successor cardinal* if $\kappa = \sigma^+$ for some σ; otherwise it is a *limit cardinal*. κ is *regular* if $\mathrm{cf}(\kappa) = \kappa$; otherwise it is *singular*. A cardinal is *weakly inaccessible* if it is regular and a limit cardinal. A cardinal κ is *strongly compact* if each κ-complete filter can be extended to a κ-complete ultrafilter.

$\mathcal{S}(X)$ denotes the power set of X. Moreover, we use the following notations:

$$[X]^\delta = \{Y \subseteq X : |Y| = \delta\};$$

$$[X]^{\geqq\delta} = \{Y \subseteq X : |Y| \geqq \delta\};$$

$$[X]^{<\delta} = \{Y \subseteq X : |Y| < \delta\} \quad \text{and}$$

$$S^\delta(X) = \{Y \subseteq X : |X - Y| < \delta\}.$$

Let $\mathcal{F}$ be a filter on κ. $\mathcal{F}$ is *proper* if $0 \notin \mathcal{F}$; $\mathcal{F}$ is *uniform* if $|X| = \kappa$ for every $X \in \mathcal{F}$ and $\mathcal{F}$ is *non-trivial* if $\mathcal{S}^\omega(\kappa) \subseteq \mathcal{F}$. A filter $\mathcal{F}$ is λ*-complete* if for every $\alpha < \lambda$ and every $\{F_\xi : \xi < \alpha\} \subseteq \mathcal{F}$ we have $\bigcap_{\xi<\alpha} F_\xi \in \mathcal{F}$. All filters considered in our paper are proper and non-trivial.

We use similar terminology for fields of sets, for example $\mathcal{B} \subseteq \mathcal{S}(\kappa)$ is a *proper* field if $\mathcal{B} \neq \mathcal{S}(\kappa)$. It is λ*-complete* if for every $\alpha < \lambda$ and every family $\{B_\xi : \xi < \alpha\} \subseteq \mathcal{B}$ we have $\bigcap_{\xi<\alpha} B_\xi \in \mathcal{B}$. A set $X \subseteq \kappa$ is *unbounded* if it is cofinal in κ, that is, for all $\xi < \kappa$, there exists $\eta \in X$ such that $\xi \leqq \eta$. It is *closed* if it is closed in the order topology on κ. It is well known that if $\mathrm{cf}(\kappa) > \omega$ then the filter generated by the family of all closed unbounded subsets of κ is a proper $\mathrm{cf}(\kappa)$-complete filter on κ. We call it the *closed unbounded filter on* κ.

We shall adopt some of the terminology used for the closed unbounded filters. Let $\mathcal{F}$ be a filter on κ, a set $X \subseteq \kappa$ is $\mathcal{F}$-stationary if $X \cap F \neq 0$ for every $F \in \mathcal{F}$. A filter $\mathcal{F}$ on κ is *normal* if for each $\mathcal{F}$-stationary set $X \subseteq \kappa$ and every regressive function $f: X \to \kappa$ (f is *regressive* if for all non zero ξ belonging to X, $f(\xi) < \xi$) there exists an $\mathcal{F}$-stationary set $Y \subseteq X$ and $\beta \in \kappa$ such that $f(\xi) = \beta$ for all $\xi \in Y$. The well known Fodor's Theorem, (Fodor (1956)), says that for a regular cardinal $\kappa > \omega$ the closed unbounded filter on κ is κ-complete and normal. A family $\mathcal{U}$ is a *partition* of a set X if $\bigcup \mathcal{U} = X$,

$0 \notin \mathcal{U}$ and elements of $\mathcal{U}$ are pairwise disjoint. A set S is a *selector* of a family $\mathcal{A}$ of sets if $|S \cap A| = 1$ for all $A \in \mathcal{A}$ and $S \subseteq \bigcup \mathcal{A}$.

Let $\mathcal{F}$ be a filter on κ and $A, B \subseteq \kappa$. We say that A and B are *$\mathcal{F}$-almost disjoint* if $A \cap B$ is not $\mathcal{F}$-stationary. By *sat*$(\mathcal{F})$, we denote the smallest cardinal δ such that every family of pairwise $\mathcal{F}$-almost disjoint $\mathcal{F}$-stationary sets is of cardinality less than δ. If $f: \kappa \to \kappa$ then for $A \subseteq \kappa$ we denote by $f * A$ the set $\{f(\xi): \xi \in A\}$; similarly for a family $\mathcal{A}$ of subsets of κ, $f * \mathcal{A} = \{f * A : A \in \mathcal{A}\}$.

The following two combinatorial facts will play an essential role in our considerations.

FACT A. (Sierpiński). *Assume that $\mathcal{G} \subseteq [\kappa]^\kappa$ has cardinality $\leqq \kappa$. Then there exists a family $\mathcal{H} \subseteq [\kappa]^\kappa$ of pairwise disjoint sets such that $|\mathcal{H}| = \kappa$ and for every $G \in \mathcal{G}$ and $H \in \mathcal{H}$ we have $|G \cap H| = \kappa$.*

For the proof see Sierpiński (1934), page 113, Theorem 1.

FACT B. (Solovay). *Let $\mathcal{F}$ be the closed unbounded filter on a regular cardinal $\kappa > \omega$. Then every $\mathcal{F}$-stationary set A can be decomposed into κ pairwise disjoint $\mathcal{F}$-stationary sets.*

For the proof see Solovay (1971).

CH denotes the Continuum Hypothesis, that is the statement $2^\omega = \omega_1$. $\mathbf{A}\kappa$ denotes a version of Martin's Axiom from Martin and Solovay (1970). For consequences of $\mathbf{A}\kappa$ used in §4, see Martin and Solovay (1970).

1. Proof of Theorem 0

PROOF OF (a). Obviously if $\mathcal{F}$ has a basis, then $\mathcal{F}$ has a pseudobasis. Suppose that $\mathcal{A}$ is a pseudobasis for $\mathcal{F}$. By Fact A., there exists a family $\mathcal{H} \subseteq [\kappa]^\kappa$, $|\mathcal{H}| = \kappa$, of pairwise disjoint sets, such that for every $A \in \mathcal{A}$ and $H \in \mathcal{H}$, we have $|A \cap H| = \kappa$. Let f be a mapping of $\mathcal{H}$ onto $\mathcal{A}$. Clearly, the family $\mathcal{G} = \{H \cap f(H): H \in \mathcal{H}\}$ is a pseudobasis for $\mathcal{F}$ consisting of pairwise disjoint sets. But then every selector of $\mathcal{G}$ is $\mathcal{F}$-stationary and hence $\mathcal{F}$ has the property $\mathbf{U}(\kappa)$.

Suppose that $\mathcal{F}$ has the property $\mathbf{U}(\kappa)$. Let $\mathcal{U} \subseteq [\kappa]^\kappa$ be a family of pairwise disjoint sets such that every selector of $\mathcal{U}$ is $\mathcal{F}$-stationary. Obviously $|\mathcal{U}| \leqq \kappa$. We claim that $\mathcal{U}$ is a pseudobasis for $\mathcal{F}$. If not, then there exists $F \in \mathcal{F}$ such that for each $U \in \mathcal{U}$ we have $U - F \neq 0$. Let S be a selector of the family $\{U - F: U \in \mathcal{U}\}$. Then $S \cap F = 0$, so S is not $\mathcal{F}$-stationary which is imposible since S is a selector of $\mathcal{U}$.

PROOF OF (b). Let L be a Lusin set for $\mathcal{F}$. Let $\mathcal{A}$ be any partition of L such that $\mathcal{A} \subseteq [L]^\kappa$ and $|\mathcal{A}| = \kappa$. We claim that $\mathcal{A}$ is a pseudobasis for $\mathcal{F}$.

Indeed, take any $F \in \mathcal{F}$. Since $|L - F| < \kappa$, the family $\{A \in \mathcal{A} : A - F \neq 0\}$ has cardinality less than κ. Thus there exists $A \in \mathcal{A}$ such that $A \subseteq F$.

PROOF OF (c). (Compare with P_8 and P_8 a of Sierpiński (1934).) Suppose that $\mathcal{F}$ is a κ-complete filter with a basis $\mathcal{A} = \{A_\xi : \xi < \kappa\}$. We can assume that $X_\xi = \bigcap_{\eta < \xi} A_\eta - A_\xi \neq 0$, for all $\xi < \kappa$. Then any selector of $\{X_\xi : \xi < \kappa\}$ is a Lusin set for $\mathcal{F}$.

PROOF OF (d). It is obvious by the definition.

PROOF OF (e). Let κ be regular and uncountable. Let $\mathcal{F}$ be a κ-complete normal filter on κ. We claim that $\mathcal{F}$ does not have the property U(2). Suppose to the contrary that $\mathcal{F}$ has the property U(2). Hence there exists a family $\mathcal{U} \subseteq [\kappa]^{\geqq 2}$ of pairwise disjoint sets such that every selector of $\mathcal{U}$ is $\mathcal{F}$-stationary. Choose, for each $U \in \mathcal{U}$, two distinct elements ξ_U, η_U of U such that $\xi_U < \eta_U$. Let $S = \{\xi_U : U \in \mathcal{U}\}$ and $T = \{\eta_U : U \in \mathcal{U}\}$. Note that both S and T as selectors of $\mathcal{U}$ are $\mathcal{F}$-stationary sets. Hence define a function $f : T \to \kappa$ by $f(\eta_U) = \xi_U$, for all $U \in \mathcal{U}$.
Then f is regressive and one-to-one, which contradicts the normality of $\mathcal{F}$.

PROOF OF (f). By (e), we can restrict ourselves to the case where κ is singular. In the case $\kappa = \omega$ any uniform ultrafilter is sufficient. The proof below works in the case where κ is limit and uncountable.

Suppose that κ is singular and $\lambda < \kappa$ is a regular cardinal. We show that there exists a λ-complete uniform filter on κ without the property U(2). Let $\kappa = \bigcup_{\xi < \mathrm{cf}(\kappa)} A_\xi$, where $\{A_\xi : \xi < \mathrm{cf}(\kappa)\}$ are pairwise disjoint and for every $\xi < \eta < \mathrm{cf}(\kappa)$, $|A_\xi|$ is regular, $\lambda \leqq |A_\xi|$ and $\Sigma_{\zeta < \eta} |A_\zeta| < |A_\eta|$. For every $\xi < \mathrm{cf}(\kappa)$, pick an $|A_\xi|$-complete uniform filter $\mathcal{F}_\xi$ on A_ξ without the property U(2). Define a filter $\mathcal{F}$ on κ by: $X \in \mathcal{F}$ if for each $\xi < \mathrm{cf}(\kappa)$, $A_\xi \cap X \in \mathcal{F}_\xi$. Note the following two facts:

(1) $\mathcal{F}$ is a uniform λ-complete filter on κ (in fact $\mathcal{F}$ is $|A_0|$-complete).

(2) A set $X \subseteq \kappa$ is $\mathcal{F}$-stationary if and only if for some $\xi < \mathrm{cf}(\kappa)$, $A_\xi \cap X$ is $\mathcal{F}_\xi$-stationary.

We check that $\mathcal{F}$ does not have U(2). Let $\mathcal{U} = \{U_\xi : \xi < \gamma\} \subseteq [\kappa]^2$ be any family of pairwise disjoint sets. Put $T_\xi = \{\eta < \gamma : U_\eta \subseteq A_\xi\}$ and $R_\xi = \{\eta < \gamma : U_\eta \cap A_\xi \neq 0$ and $U_\eta \cap A_\zeta \neq 0$ for some $\zeta < \xi\}$. Let $T = \bigcup_{\xi < \mathrm{cf}(\kappa)} T_\xi$ and $R = \bigcup_{\xi < \mathrm{cf}(\kappa)} R_\xi$. Note that $T \cap R = 0$ and $T \cup R = \gamma$.

Since $\mathcal{F}_\xi$ does not have U(2), for all $\xi < \mathrm{cf}(\kappa)$, the family $\{U_\eta : \eta \in T_\xi\} \subseteq [A_\xi]^2$ has a selector $S_\xi \subseteq A_\xi$ which is not $\mathcal{F}_\xi$-stationary. Consequently, by (2), $S = \bigcup_{\xi < \mathrm{cf}(\kappa)} S_\xi$ is a selector of the family $\{U_\eta : \eta \in T\}$ which is not $\mathcal{F}$-stationary.

For each $\xi < \mathrm{cf}(\kappa)$, let $Y_\xi = \bigcup_{\eta \in R_\xi} U_\eta \cap A_\xi$. By the definition of R_ξ, it follows that Y_ξ is a selector of $\{U_\eta : \eta \in R_\xi\}$. Moreover $|Y_\xi| \leqq \Sigma_{\zeta<\xi} |A_\zeta| < |A_\xi|$. Thus Y_ξ is not $\mathcal{F}_\xi$-stationary. Whence, by (2), $Y = \bigcup_{\xi<\mathrm{cf}(\kappa)} Y_\xi$ is a selector of the family $\{U_\eta : \eta \in R\}$ which is not $\mathcal{F}$-stationary. Thus, $S \cup Y$ is a selector of $\mathcal{U}$ which is not $\mathcal{F}$-stationary.

Observe that in the proof above we used only the fact that κ is a limit uncountable cardinal.

Proof of (g). Let $\mathcal{F} = \{F \subseteq \kappa \times \kappa : |\{\xi : (\eta, \xi) \notin F\}| < \lambda \text{ for all } \eta < \kappa\}$. Then $\mathcal{F}$ is a λ-complete uniform filter on $\kappa \times \kappa$. We claim that $\mathcal{F}$ does not have any basis. Suppose to the contrary that $\mathcal{A} = \{A_\xi : \xi < \kappa\}$ is a basis for $\mathcal{F}$. For $\eta < \kappa$, let $B_\eta = A_\eta \cap \{(\eta, \xi) : \xi < \kappa\}$. Since $A_\eta \in \mathcal{F}$, we have $B_\eta \neq 0$. Pick $(\eta, \xi_\eta) \in B_\eta$ and put $C = (\kappa \times \kappa) - \{(\eta, \xi_n) : \eta < \kappa\}$. Then $C \in \mathcal{F}$. It is clear that $(\eta, \xi_\eta) \in A_\eta - C$. Thus $\mathcal{A}$ is not a basis for $\mathcal{F}$. On the other hand $\mathcal{F}$ has a Lusin set. Indeed, $L = \{(0, \xi) : \xi < \kappa\}$ is a Lusin set for $\mathcal{F}$.

The example above was suggested to us by E. Marczewski.

Proof of (h). Let $\mathcal{A} \subseteq [\kappa]^\kappa$ be a maximal family of almost disjoint subsets of κ such that $|\mathcal{A}| > \kappa$. That is, for every distinct $A_0, A_1 \in \mathcal{A}$ we have $|A_0 \cap A_1| < \kappa$, and for every $X \in [\kappa]^\kappa$ there is $A \in \mathcal{A}$ such that $|X \cap A| = \kappa$. The existence of such a family follows from Sierpiński (1938). Let $\mathcal{F} = \{X \subseteq \kappa : \text{there exists } \mathcal{A}' \subseteq \mathcal{A} \text{ such that } |\mathcal{A}'| < \lambda \text{ and } X \supseteq (\kappa - \cup \mathcal{A}')\}$. Obviously $\mathcal{F}$ is a λ-complete uniform filter without any Lusin set. Moreover, $\mathcal{F}$ has a pseudobasis. Indeed, fix $\mathcal{A}_0 \subseteq \mathcal{A}$ with cardinality κ. For each $A \in \mathcal{A}_0$, let $\mathcal{P}_A \subseteq [A]^\kappa$ be a partition of A such that $|\mathcal{P}_A| = \kappa$. It is easy to see that $\mathcal{P} = \cup\{\mathcal{P}_A : A \in \mathcal{A}_0\}$ is a pseudobasis for $\mathcal{F}$.

The example above was suggested to us by B. Balcar (compare Balcar and Vopěnka (1972)). Our original example was the following: Let

$$\mathcal{E} = \{E \subseteq \kappa \times \kappa : |\{\eta : (\eta, \xi) \notin E \text{ for some } \xi < \kappa\}| < \lambda\}$$

and let $\mathcal{F}$ be the filter defined in the proof of (g). Let $\mathcal{G}$ be the filter generated by $\mathcal{E}$ and $\mathcal{F}$. Taking a suitably large λ in the definition of $\mathcal{E}$ we get that $\mathcal{G}$ has no Lusin set but has a pseudobasis.

Proof of (i). Consider the following two cases.

Case I. $\kappa = cf(\kappa)$. Let λ be a regular cardinal less than κ. Let $\{C_\xi : \xi < \lambda\} \subseteq [\kappa]^\kappa$ be a partition of κ. Define a filter $\mathcal{G} = \{G \subseteq \kappa : C_\xi \subseteq G \text{ for some } \eta < \lambda \text{ and all } \xi \text{ such that } \eta < \xi < \lambda\}$. Then $\mathcal{G}$ is a λ-complete uniform filter on κ with a basis. Let X be an arbitrary subset of κ of cardinality κ. We claim that X is not a Lusin set for $\mathcal{G}$. Indeed, since $\lambda < \kappa = \mathrm{cf}(\kappa)$, there exists $\xi_0 < \lambda$ such that $|X \cap C_{\xi_0}| = \kappa$ and hence $|X - \bigcup_{\xi_0<\xi<\lambda} C_\xi| = \kappa$. But $\bigcup_{\xi_0<\xi<\lambda} C_\xi \in \mathcal{G}$ which proves our claim.

Case II. $cf(\kappa) < \kappa$. Let $\lambda < \kappa$ be a regular cardinal and take a regular $\lambda_1 \geqq \lambda$ such that $\mathrm{cf}(\kappa) < \lambda_1 < \kappa$. Let $\{C_\xi : \xi < \lambda_1\} \subseteq [\kappa]^\kappa$ be a partition of κ. Define a filter $\mathscr{G} = \{G \subseteq \kappa : C_\xi \subseteq G$ for some $\eta < \lambda_1$ and all ξ such that $\eta < \xi < \lambda_1\}$. Then $\mathscr{G}$ is a λ_1-complete uniform filter on κ with a basis. Let X be an arbitrary subset of κ of cardinality κ. We claim that X is not a Lusin set for $\mathscr{G}$. Indeed, since $|X| = \kappa$ and $\mathrm{cf}(\kappa) < \lambda_1$, there exists $\eta_0 < \lambda_1$ such that $|\bigcup_{\xi \leqq \eta_0} C_\xi \cap X| = \kappa$. But $\bigcup_{\eta_0 < \xi < \lambda_1} C_\xi \in \mathscr{G}$ and $|X - \bigcup_{\eta_0 < \xi < \lambda_1} C_\xi| = |X \cap \bigcup_{\xi \leqq \eta_0} C_\xi| = \kappa$ which proves our claim.

Proof of (j). Suppose that a cardinal δ with $2 \leqq \delta < \kappa$ and a regular cardinal $\lambda \leqq \kappa$ are given. Without loss of the generality we can assume $\delta < \lambda$. By (*f*), there is a λ-complete uniform filter $\mathscr{F}$ on κ without the property $\mathbf{U}(2)$. Let $\mathscr{G} = \{X \subseteq \kappa \times \delta : Y \times \delta \subseteq X$ for some $Y \in \mathscr{F}\}$. Of course $\mathscr{G}$ is a λ-complete uniform filter on $\kappa \times \delta$. We claim that $\mathscr{G}$ has the property $\mathbf{U}(\delta)$. Indeed, let $A_\xi = \{\xi\} \times \delta$ for $\xi < \kappa$. Then $\kappa \times \delta = \bigcup_{\xi < \kappa} A_\xi$ and $|A_\xi| = \delta$ for all $\xi < \kappa$. Let S be any selector of the family $\{A_\xi : \xi < \kappa\}$. If S is not $\mathscr{G}$-stationary then there exists some $F \in \mathscr{F}$ such that $S \cap (F \times \delta) = 0$. But $F \times \delta = \bigcup_{\xi \in \mathscr{F}} A_\xi$, which gives a contradiction. Thus $\mathscr{G}$ has the property $\mathbf{U}(\delta)$.

Now, we show that $\mathscr{G}$ does not have the property $\mathbf{U}(\delta^+)$. Let $\{Y_\xi : \xi < \gamma\}$ be any family of pairwise disjoint subsets of $\kappa \times \delta$ such that $|Y_\xi| \geqq \delta^+$ for all $\xi < \gamma$. For a given $\xi < \gamma$, let $Y_{\xi\eta} = Y_\xi \cap (\kappa \times \{\eta\})$. Then $Y_\xi = \bigcup_{\eta < \delta} Y_{\xi\eta}$ and so there exists $\eta_\xi < \delta$ such that $|Y_{\xi\eta_\xi}| \geqq 2$. Let $A_\eta = \{Y_{\xi\eta_\xi} : \xi < \gamma$ and $\eta_\xi = \eta\}$. Since $\mathscr{F}$ does not have the property $\mathbf{U}(2)$, there exists a set $S_\eta \subseteq \kappa$ such that $S_\eta \times \{\eta\}$ is a selector of A_η and S_η is not $\mathscr{F}$-stationary. But then $S = \bigcup_{\eta < \delta} (S_\eta \times \{\eta\})$ is a selector of $\{Y_\xi : \xi < \gamma\}$ which is not $\mathscr{G}$-stationary by the λ-completeness of $\mathscr{G}$.

2. Extensions of filters

Theorem 1. *Suppose that $\mathscr{S}^\kappa(\kappa) \subseteq \mathscr{F}_1 \subseteq \mathscr{F}_2$, $\mathscr{F}_1 \neq \mathscr{F}_2$ and $\mathscr{F}_1$ is a λ-complete filter with a basis. Then there exists a λ-complete uniform filter $\mathscr{G}$ such that $\mathscr{F}_1 \subseteq \mathscr{G} \subseteq \mathscr{F}_2$ and $\mathscr{G}$ has no basis.*

Proof. Let A be an $\mathscr{F}_1$-stationary set such that $\kappa - A \in \mathscr{F}_2$. Since $\mathscr{S}^\kappa(\kappa) \subseteq \mathscr{F}_1$, and $\mathscr{F}_1$ is λ-complete we have $\lambda \leqq \mathrm{cf}(\kappa)$, $|A| = \kappa$ and $\mathscr{F}_1 \upharpoonright A$ is a uniform λ-complete filter on A. Let $\mathscr{B}$ be a basis for $\mathscr{F}_1$ and take $\mathscr{B} \upharpoonright A = \{B \cap A : B \in \mathscr{B}\}$. Then $\mathscr{B} \upharpoonright A$ is a basis for $\mathscr{F}_1 \upharpoonright A$ and every element of $\mathscr{B} \upharpoonright A$ has cardinality κ. Now, by Fact A, there exists a partition $\{X_\xi : \xi < \kappa\}$ of A such that each X_ξ is $\mathscr{F}_1 \upharpoonright A$-stationary, and therefore $\mathscr{F}_1$-stationary.

By Theorem 0.(g), there is a $\mathrm{cf}(\kappa)$-complete uniform filter $\mathscr{F}_0$ on κ without any basis. We put

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$$\tilde{\mathscr{F}}_0 = \left\{ Y \subseteq \kappa : Y \supseteq \left(\bigcup_{\xi \in C} X_\xi \right) \cup (\kappa - A) \text{ for some } C \in \mathscr{F}_0 \right\}.$$

Put $\mathscr{G} = \{F_0 \cap F_1 : F_0 \in \tilde{\mathscr{F}}_0$ and $F_1 \in \mathscr{F}_1\}$. Obviously $\mathscr{G}$ is a λ-complete uniform filter on κ and $\mathscr{F}_1 \subseteq \mathscr{G}$. To check that $\mathscr{G} \subseteq \mathscr{F}_2$. Take any $G \in \mathscr{G}$. Then $G = F_0 \cap F_1$ for some $F_0 \in \tilde{\mathscr{F}}_0$ and $F_1 \in \mathscr{F}_1$. Since $F_0 \supseteq \kappa - A$, we have $F_0 \in \mathscr{F}_2$ and $F_1 \in \mathscr{F}_2$ and so $G \in \mathscr{F}_2$.

We show that $\mathscr{G}$ does not have a basis. Suppose to the contrary that $\{D_\xi : \xi < \kappa\}$ is a basis of $\mathscr{G}$. Then, for every $\xi < \kappa$, there are $D_\xi^{(0)} \in \tilde{\mathscr{F}}_0$ and $D_\xi^{(1)} \in \mathscr{F}_1$ such that $D_\xi = D_\xi^{(0)} \cap D_\xi^{(1)}$. Also, for each $\xi < \kappa$, there exists $C_\xi \in \mathscr{F}_0$ such that $D_\xi^{(0)} \supseteq \bigcup_{\eta \in C_\xi} X_\eta \cup (\kappa - A)$. Since $\mathscr{F}_0$ does not have a basis, the family $\{C_\xi : \xi < \kappa\}$ is not a basis of $\mathscr{F}_0$. Consequently there exists $C \in \mathscr{F}_0$ such that $C_\xi - C \neq 0$ for all $\xi < \kappa$. Put $E = \bigcup_{\eta \in C} X_\eta \cup (\kappa - A)$. Obviously $E \in \tilde{\mathscr{F}}_0$, so $E \in \mathscr{G}$. Hence, there is $\xi < \kappa$ such that $D_\xi \subseteq E$. Let $\zeta \in C_\xi - C$. Then obviously $X_\zeta \cap E = 0$. On the other hand $X_\zeta \cap D_\xi = X_\zeta \cap D_\xi^{(0)} \cap D_\xi^{(1)} = X_\zeta \cap D_\xi^{(1)} \neq 0$ since X_ζ is $\mathscr{F}_1$-stationary. This gives a contradiction and proves Theorem 1.

In proofs of other results about extensions of filters the theorem below is useful.

THEOREM 2. *Suppose that $\mathscr{F}$ is a uniform filter on κ with a pseudobasis and $\mathscr{G}$ is a filter on κ for which there exists a partition of κ into κ $\mathscr{G}$-stationary sets. Then there exists a permutation f of κ such that the family $\{F \cap f * G : F \in \mathscr{F}$ and $G \in \mathscr{G}\}$ is a uniform filter on κ.*

PROOF. Since $\mathscr{F}$ has a pseudobasis there exists a partition $\{Y_\xi : \xi < \kappa\}$ of κ which is a pseudobasis for $\mathscr{F}$ (see the proof of Theorem 0(a)). By the assumption on $\mathscr{G}$, using the fact that $|\kappa^2| = \kappa$, there exists a partition $\{V_\xi : \xi < \kappa\}$ of κ into κ $\mathscr{G}$-stationary sets such that $|G \cap V_\xi| = \kappa$ for every $G \in \mathscr{G}$ and every $\xi < \kappa$. Define $f \restriction Y_\xi$ to be any one-to-one mapping of Y_ξ onto V_ξ for all $\xi < \kappa$. It is easy to see that f fulfils all our requirements.

Actually we shall need the following stronger version of Theorem 2.

THEOREM 2'. *Suppose that $\mathscr{F}$ and $\mathscr{G}$ satisfy all the assumptions of Theorem 2. Then there exists a permutation f of κ such that the family $\mathscr{H} = \{F \cap f * G : F \in \mathscr{F}$ and $G \in \mathscr{G}\}$ is a uniform filter on κ and in addition there exists a partition $\{Z_\xi : \xi < \kappa\}$ of κ into $\mathscr{H}$-stationary sets.*

PROOF. Let $\mathscr{A}$ be a pseudobasis for $\mathscr{F}$. By Fact A, there exists a partition $\{Y_\xi : \xi < \kappa\}$ of κ such that $|Y_\xi \cap A| = \kappa$ for every $\xi < \kappa$ and every $A \in \mathscr{A}$. Consequently $\mathscr{F} \restriction Y_\xi$ is a filter on Y_ξ with a pseudobasis.

By the assumptions on $\mathscr{G}$, using the fact that $|\kappa^2| = \kappa$, there exists a partition $\{X_\xi : \xi < \kappa\}$ of κ such that each X_ξ is the union of κ $\mathscr{G}$-stationary sets

$\{X_{\xi\eta}: \eta < \kappa\}$. Hence $G \upharpoonright X_\xi$ is a filter on X_ξ for which there is a partition $V_\xi = \{X_{\xi\eta}: \eta < \kappa\}$ of X_ξ into $(\mathscr{G} \upharpoonright X_\xi)$-stationary sets.

For $\xi < \kappa$, let g_ξ be a one-to-one mapping of X_ξ onto Y_ξ. From our construction of $\mathscr{F} \upharpoonright Y_\xi$ and $\mathscr{G} \upharpoonright X_\xi$ it follows that the filters $\mathscr{F} \upharpoonright Y_\xi$ and $g_\xi * (\mathscr{G} \upharpoonright X_\xi)$ satisfy the assumptions of Theorem 2. Hence there exists a permutation h_ξ of Y_ξ such that the family

$$\mathscr{H}_\xi = \{F \cap h_\xi g_\xi * G : F \in \mathscr{F} \upharpoonright Y_\xi \quad \text{and} \quad G \in \mathscr{G} \upharpoonright X_\xi\}$$

is a uniform filter on Y_ξ.

Let $f: \kappa \to \kappa$ be a function such that $f \upharpoonright X_\xi = h_\xi g_\xi$ for all $\xi < \kappa$. Notice that f is a permutation of κ such that $f * X_\xi = Y_\xi$.

Let $\mathscr{H} = \{F \cap f * G : F \in \mathscr{F}$ and $G \in \mathscr{G}\}$. To show that $\mathscr{H}$ is a filter on κ it is enough to check that, for all $\xi < \kappa$, $F \in \mathscr{F}$ and $G \in \mathscr{G}$, we have $Y_\xi \cap F \cap f * G \neq 0$. But, this follows from the fact that the right hand side of the equality $Y_\xi \cap F \cap f * G = (Y_\xi \cap F) \cap f * (G \cap X_\xi)$ is an element of the filter $\mathscr{H}_\xi$. Moreover, for every $\xi < \kappa$, the set Y_ξ is $\mathscr{H}$-stationary.

COROLLARY 1. *Let $\mathscr{F}$ be a λ-complete uniform filter on uncountable κ with a pseudobasis. Then there exists a uniform filter $\mathscr{F}_1 \supseteq \mathscr{F}$ such that:*

(a) *$\mathscr{F}_1$ does not have the property* $\mathbf{U}(2)$;

(b) *there exists a partition of κ into κ $\mathscr{F}_1$-stationary sets*;

(c) *$\mathscr{F}_1$ is λ-complete*;

(d) *if κ is regular then we can require that $\mathscr{S}^\kappa(\kappa) \subseteq \mathscr{F}_1$*;

(e) *if κ is singular we can also require that $\mathscr{S}^\kappa(\kappa) \subseteq \mathscr{F}_1$, but then $\mathscr{F}_1$ will be only min $(\lambda, cf(\kappa))$-complete.*

PROOF. Suppose that κ is regular. Let $\mathscr{G}$ be the closed unbounded filter on κ. Then $\mathscr{G}$ is κ-complete and $\mathscr{S}^\kappa(\kappa) \subseteq \mathscr{G}$. By Fact B, there exists a partition of κ into κ $\mathscr{G}$-stationary sets. By Theorem 2′, there is a permutation f on κ and a uniform filter $\mathscr{H}$ such that $\mathscr{H}$ extends $\mathscr{F}$ and $f * \mathscr{G}$. Hence, we have $\mathscr{S}^\kappa(\kappa) \subseteq \mathscr{H}$. Moreover $\mathscr{H}$ satisfies (b) and (c). Finally, since $\mathscr{G}$ is normal, by Theorem 0.(e), $\mathscr{G}$ does not have the property $\mathbf{U}(2)$. Consequently $\mathscr{H}$ does not have the property $\mathbf{U}(2)$ and so (a) holds.

Suppose that κ is singular. Then $\lambda < \kappa$. Let $\kappa = \bigcup_{\xi < \mathrm{cf}(\kappa)} A_\xi$ where $\{A_\xi : \xi < \mathrm{cf}(\kappa)\}$ are pairwise disjoint and for all $\xi < \eta < \mathrm{cf}(\kappa)$, $|A_\xi|$ is regular, $\lambda \leqq |A_\xi|$ and $\Sigma_{\zeta < \eta} |A_\zeta| < |A_\eta| < \kappa$. Choose, for each $\xi < \mathrm{cf}(\kappa)$, an $|A_\xi|$-complete uniform filter $\mathscr{F}_\xi$ on A_ξ without the property $\mathbf{U}(2)$, such that every $\mathscr{F}_\xi$-stationary subset of A_ξ can be decomposed into $|A_\xi|$ pairwise disjoint $\mathscr{F}_\xi$-stationary sets. (To get such $\mathscr{F}_\xi$ we can take, by Theorem 0.(e), and Fact B, a copy of the closed unbounded filter on $|A_\xi|$).

Proceeding as in the proof of Theorem 0.(f), we get a λ-complete

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uniform filter $\mathscr{G}$ on κ without the property $\mathbf{U}(2)$. Let $\{S_{\xi\eta}: \xi < \mathrm{cf}(\kappa)$ and $\eta < |A_\xi|\}$ be a partition of κ such that $S_{\xi\eta}$ is $\mathscr{F}_\xi$-stationary for all $\xi < \mathrm{cf}(\kappa)$ and $\eta < |A_\xi|$. Then, by our construction, $S_{\xi\eta}$ is $\mathscr{G}$-stationary. Since $|\{S_{\xi\eta}: \xi < \mathrm{cf}(\kappa)$ and $\eta < |A_\xi|\}| = \kappa$, we have a partition of κ into κ $\mathscr{G}$-stationary sets.

Finally, as before, we can use Theorem 2′ to get a filter $\mathscr{H}$ which satisfies the conditions (a), (b), (c) of Corollary 1.

To complete this section we shall consider extensions which increase the degree of saturatedness of filters.

THEOREM. 3. *Let $\omega \leqq \lambda \leqq \kappa$ and let $\mathscr{F}$ be a filter on κ for which there exists a partition $\{X_\xi: \xi < \lambda\}$ of κ into $\mathscr{F}$-stationary sets. Let $\mathscr{G}$ be a filter on λ and define $\mathscr{F}_1$ to be the filter generated by $\mathscr{F}$ and the family $\{E \subseteq \kappa : E \supseteq \bigcup_{\xi \in G} X_\xi$ for some $G \in \mathscr{G}\}$. Then $sat(\mathscr{F}_1) \geqq sat(\mathscr{G})$.*

PROOF. Let $\delta < \mathrm{sat}(\mathscr{G})$. We construct a family, having cardinality δ, of $\mathscr{F}_1$-stationary pairwise $\mathscr{F}_1$-almost disjoint sets.

Let $\mathscr{R}$ be a family, having cardinality δ, of $\mathscr{G}$-stationary, pairwise $\mathscr{G}$-almost disjoint sets. Define $\mathscr{R}^* = \{\bigcup_{\xi \in R} X_\xi : R \in \mathscr{R}\}$. Obviously $|\mathscr{R}^*| = \delta$. We claim that each member of $\mathscr{R}^*$ is $\mathscr{F}_1$-stationary. Indeed, take $Y = \bigcup_{\xi \in R} X_\xi$ for some $R \in \mathscr{R}$ and let $A \in \mathscr{F}_1$. Then $A \supseteq F \cap \bigcup_{\xi \in G} X_\xi$ for some $F \in \mathscr{F}$ and $G \in \mathscr{G}$. Whence $A \cap Y \supseteq F \cap \bigcup_{\xi \in R \cap G} X_\xi$. Since R is $\mathscr{G}$-stationary there is some $\xi_0 \in R \cap G$. Whence $A \cap Y \supseteq F \cap X_{\xi_0}$. But X_{ξ_0} is $\mathscr{F}$-stationary and so we have $A \cap Y \neq 0$.

To complete the proof it remains to show that $\mathscr{R}^*$ consists of pairwise $\mathscr{F}_1$-almost disjoint sets. Indeed, let $X = \bigcup_{\xi \in R_1} X_\xi$ and $Y = \bigcup_{\xi \in R_2} X_\xi$, where R_1, R_2 are distinct members of $\mathscr{R}$. Then $X \cap Y = \bigcup_{\xi \in R_1 \cap R_2} X_\xi$. Since $R_1 \cap R_2$ is not $\mathscr{G}$-stationary, it follows from the definition of $\mathscr{F}_1$, that $X \cap Y$ is not $\mathscr{F}_1$-stationary.

In connection with Theorem 3, we introduce the following function:

$\mathbf{F}(\kappa) = \sup\{\mathrm{sat}(\mathscr{G}): \mathscr{G}$ is a filter on $\kappa\}$ and $\mathbf{F}(\delta) = \delta^+$ for $\delta < \omega$. The well-known result of Sierpiński (1938), states that for every cardinal κ we have $\mathrm{sat}(\mathscr{S}^\kappa(\kappa)) \geqq \kappa^{++}$. Hence $\kappa^{++} \leqq \mathbf{F}(\kappa) \leqq (2^\kappa)^+$. Thus **GCH** implies that $\mathbf{F}(\kappa) = \kappa^{++}$. Also observe that if $2^\lambda \leqq \kappa$ for some cardinal λ then $\mathbf{F}(\kappa) \geqq (2^\lambda)^+$, (see Sierpiński (1928)).

The following proposition shows that in the definition of the function $\mathbf{F}(\kappa)$ we can require that filters under consideration be uniform.

PROPOSITION. *Let $\mathbf{F}^*(\kappa) = sup\{sat(\mathscr{F}): \mathscr{F}$ is a uniform filter on $\kappa\}$. Then $\mathbf{F}^*(\kappa) = \mathbf{F}(\kappa)$.*

PROOF. Obviously $\mathbf{F}^*(\kappa) \leqq \mathbf{F}(\kappa)$. Let $\mathscr{G}$ be an arbitrary filter on κ such that $\mathrm{sat}(\mathscr{G}) = \delta$. We construct a uniform filter $\mathscr{F}$ on κ such that $\mathrm{sat}(\mathscr{F}) \geqq$

sat$(\mathscr{G})$. Let $\{X_\xi : \xi < \kappa\}$ be a partition of κ into sets of the cardinality κ. Let $\mathscr{F} = \{F \subseteq \kappa : \text{for some } G \in \mathscr{G},\ F \supseteq \bigcup_{\xi \in G} X_\xi\}$. It is easy to see that $\mathscr{F}$ satisfies all our requirements and so $\mathbf{F}^*(\kappa) \geqq \mathbf{F}(\kappa)$.

COROLLARY 2. *Let $\mathscr{F}$ be a filter on κ for which there exists a partition of κ into $\delta(\geqq \omega)$ $\mathscr{F}$-stationary sets. Then for every $\lambda < \mathbf{F}(\delta)$ there exists a filter $\mathscr{F}_1 \supseteq \mathscr{F}$ such that sat $(\mathscr{F}_1) \geqq \lambda^+$. In particular $\mathscr{F}_1$ can be choosen so that sat $(\mathscr{F}_1) \geqq \delta^{++}$.*

PROOF. If $\mathbf{F}(\delta)$ is a limit cardinal then $\lambda < \mathbf{F}(\delta)$ implies that $\lambda^+ < \mathbf{F}(\delta)$. By the definition of $\mathbf{F}(\delta)$ there exists a filter $\mathscr{G}$ on δ such that sat$(\mathscr{G}) \geqq \lambda^+$. Whence Corollary 2 follows by Theorem 3.

Suppose that $\mathbf{F}(\delta)$ is a successor cardinal. Then, by the definition of the function $\mathbf{F}$, there exists a filter $\mathscr{G}$ on δ such that sat$(\mathscr{G}) = \mathbf{F}(\delta) \geqq \lambda^+$. Whence again by Theorem 3, we get the required filter $\mathscr{F}_1$.

3. Extensions of fields of sets

THEOREM 4. *Let $\mathscr{B}$ be a field of subsets of κ, and let $\mathscr{F}$ be a filter on κ for which there is a partition $\mathscr{V}$ of κ into $\mathscr{F}$-stationary sets such that $|\mathscr{B}|^+ < \mathbf{F}(|\mathscr{V}|)$. Then the field $\mathscr{B}^*$ generated by $\mathscr{B}$ and $\mathscr{F}$ is a proper subfield of $\mathscr{S}(\kappa)$. (Note that if $\mathscr{B}$ and $\mathscr{F}$ are λ-complete then $\mathscr{B}^*$ is λ-complete too).*

PROOF. By Corollary 2, there exists a filter $\mathscr{F}_1 \supseteq \mathscr{F}$ such that sat$(\mathscr{F}_1) > |\mathscr{B}|^+$. Let $\mathscr{J}$ be the ideal dual for $\mathscr{F}_1$. Note that each element of $\mathscr{B}^*$ is of the form $B \Delta J$ for some $B \in \mathscr{B}$ and $J \in \mathscr{J}$.

Let $\mathscr{S}$ be a family of $\mathscr{F}_1$-almost disjoint $\mathscr{F}_1$-stationary sets such that $|\mathscr{S}| > \mathscr{B}$. We claim that $\mathscr{S} \not\subseteq \mathscr{B}$. Indeed, if not, then there are two functions $i : \mathscr{S} \to \mathscr{B}$ and $j : \mathscr{S} \to \mathscr{J}$ such that for each $S \in \mathscr{S}$ we have $S = i(S) \Delta j(S)$. Note that if S and S' are distinct members of S then $i(S) \cap i(S') \in \mathscr{J}$. Finally, since $|\mathscr{S}| > |\mathscr{B}|$, there are distinct sets S and S' in $\mathscr{S}$ such that $i(S) = i(S')$. But then $i(S) \in \mathscr{J}$ and consequently $S \in \mathscr{J}$ which is impossible.

COROLLARY 3. *If $\mathscr{F}$ is a κ-complete filter on κ such that sat $(\mathscr{F}) \geqq \kappa^+$ then for every field $\mathscr{B}$ of subsets of κ such that $|\mathscr{B}| = \kappa$ there exists a proper subfield $\mathscr{B}^*$ of $\mathscr{S}(\kappa)$ which contains $\mathscr{B}$ and $\mathscr{F}$.*

PROOF. If sat$(\mathscr{F}) \geqq \kappa^+$ then there exists a partition $\mathscr{V}$ of κ into κ $\mathscr{F}$-stationary sets. Hence Theorem 4 can be applied.

THEOREM 5. *Let $\mathscr{B}$ be any λ-complete field of subsets of $\kappa \geqq \omega_1$ such that $|B|^+ < \mathbf{F}(\kappa)$ and let $\mathscr{F}$ be a λ-complete filter on κ with a pseudobasis. Then there exists a λ-complete filter $\mathscr{F}^* \supseteq \mathscr{F}$ such that the field $\mathscr{B}^*$ generated by $\mathscr{B}$ and $\mathscr{F}^*$*

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is a proper subfield of $\mathcal{S}(\kappa)$ and for each family $\mathcal{V} \subseteq [\kappa]^{\geq 2}$ of pairwise disjoint sets there exists a selector of $\mathcal{V}$ in $\mathcal{B}^*$.

We can additionally require that the filter $\mathcal{F}^*$ above contains $\mathcal{S}^\kappa(\kappa)$ but then it will be only $\min(\lambda, \mathrm{cf}(\kappa))$-complete.

PROOF. By Corollary 1, there exists a filter $\mathcal{F}_1 \supseteq \mathcal{F}$ which is λ-complete without the property $\mathbf{U}(2)$ and a partition $\mathcal{U}$ of κ into κ $\mathcal{F}_1$-stationary sets. Let $\mathcal{B}^*$ be the subfield of $\mathcal{S}(\kappa)$ generated by $\mathcal{B}$ and $\mathcal{F}_1$. Clearly by Theorem 4, $\mathcal{B}^*$ satisfies all the requirements.

To get our additional requirement note that if $\mathcal{F}$ is a λ-complete filter with a pseudobasis then the smallest filter $\mathcal{F}'$ containing $\mathcal{F}$ and $\mathcal{S}^\kappa(\kappa)$ is $\min(\lambda, \mathrm{cf}(\kappa))$-complete and also has a pseudobasis. So we can apply Theorem 5, for $\mathcal{B}$ and $\mathcal{F}'$.

4. An application to Ulam Problems

We deal here with two problems formulated by Ulam:

(I) Let $\mathcal{B}$ be an ω_1-complete field of subsets of the real line 2^ω, which contains all Lebesgue measurable sets. Suppose that, for every uncountable partition $\mathcal{V}$ of 2^ω such that $\mathcal{V} \subseteq \mathcal{B}$ and each member of $\mathcal{V}$ is uncountable, there exists a selector of $\mathcal{V}$ in $\mathcal{B}$. Does $\mathcal{B} = \mathcal{S}(2^\omega)$?

[Ulam (1935–1940), Problem 34].

(II) Let $\mathcal{B}$ be an ω_1-complete field of subsets of the real line 2^ω, which contains all Borel sets. Suppose that for every partition $\mathcal{V}$ of 2^ω into two-elements sets there exists a selector of $\mathcal{V}$ in $\mathcal{B}$. Does $\mathcal{B} = \mathcal{S}(2^\omega)$?

[Ulam (1960), page 15].

Using the results of §3, we get the following.

PROPOSITION 1. *There exists an ω_1-complete field $\mathcal{B}^*$ of subsets of the real line 2^ω such that:*

(a) all Lebesgue measurable sets are in $\mathcal{B}^*$;

(b) $[2^\omega]^{<2^\omega} \subseteq \mathcal{B}^*$ and $\mathcal{B}^* \neq \mathcal{S}(2^\omega)$;

(c) for every family $\mathcal{V}$ of pairwise disjoint two-elements subsets of 2^ω there exists a selector of $\mathcal{V}$ in $\mathcal{B}^*$.

PROOF. Put in Theorem 5: $\kappa = 2^\omega$, $\mathcal{B}$ = Borel subsets of 2^ω; $\mathcal{F}$ = the filter dual to the ideal of the sets of the Lebesgue measure zero.

PROPOSITION 2. (*Assuming Martin Axiom $\mathbf{A}_\kappa$ for some $\kappa \leqq 2^\omega$*). *Under the hypotheses of Proposition 1, there exists a κ-complete field $\mathcal{B}^*$ which has the properties (a), (b) and (c) of Proposition 1.*

Proof. Let $\mathscr{B}$ be the field of Borel sets and let $\mathscr{F}$ be the filter dual to the ideal of the sets of Lebesgue measure zero. From $\mathbf{A}_\kappa$ it follows that $\mathrm{cf}(2^\omega) \geqq \kappa$, $\mathscr{F}$ is κ-complete and moreover the field $\mathscr{L}$ of Lebesgue measurable sets is κ-complete. Consider $\mathscr{F}^*$ and $\mathscr{B}^*$ which exist by Theorem 5. Then $\mathscr{B}^*$ has the properties (a), (b) and (c) of Proposition 1. To show that $\mathscr{B}^*$ is κ-complete observe that the field $\mathscr{B}^*$ is generated by $\mathscr{L}$ and $\mathscr{F}^*$ as well by $\mathscr{B}$ and $\mathscr{F}^*$.

5. Miscellaneous remarks

1.1 If $\mathscr{F}$ is a κ-complete filter on κ with a basis then there exists a partition $\{C_\xi : \xi < \kappa\}$ of κ such that $X \in \mathscr{F}$ if and only if $|\{\xi \in \kappa : C_\xi \not\subseteq X\}| < \kappa$. Clearly nothing can be said about the cardinality of the C_ξ's.

On the other hand the following two conditions are equivalent for κ-complete filter $\mathscr{F}$ with a basis:

(*) there exists a partition $\{C_\xi : \xi < \kappa\}$ of κ such that $|C_\xi| = \kappa$ for all $\xi < \kappa$, with the property:

$$X \in \mathscr{F} \text{ if and only if } |\{\xi \in \kappa : C_\xi \not\subseteq X\}| < \kappa;$$

(**) there is no Lusin set for $\mathscr{F}$ in $\mathscr{F}$.

Thus we get the following fact about isomorphism of filters.

If $\mathscr{F}_1$ and $\mathscr{F}_2$ are two κ-complete filters on κ with bases which satisfy the condition (**) above then there exists a permutation f of κ such that $f * \mathscr{F}_1 = \mathscr{F}_2$. Moreover, if additionally there is a set $A \in \mathscr{F}_1$ such that $(\kappa - A) \in \mathscr{F}_2$, then we can choose f such that $f = f^{-1}$.

For the case $\kappa = \omega_1$ this is well-known. (see Sierpiński (1934a), Erdös (1943), Marczewski (1946) and Oxtoby (1971), page 76). Moreover the same proof works.

2.1. It follows from Theorem 1 that for a regular cardinal κ:

(*) for each uniform ultrafilter $\mathscr{U}$ on κ there exists a κ-complete filter $\mathscr{F} \subseteq \mathscr{U}$ without any basis.

Let us compare (*) with the following result by Prikry (Prikry (1974), Added in proof (2)):

(**) for each uniform ultrafilter $\mathscr{U}$ on κ there exists a κ-complete filter $\mathscr{F} \subseteq \mathscr{U}$ without any Lusin set.

Obviously (**) implies (*). But, observe that Theorem 1 allows us to extend a uniform filter with a basis which contains $\mathscr{S}^\kappa(\kappa)$ within any greater filter to a filter without any basis.

2.2. From Corollary 1, §2, it follows that each λ-complete filter $\mathscr{F}$ with a pseudobasis can be extended to a λ-complete uniform filter without the property $\mathbf{U}(2)$. Hence, each λ-complete uniform filter on κ can be extended to a λ-complete filter without any pseudobasis. We can not prove in general that each λ-complete filter on κ can be extended to a λ-complete filter without the property $\mathbf{U}(2)$, even in the case $\kappa = \lambda = \omega_1$.

Below we give two partial results:

(1) *If λ is strongly compact (or ω), then each λ-complete filter on κ can be extended to a λ-complete filter without the property* $\mathbf{U}(2)$.

(2) *If a λ-complete filter $\mathcal{F}$ on κ does not have the property* $\mathbf{U}(n)$ *for some $n < \omega$, $n \geqq 2$, then there exists a λ-complete filter $\mathcal{G}$ on κ such that $\mathcal{G} \supseteq \mathcal{F}$ and $\mathcal{G}$ does not have the property* $\mathbf{U}(2)$.

PROOF. Let $m + 1$ be the smallest natural number such that $\mathcal{F}$ does not have $U(m+1)$. Clearly we can assume that $m \geqq 2$.

Let $\mathcal{X} = \{X_\xi : \xi < \kappa\} \subseteq [\kappa]^m$ be a partition of κ such that each selector of $\mathcal{X}$ is $\mathcal{F}$-stationary. For each $F \in \mathcal{F}$ put $F^* = \{\xi < \kappa : X_\xi \subseteq F\}$ and $\tilde{F} = \bigcup_{\xi \in F^*} X_\xi$. Obviously $\tilde{F} \subseteq F$ for each $F \in \mathcal{F}$ and since X has only $\mathcal{F}$-stationary selector, we have $\tilde{F} \neq 0$. Let $\mathcal{F}_0$ be the filter generated by $\{\tilde{F} : F \in \mathcal{F}\}$. Since the family $\{\tilde{F} : F \in \mathcal{F}\}$ is λ-multiplicative, $\mathcal{F}_0$ is a proper λ-complete filter and $\mathcal{F} \subseteq \mathcal{F}_0$. Let S be any selector of $\mathcal{X}$. Clearly S is $\mathcal{F}_0$-stationary. Let $\mathcal{G}$ be the filter generated by $\mathcal{F}_0$ and S. We show that $\mathcal{G}$ does not have $\mathbf{U}(2)$.

Let $\mathcal{A} \subseteq [\kappa]^2$ be any family of pairwise disjoint sets. Let $\mathcal{A}_0 = \{A \in \mathcal{A} : A - S \neq 0\}$. Obviously $\mathcal{A}_0$ has a selector which is not $\mathcal{G}$-stationary. To show that $\mathcal{G}$ does not have $\mathbf{U}(2)$ it is enough to find a selector of the family $\mathcal{B} = \mathcal{A} - \mathcal{A}_0$ which is not $\mathcal{F}_0$-stationary. Let $C_B = \cup\{X_\xi : X_\xi \cap B \neq 0\}$ for each $B \in \mathcal{B}$. Clearly $|C_B| = 2m > m$ for all $B \in \mathcal{B}$ and the family $\{C_B : B \in \mathcal{B}\}$ consists of pairwise disjoint sets. Since $\mathcal{F}$ does not have the property $U(2m)$ there exists a selector T of $\{C_B : B \in \mathcal{B}\}$ which is not $\mathcal{F}$-stationary. Moreover, for each $B \in \mathcal{B}$ we have $B \not\subseteq \widetilde{\kappa - T}$, Whence there exists a selector of $\mathcal{B}$ which is not $\mathcal{F}_0$-stationary.

2.3. We remark that the idea of the proof of Corollary 1, yields the following more general fact:

Suppose that $\mathbf{F}$ *is a family of filters on κ such that*:

(1) *there exists a filter $\mathcal{F} \in \mathbf{F}$ for which there exists a partition $\mathcal{V}$ of κ into κ $\mathcal{F}$-stationary sets*;

(2) *if f is any permutation of κ and $\mathcal{F} \in \mathbf{F}$ then $f * \mathcal{F} \in \mathbf{F}$; and*

(3) *if $\mathcal{F} \in \mathbf{F}$ and $\mathcal{F} \subseteq \mathcal{G}$ then $\mathcal{G} \in \mathbf{F}$.*

Then each filter on κ with a pseudobasis has an extension in $\mathbf{F}$.

2.4. We remark that if $\kappa = \delta^+$ then in the proof of Corollary 1, Ulam matrices (see Ulam (1930)), can be used to get a partition $\mathcal{V}$ of κ into κ

$\mathcal{G}$-stationary sets. (Compare our proof of Corollary 1.) If κ is limit then we can adapt the method of the proof of Theorem 0.(f). In the weakly inaccessible case we get by this method only $\min(\lambda, \lambda')$-completeness of $\mathcal{F}_1$ for any $\lambda' < \kappa$.

3.1. Note that in both Propositions 1 and 2 of §4, we can replace everywhere "Lebesgue measurability" by "Baire property". Other results of this type can be easily obtained from our Theorem 5.

ADDED IN PROOF. (Dec. 10, 1975). We can replace (c) in Proposition 1 of §4, by the following stronger condition:

(c') for every partition $\mathcal{V} \subseteq [2^\omega]^{\leq\omega}$ *there is a selector of* $\mathcal{V}$ *in* $\mathcal{B}^*$.

A corresponding strengthening of Proposition 2 can be also proved.

1. AN OPEN PROBLEM. Let $\mathcal{C}$ be an ω_1-complete field of subsets of the real line 2^ω, which contains all Lebesgue measurable sets. Suppose that for every partition $\mathcal{V} \subseteq \mathcal{C}$ of 2^ω there exists a selector of $\mathcal{V}$ in $\mathcal{C}$. Does $\mathcal{C} = \mathcal{S}(2^\omega)$? Our conjecture is NO, at least in **ZFC + CH**.

2. ALAIN LOUVEAU has proved in 1975, answering one of our questions, that there exists a filter $\mathcal{F}$ on ω such that $\mathcal{F}$ does not have the property **U**(2) and there exists a partition of ω into ω $\mathcal{F}$-stationary sets. It easily follows from Louveau's result that our Corollary 1 and Theorem 5 formulated for $\kappa > \omega$ are also true for $\kappa = \omega$.

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