BULL. AUSTRAL. MATH. SOC. 46A08
VOL. 48 (1993) [1–6]

COUNTABLY QUASI-SUPRABARRELLED SPACES

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In this paper we obtain some permanence properties of a class of locally convex spaces located between quasi-suprabarrelled spaces and quasi-totally barrelled spaces, for which a closed graph theorem is given.

1. INTRODUCTION

Throughout this paper the word "space" will stand for "Hausdorff locally convex topological vector space defined over the field $\mathbb{K}$ of real or complex numbers". Let us recall a space E is quasi-suprabarrelled [1] if, given an increasing sequence of subspaces of E covering E, there is one which is barrelled; E satisfies condition (G) [4] if, given a sequence of subspaces of E covering E, there is one which is barrelled; E is quasi-totally barrelled [2] if, given a sequence of subspaces of E covering E, there is one which is barrelled and its closure has countable codimension in E; E is totally barrelled [12] if, given a sequence of subspaces of E covering E, there is one which is barrelled and its closure has finite codimension in E; E is unordered Baire-like [6] if, given a sequence of closed absolutely convex subsets of E covering E, there is one which is a neighbourhood of the origin; and E is suprabarrelled [9] ((bd) in [5]) if, given an increasing sequence of subspaces of E covering E, there is one which is barrelled and dense in E. The relationship among these classes of spaces is the following:

$$\text{unordered Baire-like} \Rightarrow \text{totally barrelled} \Rightarrow \text{suprabarrelled} \Rightarrow \text{barrelled}.$$

and

$$\text{totally barrelled} \Rightarrow \text{quasi-totally barrelled} \Rightarrow (G) \Rightarrow \text{quasi-suprabarrelled} \Rightarrow \text{barrelled}.$$

In this paper we shall introduce a class of spaces located between quasi-totally barrelled spaces and quasi-suprabarrelled spaces, which enjoys good permanence properties, and satisfies a closed graph theorem.

Our notation is standard, so if A is a subset of a linear space, $\langle A\rangle$ will denote its linear span and if $\{E_i : i \in I\}$ is a family of spaces, $E = \Pi\{E_i : i \in I\}$ and J is a subset of I, $E(J)$ will denote the subspace of E consisting of those elements of support J.

Received 30 June 1992

2. Countably quasi-suprabarrelled spaces

DEFINITION: We shall say a space E is countably quasi-suprabarrelled if, given an increasing sequence of subspaces $\{E_n : n \in \mathbb{N}\}$ covering E, there is one of them, say E_p, which is barrelled and its closure, $\overline{E_p}$, has countable codimention in E.

Clearly, quasi-totally barrelled $\Rightarrow$ countably quasi-suprabarrelled $\Rightarrow$ quasi-suprabarrelled, and suprabarrelled $\Rightarrow$ countably quasi-suprabarrelled.

It is easy to check that if c is the cardinal of the continuum, φ_c verifies condition (G) and, consequently, is quasi-suprabarrelled but not countably quasi-suprabarrelled, [**2**, Example 2]. On the other hand, φ is a non-suprabarrelled countably quasi-suprabarrelled space since every linear subspace of it has countable codimension. Moreover, if I is any index set, it follows from Theorem 1 below that φ^I is also a non-suprabarrelled countably quasi-suprabarrelled space, and, in general, each countably quasi-suprabarrelled space containing a complemented copy of φ is not suprabarrelled.

EXAMPLES. *Countably quasi-suprabarrelled spaces which are not quasi-totally barrelled.*

1. Let E be a locally convex space and $m_0(E)$ the space of the $2^{\mathbb{N}}$-simple functions defined over $\mathbb{N}$ with values in E endowed with the uniform convergence topology. From [**3**] it follows that if E is nuclear and unordered Baire-like, then $m_0(E)$ is suprabarrelled and, consequently, countably quasi-suprabarrelled. If $\{A_n : n \in \mathbb{N}\}$ denotes the sequence of all the subsets of two different positive integers of $\mathbb{N}$ and L_n is the linear subspace of $m_0(E)$ of all the $f \in m_0(E)$ which are constant on A_n, it is clear that each L_n is closed in $m_0(E)$, that $\{L_n : n \in \mathbb{N}\}$ covers $m_0(E)$ and that if $\dim E$ is uncountable, then each L_n has uncountable codimension in $m_0(E)$. Hence, if E is a nuclear unordered Baire-like space of uncountable dimension, $m_0(E)$ is a countably quasi-suprabarrelled space which is not quasi-totally barrelled. $\square$

2. Let E be a Banach space containing a sequence of closed linear subspaces $\{X_n : n \in \mathbb{N}\}$ of infinite dimension such that for each $n \in \mathbb{N}$, the closed linear hull of $\{X_m : m > n\}$ is a topological complement of $X_1 + \ldots + X_n$ and let E_n be the closed linear hull of $\{X_m : m \in \mathbb{N} \setminus \{n\}\}$. If $\mathcal{U}$ is an ultrafilter in $\mathbb{N}$ which contains the filter of all the subsets of $\mathbb{N}$ whose complement has zero density, $L(U)$ the closure in E of the linear hull of $\bigcup\{X_n : n \in \mathbb{N} \setminus U\}$ for each $U \in \mathcal{U}$, and $L{:}\ = \bigcup\{L(U) : U \in \mathcal{U}\}$, then L is a suprabarrelled and dense subspace in E, [**11**, Proposition 12]. If each X_n has infinite dimension, then $E_n \cap L$ is a subspace of uncountable codimension in L. Finally, as each E_n is closed and $\{E_n \cap L : n \in \mathbb{N}\}$ covers L, we obtain that L is a countably quasi-suprabarrelled space which is not quasi-totally barrelled. $\square$

Clearly, the topological product of φ and any non-quasi-totally barrelled countably quasi-suprabarrelled space is an example of a countably quasi-suprabarrelled space

https://doi.org/10.1017/S0004972700015380 Published online by Cambridge University Press

which is neither suprabarrelled nor quasi-totally barrelled. On the other hand, a metrisable space E is countably quasi-suprabarrelled if and only if E is suprabarrelled. But, as we have mentioned above, there exist non-suprabarrelled countably quasi-suprabarrelled spaces. Next we shall show the following.

PROPOSITION 1. *Let E be a countably quasi-suprabarrelled space. If E is not suprabarrelled then E is not Baire-like either.*

PROOF: If E is not suprabarrelled, there exists an increasing sequence of linear subspaces $\{E_n : n \in \mathbb{N}\}$ of E covering E, such that no E_n is barrelled and dense at the same time. As E is countably quasi-suprabarrelled, we may suppose that each E_n is barrelled and its closure is of countable codimension. Hence E cannot be Baire-like since it may be covered by an increasing sequence of closed linear subspaces of infinite countable codimension. $\square$

3. Properties of countably quasi-suprabarrelled spaces

Next we shall obtain some permanence properties of countably quasi-suprabarrelled spaces.

PROPOSITION 2. *Let E be a countably quasi-suprabarrelled space. If F is a closed linear subspace of E then E/F is countably quasi-suprabarrelled.*

PROOF: Let $\{G_n : n \in \mathbb{N}\}$ be an increasing sequence of subspaces of E/F covering E/F. Let k be the canonical mapping of E onto E/F. Then $\{k^{-1}(G_n) : n \in \mathbb{N}\}$ is an increasing sequence of subspaces of E covering E, so there must be some some $p \in \mathbb{N}$ such that $k^{-1}(G_p)$ is barrelled and $\mathrm{cod}_E \overline{k^{-1}(G_p)} \leqslant \aleph_0$. Now, $G_p = k\left(k^{-1}(G_p)\right)$ is barrelled and if L is an algebraic complement of $\overline{k^{-1}(G_p)}$ in E, then $\overline{G_p} + k(L) = \overline{k(k^{-1}(G_p))} + k(L) \supset k\left(\overline{k^{-1}(G_p)}\right) + k(L) = k\left(\overline{k^{-1}(G_p)} + L\right) = k(E) = E/F$. Hence $\overline{G_p}$ has countable codimension in E/F. $\square$

PROPOSITION 3. *Let F be a dense linear subspace of E. If F is countably quasi-suprabarrelled then E is countably quasi-suprabarrelled.*

PROOF: Let $\{E_n : n \in \mathbb{N}\}$ be an increasing sequence of subspaces of E covering E. Since F is countably quasi-suprabarrelled there is some $p \in \mathbb{N}$ such that each $F \cap E_p$ is barrelled and $\mathrm{cod}_F \overline{F \cap E_p}^F \leqslant \aleph_0$. Let L be a topological complement of $\overline{F \cap E_p}^F$ in F. $\overline{F \cap E_p} \oplus L$ coincides with E since it is closed and $F \subset \overline{F \cap E_p} \oplus L$, so $\overline{E_p} + L = E$ and $\mathrm{cod}_E \overline{E_p} \leqslant \aleph_0$. Besides, $F \cap E_p \oplus_t L$ is a barrelled dense subspace of $E_p + L$. Hence E_p is barrelled. $\square$

PROPOSITION 4. *Let F be a countable codimensional subspace of E. If E is countably quasi-suprabarrelled then F is countably quasi-suprabarrelled.*

https://doi.org/10.1017/S0004972700015380 Published online by Cambridge University Press

PROOF: Let $\{F_n : n \in \mathbb{N}\}$ be an increasing sequence of subspaces of F covering F. Let G be an algebraic complement of F in E. As $\{F_n + G : n \in \mathbb{N}\}$ is an increasing sequence of subspaces of E covering E, we may assume that every F_n is barrelled. On the other hand, as $\{\overline{F_n} + G : n \in \mathbb{N}\}$ is also an increasing sequence of subspaces of E covering E we may assume that every $\overline{F_n} + G$ is barrelled. So, if L_n is a topological complement of $\overline{F_n}$ in $\overline{F_n} + G$, $L_n \cong \varphi$, [7]. Therefore, $\overline{F_n} + G = \overline{F_n} \oplus_t L_n$ is a closed subspace of E for every $n \in \mathbb{N}$ and there must be some $p \in \mathbb{N}$ so that $\overline{F_p} + G$, and consequently $\overline{F_p}$, has countable codimension in E. Hence $\overline{F_p}^F$ has countable codimension in F. ∎

PROPOSITION 5. *The topological product of finitely many countably quasi-suprabarrelled spaces is countably quasi-suprabarrelled.*

PROOF: Assume E_1 and E_2 are countably quasi-suprabarrelled and $E = E_1 \times E_2$. Let $\{F_n : n \in \mathbb{N}\}$ be an increasing sequence of subspaces of E covering E. Then there exists a subsequence $\{F_{n_p} : p \in \mathbb{N}\}$ such that $\mathrm{cod}_{E_i} \overline{F_{n_p} \cap E_i}^{E_i} \leqslant \aleph_0$, $i = 1, 2$. For each $p \in \mathbb{N}$ let $A_{p,i}$ be a cobasis of $\overline{F_{n_p} \cap E_i}^{E_i}$ in E_i, $i = 1, 2$. Set $A := \bigcup\{A_{p,1} \cup A_{p,2} : p \in \mathbb{N}\}$ and, for each $p \in \mathbb{N}$, let $L_p := \langle F_{n_p} \cup A \rangle$. If some L_p were barrelled, F_{n_p} would be barrelled and the proof would be finished since $E = E_1 \times E_2 = \left(\overline{F_{n_p} \cap E_1}^{E_1} + \langle A_{p,1} \rangle\right) \times \left(\overline{F_{n_p} \cap E_2}^{E_2} + \langle A_{p,2} \rangle\right) \subset \overline{F_{n_p}}^E + \langle A_{p,1} \cup A_{p,2} \rangle$, that is $\mathrm{cod}_E \overline{F_{n_p}}^E \leqslant \aleph_0$.

Let us suppose that none of the L_p is barrelled. Then for each $p \in \mathbb{N}$ there is a barrel, say T_p, in L_p which is not a neighbourhood of the origin in L_p. Now, since $\{L_p \cap E_i : p \in \mathbb{N}\}$ is an increasing sequence of subspaces of E_i covering E_i, $i = 1, 2$, there must be some positive integer $q \in \mathbb{N}$ such that $L_q \cap E_i$ is barrelled. Therefore, setting $V_q := \overline{T_q}^E$, $V_q \cap L_q \cap E_i$ is a neighbourhood of the origin in $L_q \cap E_i$.

On the other hand, $L_q \cap E_i$ is dense in E_i since $\overline{L_q \cap E_i}^{E_i} \supset \langle \overline{F_{n_q} \cap E_i}^{E_i} \cup A_{q,i} \rangle = E_i$. Therefore, $\overline{V_q \cap L_q \cap E_i}^{E_i}$ is a neighbourhood of the origin in E_i, $i = 1, 2$, and V_q is a neighbourhood of the origin in E since $\overline{V_q \cap L_q \cap E_1}^{E_1} \times \overline{V_q \cap L_q \cap E_2}^{E_2} \supset V_q + V_q = 2V_q$. Hence T_q is a neighbourhood of the origin in L_q, which is not possible. ∎

In order to show that this result is true for arbitrarily many spaces we shall need [**1**, Theorem 2] and [**2**, Proposition 4]:

LEMMA 1. *Let $\{E_i : i \in I\}$ be a family of spaces such that for every finite subset $H \subset I$, $E(H)$ is quasi-suprabarrelled. Then $E = \Pi\{E_i : \in I\}$ is quasi-suprabarrelled.*

LEMMA 2. *Let $\{E_i : i \in I\}$ be a family of spaces and $\mathcal{B}$ a countable family of closed absolutely convex subsets of $E = \Pi\{E_i : i \in I\}$ such that $\mathrm{cod}_E\langle B \rangle > \aleph_0$ for each $B \in \mathcal{B}$. Suppose that $\mathcal{F} := \{\langle B \rangle : B \in \mathcal{B}\}$ covers E and let $\mathcal{F}_i := \{F \in \mathcal{F} : \mathrm{cod}_{E(\{i\})} F \cap E(\{i\}) > \aleph_0\}$. If for each $F \in \mathcal{F}$ there is a finite subset $J(F)$ of I*

https://doi.org/10.1017/S0004972700015380 Published online by Cambridge University Press

such that $F \supset E(I \setminus J(F))$, then there exists some $j \in I$ such that $\mathcal{F}_j$ covers $E(\{j\})$.

THEOREM 1. *If $\{E_i : i \in I\}$ is a family of countably quasi-suprabarrelled spaces, then $E = \Pi\{E_i : i \in I\}$ is countably quasi-suprabarrelled.*

PROOF: By Lemma 1, E is quasi-suprabarrelled. So, if E is not countably quasi-suprabarrelled, there exists an increasing sequence of barrelled subspaces of E covering E, $\{F_n : n \in \mathbb{N}\}$, such that $\mathrm{cod}_E \overline{F_n}^E > \aleph_0$ for every $n \in \mathbb{N}$. Then $\{\overline{F_n} : n \in \mathbb{N}$ and $\overline{F_n} \supset E(I \setminus J_n)$ with J_n a finite subset of $I\}$ is also an increasing sequence of barrelled subspaces of E covering E, [**12**, Proposition 4].

Now, by Lemma 2, there exists some $j \in I$ such that $\{\overline{F_n} : n \in \mathbb{N}$ and $\mathrm{cod}_{E(\{j\})} \overline{F_n} \cap E(\{j\}) > \aleph_0\}$ covers $E(\{j\})$, which is not possible since $E(\{j\})$ is countably quasi-suprabarrelled. $\square$

Finally let us recall that a locally convex space E is a Γ_r-space if given any quasi-complete subspace G of $E^*(\sigma(E^*, E))$ such that $G \cap E'$ is dense in $E'(\sigma(E', E))$, then G contains E', and that Γ_r-spaces are the maximal class of locally convex spaces satisfying the closed graph theorem when barrelled spaces are considered as the domain, (see [**8**] and [**10**, Chapter 1, Section 6.2]). Moreover [**8**, Corolario 1.14] provides:

LEMMA 3. *Let f be a continuous linear mapping from a barrelled space E into F. If F is a Γ_r-space then f has a continuous extension from the completion of E into F.*

THEOREM 2. *Let E be a countably quasi-suprabarrelled space and suppose $\{F_n : n \in \mathbb{N}\}$ is an increasing sequence of subspaces of F such that on each F_n there exists a topology, τ_n, finer than the original one so that $F_n(\tau_n)$ is a Γ_r-space. If f is a linear mapping from E into F with closed graph then either there is some $p \in \mathbb{N}$ such that $f(E) \subset F_p$ and f is continuous or there is a topological complement H of φ in E such that $f(H) \subset F_p$, f being continuous.*

PROOF: The sequence of subspaces $\{f^{-1}(F_n) : n \in \mathbb{N}\}$ of E is increasing and covers E, so there must to some $p \in \mathbb{N}$ such that $f^{-1}(F_p)$ is barrelled and its closure, H, has countable codimension in E. Let L be a topological complement of H in E. If $\dim L < \aleph_0$, then the restriction of f to L is continuous. If $\dim L = \aleph_0$, then $L \cong \varphi$ and the restriction of f to L is continuous, too. Thus in order to see that f is continuous it is enough to show that the restriction $f|_H$ of f to H is continuous. The restriction g of f to $f^{-1}(F_p)$ has closed graph in $f^{-1}(F_p) \times F_p(\tau_p)$ and thus is continuous. Now Lemma 3 allows us to extend g to a continuous linear mapping $h : H \to F_p(\tau_p)$. Let us show that $h = f|_H$. Given $x \in H$, let $\{x_i : i \in I\}$ be a net in $f^{-1}(F_p)$ converging to x in H. Then the net $\{f(x_i) : i \in I\}$ converges to $h(x)$ in $F_p(\tau_p)$ and, consequently, in F. Hence $f(x) = h(x)$ since $f|_H$ has closed graph in $H \times F$.

As $f(H) \subset F_p$, the proof is complete if $\dim L = \aleph_0$. If $\dim L < \aleph_0$, it is clear that there is some $q \in \mathbb{N}$ so that $f(L) \subset F_q$, and therefore $r = \max\{p, q\}$ gives $f(E) \subset F_r$. $\square$

References

[1] J.C. Ferrando and M. López-Pellicer, 'Quasi-suprabarrelled spaces', *J. Austral. Math. Soc. Ser. A* **46** (1989), 137–145.

[2] J.C. Ferrando and J. Más, 'Quasi totally barrelled spaces', *Portugal Math.* **47** (1990), 205–213.

[3] J.C. Ferrando and L.M. Sánchez Ruiz, 'Algunos espacios tonelados de clase $\aleph_0$', in *Actas XV Jornadas Luso-Espanholas de Matematica*, Vol II (Evora, Portugal, 1990), pp. 267–271.

[4] P. Pérez Carreras and J. Bonet, 'Remarks and examples concerning suprabarrelled and totally barrelled spaces', *Arch. Math.* **39** (1982), 340–347.

[5] W.J. Robertson, I. Tweddle and F.E. Yeomans, 'On the stability of barrelled topologies III', *Bull. Austral. Math. Soc.* **22** (1980), 99–112.

[6] S.A. Saxon, 'Nuclear and product spaces, Baire-like spaces and the strongest locally convex topology', *Math. Ann.* **197** (1972), 87–106.

[7] S.A. Saxon and M. Levin, 'Every countable-codimensional subspace of a barrelled space is barrelled', *Proc. Amer. Math. Soc.* **29** (1971), 91–96.

[8] M. Valdivia, 'Sobre el teorema de la gráfica cerrada', *Collect. Math.* **22** (1971), 51–72.

[9] M. Valdivia, 'On suprabarrelled spaces', in *Func. Anal. Holomorphy and Approximation Theory. Rio de Janeiro 1978*, Lecture Notes in Math. **843** (Springer-Verlag, Berlin, Heidelberg, New York, 1981), pp. 572–580.

[10] M. Valdivia, *Topics in locally convex spaces*, Math. Studies **67** (North Holland, Amsterdam, 1982).

[11] M. Valdivia, 'Three-space problems and projective tensor products of locally convex spaces', *Proc. Royal Irish Acad.* **86A** (1986), 127–141.

[12] M. Valdivia and P. Pérez Carreras, 'On totally barrelled spaces', *Math. Z.* **178** (1981), 263–269.

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