THE STUDY OF PLANETARY SECULAR PERTURBATIONS*

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Summary. The methods of the first-order planetary secular perturbations are discussed in this paper. On the basis of Gauss' method, some concrete applicable formulae are derived. And then, orbital evolutions of the nine major planets in solar system during about 2,100,000 years are investigated numerically. The method applied here is in principle suitable for the orbits of arbitrary eccentricities and inclinations.

FOREWORD

In order to make an approach to the secular perturbations which some celestial bodies with rather special orbits (for instances, the Mercury, the Pluto, asteroids and comets) have undergone, the way in the classical method to solve the non-periodic part of disturbing function is not suitable, because it depends on developing the eccentricities and inclinations regarded as small quantities. We tried to introduce here the Gauss' method. Making use of its complete analytical solution for part of the problem and also the combination of the analytical and numerical method, we study the orbital evolution of this kind of special celestial bodies during a comparatively long period.

I. ON THE SECULAR PERTURBATIONS

There is no secular perturbation on semi-major axis, and the secular inequality of mean anomaly is of no consequence. Therefore, only the variations of the remaining four elements need to be considered. Let

$$\begin{aligned} h &= e\sin(\omega+\Omega), \\ k &= e\cos(\omega+\Omega), \\ p &= \sin i \sin\Omega \\ q &= \sin i \cos\Omega \end{aligned} \qquad (1)$$

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For the sake of simplicity, we limit the problem to the disturbed planet and the disturbing planet. Their masses be designated by m and m_1 respectively. Developed up to square terms of the eccentricities and inclinations, the related non-periodic part of the disturbing function is[1]

$$N = Gm_1D[h^2+k^2+h_1^2+k_1^2-p^2-q^2-p_1^2-q_1^2+2pp_1+2qq_1] \\ -2Gm_1E(hh_1+kk_1). \tag{2}$$

Accordingly, the secular inequalities of the related elements can be expressed in the following simple form:

$$[\dot{h}] = \frac{2Gm_1}{na^2}(kD-k_1E); \quad [\dot{k}] = -\frac{2Gm_1}{na^2}(hD-h_1E);$$
$$[\dot{p}] = \frac{2Gm_1D}{na^2}(q_1-q); \quad [\dot{q}] = \frac{2Gm_1D}{na^2}(p-p_1); \tag{3}$$

In which G is the gravitational constant, n the mean motion of disturbed planet; D and E are symmetrical functions of the two semi-major axes.

If we exchange m, m_1 and their corresponding elements, the $[\dot{h}_1]$, $[\dot{k}_1]$, $[\dot{p}_1]$ and $[\dot{q}_1]$ may be obtained in the same form as (3).

With the simple form of (3), replacing h, k, p, q by [h], [k], [p] and [q], we can solve simultaneously the perturbation equations of all the major planets analytically, though it is very long and complicated. That is why for a long time it has been used as the classical method for studying secular perturbations of the major planets.

The main disadvantage of this method is that only terms of second order in the eccentricities and inclinations are considered. It seems not accurate enough for rather special orbits.

Gauss proposed another way to calculate the secular perturbations of elements avoiding the complication of dependence on the development of small quantities[1,2].

We denote by σ any one of the elements. Its perturbation equation is known as

$$\frac{d\sigma}{dt} = A_0 + \sum_i \sum_j B \cos(iM + jM_1 + Q),$$

Where A_0 corresponds to the secular perturbation on the left of (3), the second part on the right being periodic terms, M and M_1 are mean anomalies of the two planets and Q is a function of other angular elements.

The basic concept of Gauss' method is to calculate directly the $[\dot{\sigma}]$ according to the following formula:

$$[\dot{\sigma}] = A_0 = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\frac{d\sigma}{dt}\,dM\,dM_1. \tag{4}$$

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Regarding the mass of disturbing planet m_1 as a velocity-density distribution along its orbit, i.e., every mass $d\mu$ on the element arc of the orbit is proportional to the time dt taken by the planet to pass through the very arc, using coordinate transformation and introducing elliptical integral, Gauss obtained the analytical solution in (4) for dM_1, i.e. the first integral. As for the second integral, it can be integrated by using numerical method of harmonic analysis.

In order to compare the classical method with Gauss' method, taking a planet of Jupiter type (m_1=0.001, a_1=5.20, e_1=0.05, i_1=1°.2) as a disturbing body, we have made many kinds of computation for a disturbed body (a=2.75) with various eccentricities and inclinations according to the two methods respectively (the concrete formulae of Gauss' method are mentioned below). Now, we list briefly the results about $[\dot{h}]$ in Table 1, in each row and column, the value above is obtained from classical method (3) while the result in parenthesis is from Gauss' method. The latter is correct.

It is thus evident that both results agree well in the case of small eccentricities and inclinations; the larger they are, the divergence becomes more prominent until beyond recognition.

Table 1 $10^6 \times [\dot{h}]$ (for each day)

e \ i	0°.000	5°.730	11°.459	17°.189	22°.918	28°.648	34°.377
0.0	-0.024 (-0.024)	-0.024 (-0.022)	-0.024 (-0.017)	-0.024 (-0.011)	-0.024 (-0.005)	-0.024 (0.000)	-0.024 (+0.004)
0.1	-0.056 (-0.058)	-0.056 (-0.051)	-0.056 (-0.032)	-0.056 (-0.006)	-0.056 (+0.021)	-0.056 (+0.044)	-0.056 (+0.062)
0.2	-0.088 (-0.094)	-0.088 (-0.082)	-0.088 (-0.045)	-0.088 (+0.002)	-0.088 (+0.049)	-0.088 (+0.089)	-0.088 (+0.121)
0.3	-0.121 (-0.134)	-0.121 (-0.114)	-0.121 (-0.056)	-0.121 (+0.015)	-0.121 (+0.082)	-0.121 (+0.136)	-0.121 (+0.178)
0.4	-0.153 (-0.180)	-0.153 (-0.149)	-0.153 (-0.062)	-0.153 (+0.036)	-0.153 (+0.120)	-0.153 (+0.185)	-0.153 (+0.233)
0.5	-0.185 (-0.237)	-0.185 (-0.187)	-0.185 (-0.056)	-0.185 (+0.070)	-0.185 (+0.165)	-0.185 (+0.233)	-0.185 (+0.282)
0.6	-0.218 (-0.314)	-0.218 (-0.223)	-0.218 (-0.029)	-0.218 (+0.118)	-0.218 (+0.213)	-0.218 (+0.277)	-0.218 (+0.321)

II. SOME COMPUTATION FORMULAE

We adopt an appropriate system of units so that the gravitational

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constant is equal to 1.

It was proved long ago that indirect perturbation doesn't contain secular terms. Let S, T, and W denote respectively the direct perturbations along the disturbed planet's radius direction, transverse direction and normal to its orbital plane. Gauss' method can be summed up in calculating the following integrals:

$$S_0 = \frac{m_1}{2\pi}\int_0^{2\pi} S\,dM_1 , \qquad T_0 = \frac{m_1}{2\pi}\int_0^{2\pi} T\,dM_1 , \qquad W_0 = \frac{m_1}{2\pi}\int_0^{2\pi} W\,dM_1 . \tag{5}$$

According to the principle of Gauss method, we made many trial computations and arrived at the conclusion that the following method is correct and practicable.

1. General condition

Let $\boldsymbol{P}, \boldsymbol{Q}, \boldsymbol{R}$ and $\boldsymbol{P}_1, \boldsymbol{Q}_1, \boldsymbol{R}_1$ denote the unit vectors along the directions of the orbital principal axes of the disturbed planet and the disturbing planet respectively. We refer to the heliocentric system corresponding to the directions thus mentioned as orbital coordinate system; denote by $\boldsymbol{r}$ the vector of radius direction of the disturbed planet; let a_1 and b_1 be the semi-major axis and semi-minor axis of the disturbing planet respectively and e_1 its eccentricity. Let

$$\alpha = \boldsymbol{r}\cdot\boldsymbol{P}_1 , \quad \beta = \boldsymbol{r}\cdot\boldsymbol{Q}_1 , \quad \gamma = \boldsymbol{r}\cdot\boldsymbol{R}_1 ,$$
$$\psi_0 = a_1^2 b_1^2 \gamma^2 ,$$
$$\psi_1 = a_1^2 b_1^2 - a_1^2\beta^2 - b_1^2(\alpha + a_1e_1)^2 - (a_1^2 + b_1^2)\gamma^2 ,$$
$$\psi_2 = (\alpha + a_1e_1)^2 + \beta^2 + \gamma^2 - a_1^2 - b_1^2 .$$

With the basic parameter λ , the following equation should be satisfied:

$$\lambda^3 + \psi_2\lambda^2 + \psi_1\lambda + \psi_0 = 0 . \tag{6}$$

After solving the cube equation (6) by usual method, we approach it once again in order to be more accurate:

$$\lambda_i = \lambda_{i,0} - \frac{\lambda_{i,0}^3 + \psi_2\lambda_{i,0}^2 + \psi_1\lambda_{i,0} + \psi_0}{3\lambda_{i,0}^2 + 2\psi_2\lambda_{i,0} + \psi_1} , \quad i = 1,2,3. \tag{7}$$

Of the three values of λ two are positive, and one negative. According to an order from large to small, we write them as λ_1 , λ_2 , λ_3.

To increase the accuracy, the smallest in absolute value should be calculated once again according to $\lambda_2 = -\psi_0/\lambda_1\lambda_3$ or $\lambda_3 = -\psi_0/\lambda_1\lambda_2$.

In the orbital coordinate system of disturbing planet, the unit vectors along directions of the principal axes of the transformed coordinate system that can be integrated are

$$\boldsymbol{d}_i = [-C\lambda_i(b_1^2-\lambda_i)(\alpha+a_1e_1),\ -C\lambda_i(a_1^2-\lambda_i)\beta,\ C(a_1^2-\lambda_i)(b_1^2-\lambda_i)\gamma] , \tag{8}$$

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here

$$C = \{[\lambda_i(b_i^2-\lambda_i)(\alpha+a_1e_1)]^2+[\lambda_i(a_i^2-\lambda_i)\beta]^2 + [(a_i^2-\lambda_i)(b_i^2-\lambda_i)\gamma]^2\}^{-1/2}, \quad i=1,2,3.$$

In order to be accurate and reliable, the $\boldsymbol{d}$ corresponding to the smallest λ in absolute value should also be computed once again according to $\boldsymbol{d}_2 = \boldsymbol{d}_3 \times \boldsymbol{d}_1$ or $\boldsymbol{d}_3 = \boldsymbol{d}_1 \times \boldsymbol{d}_2$

Let $k^2 = (\lambda_1-\lambda_2)/(\lambda_1-\lambda_3)$, the corresponding effective integrals are

$$\left.\begin{aligned} G_x &= 4[E(k)-F(k)]/[(\lambda_1-\lambda_2)\sqrt{(\lambda_1-\lambda_3)}],\\ G_y &= 4\sqrt{(\lambda_1-\lambda_3)}\cdot[(1-k^2)F(k)-E(k)]/[(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)],\\ G_z &= 4\cdot E(k)/[(\lambda_2-\lambda_3)\sqrt{(\lambda_1-\lambda_3)}] = -(G_x+G_y), \end{aligned}\right\} \tag{9}$$

where

$$F(k) = \int_0^{\pi/2}\frac{d\theta}{\sqrt{1-k^2\sin\theta}} = \frac{\pi}{2}\Big[1+\Big(\frac{1}{2}\Big)^2k^2+\Big(\frac{1\cdot3}{2\cdot4}\Big)^2k^4 + \Big(\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\Big)^2k^6+\cdots\cdots\Big],$$

$$E(k) = \int_0^{\pi/2}\sqrt{1-k^2\sin\theta}\,d\theta = \frac{\pi}{2}\Big[1-\Big(\frac{1}{2}\Big)^2k^2 - \Big(\frac{1\cdot3}{2\cdot4}\Big)^2\cdot\frac{1}{3}k^4-\Big(\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\Big)^2\cdot\frac{1}{5}k^6-\cdots\cdots\Big]$$

are the first and second standard elliptical integrals respectively. Let

$$\left.\begin{aligned} \boldsymbol{S}^* &= [\alpha/\gamma,\ \beta/\gamma,\ \gamma/\gamma],\\ \boldsymbol{T}^* &= \boldsymbol{W}^*\times\boldsymbol{S}^*,\\ \boldsymbol{W}^* &= [\boldsymbol{R}\cdot\boldsymbol{P}_1,\ \boldsymbol{R}\cdot\boldsymbol{Q}_1,\ \boldsymbol{R}\cdot\boldsymbol{R}_1]. \end{aligned}\right\} \tag{10}$$

Finally we have

$$[S_0, T_0, W_0] = -\frac{m_1}{2\pi}[G_x\boldsymbol{r}\cdot\boldsymbol{d}_1\ G_y\boldsymbol{r}\cdot\boldsymbol{d}_2\ G_z\boldsymbol{r}\cdot\boldsymbol{d}_3]\begin{bmatrix}\boldsymbol{S}^*\cdot\boldsymbol{d}_1 & \boldsymbol{T}^*\cdot\boldsymbol{d}_1 & \boldsymbol{W}^*\cdot\boldsymbol{d}_1\\ \boldsymbol{S}^*\cdot\boldsymbol{d}_2 & \boldsymbol{T}^*\cdot\boldsymbol{d}_2 & \boldsymbol{W}^*\cdot\boldsymbol{d}_2\\ \boldsymbol{S}^*\cdot\boldsymbol{d}_3 & \boldsymbol{T}^*\cdot\boldsymbol{d}_3 & \boldsymbol{W}^*\cdot\boldsymbol{d}_3\end{bmatrix} \tag{11}$$

2. Special condition

When $a_1/r \ll 1$ or $r/a_1 \ll 1$ (e.g. the mutual condition beteen Mercury and Pluto), the solution of Gauss' method given above will be determined with great difficulty and the results thus obtained might be distorted seriously. Actually, the two bodies' orbits this time are a great distance apart. Compared with the attractive influence of their adjacent planets, the relative contribution of perturbations among them are very small. In this condition the two heliocentric radius vectors are very different in length, it is easy for using developing method to deal with.

Being equivalent to the first integral (5) in Gauss' method, the three components along the orbital coordinate system of disturbing planet can be written as

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$$F_\alpha = \frac{m_1}{2\pi}\frac{\partial}{\partial\alpha}\int_0^{2\pi}\frac{dM_1}{\Delta}, \quad F_\beta = \frac{m_1}{2\pi}\frac{\partial}{\partial\beta}\int_0^{2\pi}\frac{dM_1}{\Delta}, \quad F_\gamma = \frac{m_1}{2\pi}\frac{\partial}{\partial\gamma}\int_0^{2\pi}\frac{dM_1}{\Delta}, \tag{12}$$

here α, β and γ, as defined before, are the heliocentric coordinates of the disturbed planet in the orbital coordinate system of disturbing planet; Δ is the distance between them.

Expanding $1/\Delta$ up to the cube of r_1/r or r/r_1, we integrate the expansion, partially differentiate it, put it in order and finally obtain the result as follows:
When $a_1/r \ll 1$ (the disturbing attraction of an inner planet on a far outer planet),

$$\left.\begin{aligned}
F_\alpha/m_1 = & -(\alpha + \tfrac{3}{2}a_1e_1)/r^3 + [\tfrac{15}{4}a_1^3e_1(1+\tfrac{3}{4}e_1^2) \\
& + 3a_1^2(1+\tfrac{11}{4}e_1^2)\alpha + \tfrac{9}{2}a_1e_1\alpha^2]/r^5 \\
& - \{\tfrac{75}{16}a_1^3e_1[7(1+e_1^2)\alpha^2 + (1-e_1^2)\beta^2] \\
& + \tfrac{15}{4}[(a_1^2\alpha^2 + b_1^2\beta^2) + 4a_1^2e_1^2\alpha^2]\alpha\}/r^7 \\
& + \tfrac{175}{16}a_1e_1\alpha^2[3(a_1^2\alpha^2 + b_1^2\beta^2) + 4a_1^2e_1^2\alpha^2]/r^9, \\
F_\beta/m_1 = & -\beta/r^3 + \beta[\tfrac{3}{2}(a_1^2+b_1^2) + \tfrac{9}{4}a_1^2e_1^2 + \tfrac{9}{2}a_1e_1\alpha]/r^5 \\
& - \beta\{\tfrac{75}{16}a_1^3e_1(6+e_1^2)\alpha + \tfrac{15}{4}[(a_1^2\alpha^2+b_1^2\beta^2) \\
& + 4a_1^2e_1^2\alpha^2]\}/r^7 + \tfrac{175}{16}a_1e_1\alpha\beta[3(a_1^2\alpha^2+b_1^2\beta^2) \\
& + 4a_1^2e_1^2\alpha^2]/r^9, \\
F_\gamma/m_1 = & -\gamma/r^3 + 3\gamma(\tfrac{1}{2}a_1^2 + \tfrac{3}{4}a_1^2e_1^2 + \tfrac{3}{2}a_1e_1\alpha)/r^5 \\
& - \gamma\{\tfrac{75}{4}a_1^3e_1(1+\tfrac{3}{4}e_1^2)\alpha + \tfrac{15}{4}[(a_1^2\alpha^2+b_1^2\beta^2) \\
& + 4a_1^2e_1^2\alpha^2]\}/r^7 + \tfrac{175}{16}a_1e_1\alpha\gamma[3(a_1^2\alpha^2+b_1^2\beta^2) \\
& + 4a_1^2e_1^2\alpha^2]/r^9;
\end{aligned}\right\} \tag{13}$$

When $r/a_1 \ll 1$ (the disturbing attraction of a far outer planet on an inner planet),

$$\begin{aligned}
F_\alpha/m_1 &= \alpha/2b_1^3 + 3e_1(3\alpha^2+\beta^2-4\gamma^2)/[8a_1b_1^3(1-e_1^2)], \\
F_\beta/m_1 &= \beta/2b_1^3 + 3e_1\alpha\beta/[4a_1b_1^3(1-e_1^2)], \\
F_\gamma/m_1 &= -\gamma/b_1^3 - 3e_1\alpha\gamma/[a_1b_1^3(1-e_1^2)].
\end{aligned} \tag{14}$$

It is easy from (13) or (14) to obtain

$$[S_0,\ T_0,\ W_0] = [\mathbf{F}\cdot\mathbf{S}^*, \mathbf{F}\cdot\mathbf{T}^*, \mathbf{F}\cdot\mathbf{W}^*], \tag{15}$$

in which $\boldsymbol{S}^*, \boldsymbol{T}^*, \boldsymbol{W}^*$ are defined as before, and

$$\boldsymbol{F} = [F_\alpha, F_\beta, F_\gamma].$$

After S_0, T_0 and W_0 have been obtained, there is only an integral for dM left in (4):

$$[\dot{\sigma}] = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{d\sigma}{dt}\right)_0 dM, \tag{16}$$

here $(d\sigma/dt)_0$ represents $d\sigma/dt$ in which S_0, T_0 and W_0 are used in place of the usual S, T and W.

Let P and f denote the semi-parameter and the true anomaly of the disturbed planet respectively. Write

$$\begin{aligned} S_1 &= S_0 \sin f, & T_1 &= T_0(P+r)\sin f, & W_1 &= W_0 r \sin f, \\ S_2 &= S_0 \cos f, & T_2 &= T_0(P+r)\cos f, & W_2 &= W_0 r \cos f, \\ & & T_3 &= T_0 r. & & \end{aligned}$$

The perturbation equation of the elements can be changed as

$$\left.\begin{aligned}
\left(\frac{dh}{dt}\right)_0 = {} & S_1\sqrt{P}\,\sin(\omega+\Omega) - S_2\sqrt{P}\,\cos(\omega+\Omega) \\
& + [T_1\cos(\omega+\Omega) + T_2\sin(\omega+\Omega) + T_3 e\sin(\omega \\
& +\Omega)]/\sqrt{P} + [W_1 e\cos(\omega+\Omega)\cos\omega\tan\frac{i}{2} \\
& + W_2 e\cos(\omega+\Omega)\sin\omega\tan\frac{i}{2}]/\sqrt{P}, \\
\left(\frac{dk}{dt}\right)_0 = {} & S_1\sqrt{P}\cos(\omega+\Omega) + S_2\sqrt{P}\sin(\omega+\Omega) \\
& -[T_1\sin(\omega+\Omega) - T_2\cos(\omega+\Omega) - T_3 e\cos(\omega \\
& +\Omega)]/\sqrt{P} - [W_1 e\sin(\omega+\Omega)\cos\omega\tan\frac{i}{2} \\
& + W_2 e\sin(\omega+\Omega)\sin\omega\tan\frac{i}{2}]/\sqrt{P}, \\
\left(\frac{dp}{dt}\right)_0 = {} & \{W_1[\cos(\omega+\Omega) + 2\sin^2\frac{i}{2}\sin\Omega\sin\omega] \\
& + W_2[\sin(\omega+\Omega) - 2\sin^2\frac{i}{2}\sin\Omega\cos\omega]\}/\sqrt{P}, \\
\left(\frac{dq}{dt}\right)_0 = {} & \{W_1[-\sin(\omega+\Omega) + 2\sin^2\frac{i}{2}\cos\Omega\sin\omega] \\
& + W_2[\cos(\omega+\Omega) - 2\sin^2\frac{i}{2}\cos\Omega\cos\omega]\}/\sqrt{P}.
\end{aligned}\right\} \tag{17}$$

In the condition of having given the orbital elements of both disturbing and disturbed planets, S_0, T_0 and W_0 are functions of only the mean anomaly M of disturbed planet, and so are S_i, T_i and W_i. On the basis of (17), with suitable number of points well-distributed according to M, it is not difficult for using numerical method of harmonic analysis to solve and calculate (16) to obtain $[\dot{\sigma}]$.

With such method, the results of orbital computations for different conditions have been shown in Table 1. And now, we list an example of computation again in Table 2, which also contains the corresponding result obtained wholly by numerical method, to be a test for the entire method and formulae.

Taking

	mass	a	e	ω	Ω	i
disturbing planet	0.001	5.20	0.05	229°.18	114°.59	1°.15
disturbed planet	—	2.75	0.15	57.30	57.30	8.59

we have

Table 2 (for each day)

	$10^6[\dot{h}]$	$10^6[\dot{k}]$	$10^6[\dot{p}]$	$10^6[\dot{q}]$
Gauss' method	-0.054532696	-0.099315468	-0.092330152	+0.096448951
numerical integration	-0.054532701	-0.099315487	-0.092330147	+0.096448949

III. COMPUTATIONS, RESULTS AND DISCUSSION

In order to study the orbital evolution during a long period, by integrating (4) further, we have

$$\Delta\sigma = \int_{t_0}^{t} [\dot{\sigma}]dt = \frac{1}{4\pi^2}\int_{t_0}^{t}\int_0^{2\pi}\int_0^{2\pi} \frac{d\sigma}{dt} dM dM_1 dt. \tag{18}$$

It has been mentioned to obtain the integrals for dM_1 and dM. To obtain the integral for dt, we use Adams' method; for the nine major planets of solar system, we solve jointly the equations of order 9x4x3=108 in all. We regard whether a_1/r or r/a_1 is smaller than 1/10 or not as a discriminating basis for taking (13) - (15) or (6) - (11). For harmonic analysis, we take 48 points a circle (as e>0.15, i.e. the condition of Mercury and Pluto) and 24 points a circle (as e<0.15, for the other planets). We adopt 250,000 days as the step length of integration for dt. The initial epoch is adopted on 1941 1 6.0 E.T. (2430000.5). We take the plane of total moment of momentum of the nine major planets in heliocentric system as a fundamental reference plane (very close to the invariable plane), define the ascending node of ecliptic plane to the reference plane as the zero point of longitude and transform the orbital elements of planets to this system. Thus making integration backward, we calculate the orbital variation of the nine major planets during the past 2,100,000 years.

To check it, we have also made a computation during the same period by doubling the step length. Both results are consistent. It shows that the numbers of points and the step length adopted are permissible and the repeated computations are correct.

The main results of the whole computation can be listed in a simple

Table 3

	Eccentricity e		Inclination i	
	Max.	Min.	Max.	Min.
Mercury	0.2317 (0.2372)	0.1215 (0.1314)	9°.178 (9.968)	4°.740 (2.576)
Venus	0.0706 (0.0631)	0.0000 (0.0023)	3.272 (3.353)	* (0.097)
Earth	0.0677 (0.0571)	0.0000 (0.0024)	3.100 (2.948)	* (0.050)
Mars	0.1397 (0.1330)	0.0185 (0.0035)	5.933 (6.463)	* (0.126)
Jupiter	0.0608 (0.0625)	0.0255 (0.0266)	0.482 (0.470)	0.240 (0.251)
Saturn	0.0843 (0.0845)	0.0124 (0.0103)	1.010 (0.997)	0.788 (0.802)
Uranus	0.0780 (0.0905)	0.0118 (0.0082)	1.120 (1.119)	0.907 (0.906)
Neptune	0.0145 (0.0190)	0.0056 (0.0071)	0.788 (0.768)	0.562 (0.030)
Pluto	- (0.2525)	- (0.2172)	- (17.837)	- (15.288)

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table (Table 3), in which are given the largest and the smallest values of the planetary eccentricities ane inclinations during the 2,100,000 years. To be compared, the corresponding extreme values used all along,[1] which were obtained by J.N.Stockwell in former times by making a tedious derivation and solving secular perturbation equations according to (3), are also given here. In Table 3, listed above in each row and column are Stockwell's results; those undecided for reason of the smallness of certain coefficients in the analysis are marked by asterisk. Because Pluto had not yet been discovered that time, no results for it are there in the corresponding places. The results we obtained are in parenthesis.

In Fig. 1 and Fig. 2, are given in detail the variations of eccentricities and inclinations of all the planetary orbits. It is another important contribution of this paper.

From Table 3 and the figures, many things are clear at a glance. We need not say any more except a few points as follows:

1. So-called secular variation is actually composed of many periodic variations. Except Mercury, Pluto and Neptune, most of planets present a continuous fluctuation with a period of tens of thousand years.

2. It is worth noticing that the curves of both eccentricity and inclination of Jupiter are exactly the same as that of Saturn's respectively. Actually both amplitudes are different, but they are drawn in a ordinate of equal length. The main characteristic is that the elements concerned of both planets have the same period of variation respectively but differ from each other exactly in one phase of variation: as one falls, another rises. This takes some explanation. Due to the immense mass of Jupiter and Saturn, the disturbing effect of the other planets is relatively insignificant. In mutual perturbation of only one couple of planets, based on the approximate formula (3), it is not difficult to obtain the following integral[1]:

$$\begin{aligned} & m\sqrt{a}\,e^2 + m_1\sqrt{a_1}\,e_1^2 = \text{const}, \\ & m\sqrt{a}\,\sin^2 i + m_1\sqrt{a_1}\,\sin^2 i_1 = \text{const}. \end{aligned} \tag{19}$$

that just explains the repeated variation of the two planets' elements concerned, as discussed above.

3. Venus and Earth, for they are very close to each other, also have got similar circumstances in the main.

4. The orbital variation of Pluto differs from that of the most planets. Its eccentricity and inclination all present a period of variation of about 700,000 years and there is no small fluctuation. As seen from the curves, the peak value of eccentricity is getting higher while the valley value of inclination is getting lower.

5. As seen from recent data, the mass of Pluto (1/1812000) has been taken a little too large. It has little influence on secular variation of

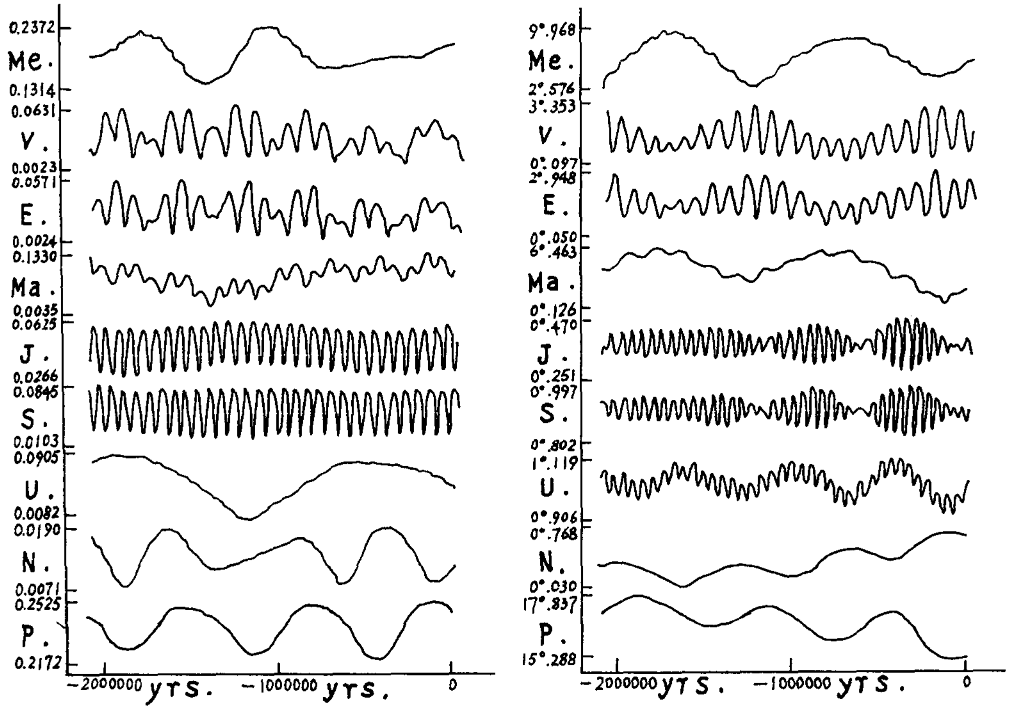

Fig. 1 e Fig. 2 i

Figures 1 and 2: Variations of eccentricities and inclinations of the planetary orbits.

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the orbit of Pluto but there may be some influence on the adjacent planet, the Neptune.

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