Yasunori Okabe
Nagoya Math. J.
Vol. 54 (1974), 191–213

ON THE STRUCTURE OF SPLITTING FIELDS OF STATIONARY GAUSSIAN PROCESSES WITH FINITE MULTIPLE MARKOVIAN PROPERTY

Dedicated to the late Machiko Okabe

YASUNORI OKABE

§ 1. Introduction

Let $\boldsymbol{X} = (X(t); t \in \boldsymbol{R})$ be a real stationary mean continuous Gaussian process with expectation zero which is purely nondeterministic. In this paper we shall investigate the structure of splitting fields of $\boldsymbol{X}$ having finite multiple Markovian property using the results in [6]. We follow the notations and terminologies in [6].

We shall remember three kinds of definitions of the N-ple Markovian property ($N \in \boldsymbol{N}$).

DEFINITION 1.1. We say that $\boldsymbol{X}$ has the N-ple Markovian property in the broad sense if the splitting field $\boldsymbol{F}_X^{+/-}(t)$ is generated by N linearly independent random variables in $\boldsymbol{M}$ for any $t \in \boldsymbol{R}$.

It is known that $\boldsymbol{X}$ has the N-ple Markovian property in the broad sense if and only if $\boldsymbol{X}$ has a rational spectral density of degree $2N$ ([1], [5]).

DEFINITION 1.2. We say that $\boldsymbol{X}$ has the N-ple Markovian property in the narrow sense if $\boldsymbol{X}$ has the N-ple Markovian property in the broad sense and $\boldsymbol{F}_X^{+/-}(t)$ is equal to the germ field $\partial \boldsymbol{F}_X(t)$ for any $t \in \boldsymbol{R}$.

It is also known that $\boldsymbol{X}$ has the N-ple Markovian property in the narrow sense if and only if its spectral density is the reciprocal of a polynomial of degree $2N$ ([1], [5], [6]).

The third definition is

DEFINITION 1.3. We say that $\boldsymbol{X}$ has the N-ple Markovian property in the sense of T. Hida if, for any $N+2$ real numbers $t_0 < t_1 < \cdots < t_{N+1}$,

Received October 29, 1973.
This study is supported in part by Yukawa Foundation.

https://doi.org/10.1017/S0027763000024673 Published online by Cambridge University Press

$\{E(X(t_n)\,|\,\boldsymbol{F}_X^-(t_0))\,;\,1 \le n \le N\}$ is linearly independent and $\{E(X(t_n)\,|\,\boldsymbol{F}_X^-(t_0))\,;\,1 \le n \le N+1\}$ is linearly dependent.

It is shown in [3] that, if $\boldsymbol{X}$ has the N-ple Markovian property in the sense of T. Hida, $\boldsymbol{X}$ has a rational spectral density of degree $2N$.

In this paper we shall consider the case where $\boldsymbol{X}$ has the N-ple Markovian property in the broad sense.

In § 2 we shall give a formula for the canonical representation kernel of our process $\boldsymbol{X}$ (Theorem 2.1). In the proof of Theorem 2.1 we shall use Theorem 8.1 in [6], which gives a formula for the canonical representation kernel of process $\boldsymbol{X}$ having the Markovian property. By the Markovian property we mean that $\boldsymbol{X}$ satisfies $\boldsymbol{F}_X^{+/-}(t) = \partial \boldsymbol{F}_X(t)$ for any $t \in \boldsymbol{R}$ ([5], [6]).

In § 3 we shall construct an N-dimensional stationary Gaussian process $\mathscr{X} = (\mathscr{X}(t)\,;\,t \in \boldsymbol{R})$ satisfying

(1.1) {the n-th component of $\mathscr{X}(t)\,;\,1 \le n \le N$} is lineary independent in $\boldsymbol{M}$ and

(1.2) $\boldsymbol{F}_X^{+/-}(t) = \boldsymbol{F}_{\mathscr{X}}^{+/-}(t) = \sigma(\mathscr{X}(t))$ for any $t \in \boldsymbol{R}$ (Theorems 3.2 (ii) and 3.3). We can give an expression of the linear predictor of $X(t)$ $(t > 0)$ using the past $\boldsymbol{F}_X^-(0)$ in terms of the process $\mathscr{X}$ (Theorem 3.2 (i)). The relation (1.2) implies that $\mathscr{X}$ has a simple Markovian property.

In § 4 we shall investigate the structure of $\mathscr{X}$ from the point of view of Markov processes, and show that a Markov process $(\mathscr{X}(t), P(\cdot\,|\,\mathscr{X}(0) = x)\,;\,t > 0, x \in \boldsymbol{R}^N)$ is a recurrent Gaussian diffusion process with transition probability density and has a unique invariant measure (Theorem 4.3).

We shall prove in § 5 that the N-dimensional stationary Gaussian process $\mathscr{X}$ satisfying (1.1) and (1.2) is unique up to multiplicative nonsingular $N \times N$-matirices (Theorem 5.1).

In § 6 we shall define a nonsingular $N \times N$-matrix T and an associated N-dimensional stationary Gaussian process $\mathscr{Y} = (\mathscr{Y}(t)\,;\,t \in \boldsymbol{R})) = (T^{-1}\mathscr{X}(t)\,;\,t \in \boldsymbol{R})$. We note that the matrix T can be definitely expressed in terms of the spectral density of $\boldsymbol{X}$. Then we shall prove that the N-th component process of $\mathscr{Y}$ $(=\boldsymbol{Y})$ has the N-ple Markovian property in the narrow sense and satisfies

$$\boldsymbol{F}_X^{+/-}(t) = \boldsymbol{F}_Y^{+/-}(t) = \partial \boldsymbol{F}_Y(t) \qquad (t \in \boldsymbol{R}) \tag{1.3}$$

(Theorem 6.2). We can also give an alternative expression of the linear

https://doi.org/10.1017/S0027763000024673 Published online by Cambridge University Press

predictor of $X(t)$ $(t>0)$ using the past $F_X^-(0)$ in terms of the process $\mathscr{Y}$ (Theorem 6.3 (i)).

Finally in §7 we shall give three applications of our results. At first we shall characterize the Markovian property of stationary Gaussian processes from the point of view of representations and then give a necessary and sufficient condition for the N-ple Markovian property in the sense of T. Hida (Theorems 7.1 and 7.2). Next we shall characterize the linear predictor of $X(t)$ $(t>0)$ using the past $F_X^-(0)$ as a unique solution of an initial value problem of a differential equation, which is derived from the spectral density of $\boldsymbol{X}$. As the third application, we shall give an expression of nonlinear predictors of $X(t)$ $(t>0)$ using the past $F_X^-(0)$ in terms of the Gaussian diffusion process $(\mathscr{X}(t), P(\cdot\,|\,\mathscr{X}(0)=x); t>0, x\in \boldsymbol{R}^N)$ defined in §4 (Theorem 7.4).

§2. Rational weights

Let N be a positive integer and let $\Delta=\Delta(\lambda)$ be a rational function of degree $2N$ which is nonnegative, symmetric and integrable. Then we have the following decomposition:

$$(2.1)\quad \begin{cases} \Delta(\lambda)=\left|\dfrac{Q(-\lambda)}{P(-\lambda)}\right|^2 \quad (\lambda\in\boldsymbol{R}), \\ V_P=\boldsymbol{C}^+, \quad V_Q\subset \boldsymbol{C}^+\cup\boldsymbol{R}, \quad V_P\cap V_Q=\phi \quad \text{and} \\ Q(z)=\sum_{n=0}^{N-1} b_n(-iz)^n, \quad P(z)=\sum_{n=0}^{N} c_n(-iz)^n, \quad b_n, c_n\in\boldsymbol{R}, c_N\neq 0, \end{cases}$$

where V_S denotes the set of zero points of a polynomial S. Such a decomposition is unique up to multiplicative constants of absolute one.

2.1. We denote by F the Fourier transform of the reciprocal of a function $P(-\cdot)$ in (2.1):

$$(2.2)\qquad F=(P(-\cdot)^{-1})^\wedge .$$

It is easy to see that $F=0$ in $(-\infty,0)$ and $F^{(n)}\in\mathscr{A}((0,\infty))\cap L^2((0,\infty))$ $(n=0,1,2,\cdots)$. By Lemmas 8.5, 8.6 (ii) and Proposition 8.1 in [6] we have

LEMMA 2.1. (i) $F^{(n)}(0+)=0\ (0\le n\le N-1)$, $F^{(N-1)}(0+)=2\pi(-1)^N c_N^{-1}$,
(ii) $F^{(n)}\in L^2(\boldsymbol{R})$ $(0\le n\le N-1)$ *(distribution derivatives)*,
(iii) $\{F^{(n)}; 0\le n\le N-1\}$ *is linearly independent in* $L^2(\boldsymbol{R})$.

https://doi.org/10.1017/S0027763000024673 Published online by Cambridge University Press

We define for any $n \in \{0, 1, \cdots, N-1\}$ an L^2-function F_n by

$$(2.3)\qquad F_n(t) = \begin{cases} (2\pi)^{-1} \sum_{k=0}^{N-n-1} c_{n+k+1}(-1)^{k+1} F^{(k)}(t) & (t > 0)\,, \\ 0 & (t \leq 0)\,. \end{cases}$$

In particular we have

$$(2.4)\qquad F_{N-1} = (-2\pi)^{-1} c_N F\,.$$

Then it follows from Lemmas 8.2, 8.3 and Proposition 8.1 in [6] that

LEMMA 2.2. (i) $F_0(0+) = 1$, $F_n(0+) = 0$ $(1 \leq n \leq N-1)$,
(ii) $F_n = (2\pi)^{-1} \sum_{k=0}^{N-n-1} c_{n+k+1} (-1)^{k+1} F^{(k)}$ $(1 \leq n \leq N-1)$,
(iii) $F_0^{(1)} = \delta - (2\pi)^{-1} c_0 F$, $F_n^{(1)} = -F_{n-1} - (2\pi)^{-1} c_n F$ $(1 \leq n \leq N-1)$,
(iv) $\{F_n; 0 \leq n \leq N-1\}$ *is linearly independent in* $L^2(\boldsymbol{R})$.

Furthermore it follows from Theorem 8.1 in [6] that

LEMMA 2.3. *For any* $s \in (-\infty, 0)$, $t \in (0, \infty)$ *and* $n \in \{0, \cdots, N-1\}$,
(i) $F(t-s) = \sum_{n=0}^{N-1} (-1)^n F^{(n)}(t) F_n(-s)$,
(ii) $F_n(t-s) = (2\pi)^{-1} \sum_{\ell=0}^{N-1} (-1)^{\ell} (\sum_{m=0}^{N-n-1} c_{n+m+1} (-1)^{m+1} F^{(\ell+m)}(t)) F_\ell(-s)$.

By using Lemmas 2.1 (i), 2.2 (i) and 2.2 (iii), we can show

LEMMA 2.4. $F_n^{(m)}(0+) = (-1)^n \delta_{mn}$ $(0 \leq m, n \leq N-1)$.

Next we shall prove

LEMMA 2.5. *There exist N positive numbers* $t_0 < t_1 < \cdots < t_{N-1}$ *such that* $\det(F^{(m)}(t_n))_{0 \leq m,n \leq N-1} \neq 0$.

Proof. Assume that $\det(F^{(m)}(t_n)) = 0$ for any N positive numbers $t_0 < t_1 < \cdots < t_{N-1}$. Differentiating it n times with respect to t_n for each $n \in \{0, 1, \cdots, N-1\}$ and then letting $t_0 < t_1 < \cdots < t_{N-1}$ tend to zero, we see from Lemma 2.1 (i) that

$$\det \begin{pmatrix} 0 & & & 1 \\ & & \cdot^{\cdot^{\cdot}} & \\ & 1 & & * \\ 1 & & & \end{pmatrix} = 0\,.$$

This is absurd. Therefore we have the desired result. (Q.E.D.)

https://doi.org/10.1017/S0027763000024673 Published online by Cambridge University Press

Finally in subsection 2.1 we shall show

LEMMA 2.6. *The following* (i) *and* (ii) *are equivalent*:
(i) $\det(F^{(m)}(t_n))_{0\le m,n\le N-1} \neq 0$ *for any* N *positive numbers* $t_0 < t_1 < \cdots < t_{N-1}$;
(ii) $V_P \subset \{z \in C^+ ; \operatorname{Re} z = 0\}$.

Proof. We decompose $P(z) = d_1 \prod_{n=0}^{N-1}(\lambda_n + iz)$, where d_1 is a constant and $\operatorname{Re}\lambda_n > 0$ $(0 \le n \le N-1)$. Denoting by f_n the Fourier transform of $(\lambda_n - i\cdot)^{-1}$ $(0 \le n \le N-1)$, we find that $f_n(t) = 2\pi(\operatorname{Re}(\lambda_n))^{-1}e^{-\lambda_n t}(t > 0)$, $f_n(t) = 0$ $(t < 0)$ and $F = d_2 f_0 * f_1 * \cdots * f_{N-1}$ with some constant d_2. At first we assume that (ii) holds and so $\lambda_n \in \boldsymbol{R}$ $(0 \le n \le N-1)$. We define $N+1$ functions v_n in $\mathscr{A}((0,\infty))$ $(0 \le n \le N)$ by

$$\begin{cases} v_0(t) = d_1^{-1}e^{\lambda_0 t}\ , \\ v_n(t) = e^{(\lambda_n - \lambda_{n-1})t} \qquad (1 \le n \le N-1)\ , \\ v_N(t) = e^{-\lambda_{N-1}t} \end{cases}$$

and then N functions G_n in $\mathscr{A}((0,\infty))$ $(1 \le n \le N)$ by

$$G_n(t) = v_N(t)\int_0^t v_{N-1}(t_1)dt_1 \int_0^{t_1} v_{N-2}(t_2)dt_2 \cdots \int_0^{t_{N-n}} v_{n-1}(t_{N-n+1})dt_{N-n+1}\ .$$

It may be easily seen that $P\left(\frac{1}{i}\frac{d}{dt}\right)G_n = 0$ in $(0,\infty)$ $(1 \le n \le N)$. Since v_n's are positive, we can apply (II, 30) in [3] to get that $\det(G_m(t_n)) \neq 0$ for any N positive numbers $t_0 < t_1 < \cdots < t_{N-1}$. Since $P\left(\frac{1}{i}\frac{d}{dt}\right)F_n = 0$ in $(0,\infty)$ $(0 \le n \le N-1)$, we see from Lemma 2.1 (iii) that there exists a nonsingular $N \times N$-matrix C satisfying $(F^{(m)}(t_n)) = C(G_m(t_n))$ and so (i) holds. Next let's assume that (ii) does not hold. Since $\overline{P(\lambda)} = P(-\lambda)$ $(\lambda \in \boldsymbol{R})$, we then may assume and do that $\lambda_0 \notin \boldsymbol{R}$ and $\lambda_1 = -\overline{\lambda_0}$. By an easy calculation it is shown that $f \equiv f_0 * f_1$ is equal to $d_3 \sin(\operatorname{Re}\lambda_0 \cdot t)e^{-iIm\lambda_0 \cdot t}$ in $(0,\infty)$ for some constant d_3. Since $f_2 * f_3 * \cdots * f_{N-1}$ is a fundamental solution of a differential operator $S\left(\frac{1}{i}\frac{d}{dt}\right)$ of order $N-2$ with constant coefficients, we find that $S\left(\frac{1}{i}\frac{d}{dt}\right)F = d_1 f$. This implies that, for any N positive numbers $t_0 < t_1 < \cdots < t_{N-1}$,

https://doi.org/10.1017/S0027763000024673 Published online by Cambridge University Press

$$\det(F^{(m)}(t_n)) = d_4 \det \begin{bmatrix} F(t_0) & \cdots\cdots\cdots & F(t_{N-1}) \\ \vdots & & \vdots \\ F^{(N-3)}(t_0) & \cdots & F^{(N-3)}(t_{N-1}) \\ f(t_0) & \cdots\cdots\cdots & f(t_{N-1}) \\ F^{(N-1)}(t_0) & \cdots & F^{(N-1)}(t_{N-1}) \end{bmatrix},$$

where d_4 is a constant. Since $f(n\pi(\mathrm{Re}\,\lambda_0)^{-1}) = 0$ $(n \in N)$, we find that (i) does not hold. Thus we have proved Lemma 2.6. (Q.E.D.)

2.2. We denote by E the Fourier transform of a function $P(-\cdot)^{-1}Q(-\cdot)$:

$$E = (P(-\cdot)^{-1}Q(-\cdot))^\wedge . \tag{2.5}$$

By (2.2) we have

$$E = Q\left(\frac{1}{i}\frac{d}{dt}\right)F . \tag{2.6}$$

We define for any $n \in \{0, 1, \cdots, N-1\}$ an L^2-function E_n by

$$E_n(t) \equiv \begin{cases} Q\left(\frac{1}{i}\frac{d}{dt}\right)F_n(t) & (t > 0) , \\ 0 & (t \leq 0) . \end{cases} \tag{2.7}$$

In particular we see from (2.4) and (2.6) that

$$E_{N-1} = (-2\pi)^{-1}c_N E . \tag{2.8}$$

Immediately from Lemma 2.3 and (2.7) we have

THEOREM 2.1. *For any $s \in (-\infty, 0)$, $t \in (0, \infty)$ and $n \in \{0, 1, \cdots, N-1\}$,*

(i) $E(t-s) = \sum_{n=0}^{N-1} (-1)^n F^{(n)}(t) E_n(-s)$,

(ii) $E_n(t-s) = (2\pi)^{-1} \sum_{\ell=0}^{N-1} (-1)^\ell (\sum_{k=0}^{N-n-1} (-1)^{k+1} c_{n+k+1} F^{(k+\ell)}(t)) E_\ell(-s)$.

Moreover it follows from Lemmas 2.2 (iii) and 2.4 that

LEMMA 2.7. (i) $E_n(0+) = b_n$ $(0 \leq n \leq N-1)$,

(ii) $E_0'(t) = (-2\pi)^{-1}c_0 E(t)$, $E_n'(t) = -E_{n-1}(t) - (2\pi)^{-1}c_n E(t)$ $(t > 0, 1 \leq n \leq N-1)$.

Finally we shall prove

LEMMA 2.8. *$\{E_n : 0 \leq n \leq N-1\}$ is linearly independent in $L^2(\mathbf{R})$.*

Proof. Let α_n $(0 \leq n \leq N-1)$ be real constants such that $\sum_{n=0}^{N-1} \alpha_n E_n$

https://doi.org/10.1017/S0027763000024673 Published online by Cambridge University Press

$= 0$. We then see from (2.7) that $Q\left(\frac{1}{i}\frac{d}{dt}\right)(\sum_{n=0}^{N-1}\alpha_n F_n) = 0$ in $\boldsymbol{R} - \{0\}$ in the sense of distributions. Therefore, there exists a polynomial Q_1 such that $Q\left(\frac{1}{i}\frac{d}{dt}\right)(\sum_{n=0}^{N-1}\alpha_n F_n) = Q_1\left(\frac{1}{i}\frac{d}{dt}\right)\delta$. By taking the inverse Fourier transform of both sides, we find that $Q(-\lambda)(\sum_{n=0}^{N-1}\alpha_n \tilde{F}_n(\lambda)) = Q_1(-\lambda)$ $(\lambda \in \boldsymbol{R})$. Since Lemma 2.2 (ii) implies that $\tilde{F}_n(\lambda) = (-2\pi)^{-1}(\sum_{m=0}^{N-n-1} c_{n+m+1}(i\lambda)^m)P(-\lambda)^{-1}(\lambda \in \boldsymbol{R})$, there exists a polynomial Q_2 of at most degree $N-1$ such that $Q(\lambda)Q_2(\lambda)P(\lambda)^{-1} = Q_1(\lambda)$ $(\lambda \in \boldsymbol{R})$. Hence we see from (2.1) that $Q_2 = 0$ and so $Q_1 = 0$. This implies that $\sum_{n=0}^{N-1}\alpha_n F_n = 0$ and so $\alpha_n = 0$ $(0 \le n \le N-1)$ by Lemma 2.2 (iv). Thus we have proved Lemma 2.8. Q.E.D.

§ 3. $F_X^{+/-}(t)$ (I)

In the sequel we shall consider a real stationary Gaussian process $\boldsymbol{X} = (X(t); t \in \boldsymbol{R})$ having the spectral density Δ of the form (2.1). We assume that $\boldsymbol{X}$ has expectation zero. Since $P(-\cdot)^{-1}Q(-\cdot)$ is an outer function of the Hardy weight Δ, we get from (2.5) the following canonical representation:

$$X(t) = \sqrt{2\pi}^{-1}\int_{-\infty}^{t} E(t-s)dB(s)\ , \tag{3.1}$$

where $(B(t); t \in \boldsymbol{R})$ is a standard Brownian motion satisfying

$$F_X^-(t) = \sigma(B(s_1) - B(s_2); s_1, s_2 < t) \qquad \text{for any } t \in \boldsymbol{R}\ . \tag{3.2}$$

Using L^2-functions E_n in (2.7) we define random variables $X_n(t)$ $(t \in \boldsymbol{R}, 0 \le n \le N-1)$ by

$$X_n(t) = \sqrt{2\pi}^{-1}\int_{-\infty}^{t} E_n(t-s)dB(s) \tag{3.3}$$

and then an N-dimensional stationary Gaussian process $\mathscr{X} = (\mathscr{X}(t); t \in \boldsymbol{R})$ by

$$\mathscr{X}(t) = (X_0(t), \cdots, X_{N-1}(t))^*\ . \tag{3.4}$$

Particularly we see from (2.8) that

$$X_{N-1}(t) = (-2\pi)^{-1}c_N X(t) \qquad (t \in \boldsymbol{R})\ . \tag{3.5}$$

We define an $N \times N$-matrix A and an N-vector $\boldsymbol{b}$ by

https://doi.org/10.1017/S0027763000024673 Published online by Cambridge University Press

$$(3.6)\qquad A = \begin{bmatrix} 0 & & & & a_0 \\ -1 & \ddots & & 0 & a_1 \\ & -1 & \ddots & & \vdots \\ & & \ddots & 0 & \vdots \\ & 0 & & \ddots & \vdots \\ & & & -1 & a_{N-1} \end{bmatrix}, \qquad \boldsymbol{b} = \begin{bmatrix} b_0 \\ \vdots \\ b_{N-1} \end{bmatrix},$$

where $a_n = c_n c_N^{-1}$ $(0 \leq n \leq N)$.

In the same way as Theorem 9.1 in [6] we can show from (2.8) and Lemma 2.7 that

THEOREM 3.1. *For almost all* ω

$$\mathscr{X}(t) - \mathscr{X}(s) = \sqrt{2\pi}^{-1}\boldsymbol{b}(B(t) - B(s)) + \int_s^t A\mathscr{X}(u)du \qquad (s < t)\,.$$

In particular $\mathscr{X}(t)$ *is continuous in* $t \in \boldsymbol{R}$.

Noting (3.2) we see from Theorem 2.1 (i) and Lemma 2.8 that

THEOREM 3.2. (i) *For any* s *and* $t \in \boldsymbol{R}$, $s < t$,

$$E(X(t)\,|\,\boldsymbol{F}_X^-(s)) = \sum_{n=0}^{N-1}(-1)^n F^{(n)}(t-s)X_n(s)\,.$$

(ii) $\{X_n(t)\,;\, 0 \leq n \leq N-1\}$ *is linearly independent in* $\boldsymbol{M}$ *for any* $t \in \boldsymbol{R}$.

We define for any $t \in \boldsymbol{R}$ an $N \times N$-matrix $A(t) = (A(t)_{mn})$ by

$$(3.7)\quad A(t)_{mn} = (2\pi)^{-1}\sum_{k=0}^{N-m-1}(-1)^{n+k+1}c_{m+k+1}F^{(n+k)}(t) \qquad (0 \leq m, n \leq N-1)\,.$$

Then we shall show

LEMMA 3.1. (i) *For any* s *and* $t \in \boldsymbol{R}$, $s < t$,

$$E(\mathscr{X}(t)\,|\,\boldsymbol{F}_X^-(s)) = A(t-s)\mathscr{X}(s)\,.$$

(ii) $A(t) = e^{tA}$ $(t > 0)$.

Proof. By Theorem 2.1 (ii) we have (i). We particularly see from Lemma 2.8 that $A(s+t) = A(s)A(t)$ $(s > 0, t > 0)$. Since $A(t)$ is continuous in $t \in (0, \infty)$ and $A(0+) = I$, this implies that there exists an $N \times N$-matrix $\tilde{A}$ satisfying $A(t) = e^{t\tilde{A}}$ $(t > 0)$. Since $B(t) - B(0)$ $(t > 0)$ are independent of $\boldsymbol{F}_X^-(0)$ and $\mathscr{X}(0)$ is $\boldsymbol{F}_X^-(0)$-measurable by (3.2), we see from Theorem 3.1 and Lemma 3.1 (i) that

$$E(\mathscr{X}(t)\,|\,\boldsymbol{F}_X^-(0)) = \left(I + \int_0^t A e^{u\tilde{A}} du\right)\mathscr{X}(0)$$
$$= e^{t\tilde{A}}\mathscr{X}(0) \qquad (t > 0)\ .$$

By Theorem 3.2 (ii) we get

$$e^{t\tilde{A}} = I + \int_0^t A e^{u\tilde{A}} du \qquad (t > 0)\ .$$

Differentiating both sides at $t = 0$, we find that $\tilde{A} = A$. Thus we have proved Lemma 3.1. (Q.E.D.)

In the same way as in the case of $\boldsymbol{X}$, we shall consider the past fields $\boldsymbol{F}_{\mathscr{X}}^-(t)$, the future fields $\boldsymbol{F}_{\mathscr{X}}^+(t)$ and the splitting fields $\boldsymbol{F}_{\mathscr{X}}^{+/-}(t)$ $(t \in \boldsymbol{R})$ associated with $\mathscr{X}$ (Definition 9.1 in [6]). We then see from (3.2), (3.3) and (3.4) that

$$\boldsymbol{F}_X^-(t) = \boldsymbol{F}_{\mathscr{X}}^-(t) \qquad (t \in \boldsymbol{R})\ . \tag{3.8}$$

Now we shall prove the following main theorem.

THEOREM 3.3. *$\boldsymbol{F}_X^{+/-}(t) = \boldsymbol{F}_{\mathscr{X}}^{+/-}(t) = \sigma(\mathscr{X}(t))$ for any $t \in \boldsymbol{R}$.*

Proof. By virtue of Lemma 2.5, we see from Theorem 3.2 that $\boldsymbol{M}^{+/-}(t)$ is equal to the closed linear hull of $\{X_n(t)\,; 0 \le n \le N-1\}$ $(t \in \boldsymbol{R})$. This implies by Lemma 2.1 (iii) in [6] that $\boldsymbol{F}_X^{+/-}(t) = \sigma(\mathscr{X}(t))$ for any $t \in \boldsymbol{R}$. It is clear that $\sigma(\mathscr{X}(t)) \subset \boldsymbol{F}_{\mathscr{X}}^-(t) \cap \boldsymbol{F}_{\mathscr{X}}^+(t) \subset \boldsymbol{F}_{\mathscr{X}}^{+/-}(t)$ since $\mathscr{X}(t)$ is continuous in $t \in \boldsymbol{R}$. On the other hand, it follows from Lemma 3.1 that, for any $t \in \boldsymbol{R}$ and any $h > 0$,

$$X_n(t+h) = A(h)\mathscr{X}(t)_n + \sqrt{2\pi}^{-1}\int_t^{t+h} E_n(t+h-s)dB(s) \qquad (0 \le n \le N-1)\ .$$

Since $B(t+z) - B(t)$ $(z > 0)$ are independent of $\boldsymbol{F}_{\mathscr{X}}^-(t)$ for any $t \in \boldsymbol{R}$ by (3.2) and (3.8), we can see that $\boldsymbol{F}_{\mathscr{X}}^-(t)$ is independent of $\boldsymbol{F}_{\mathscr{X}}^+(t)$ under the condition that $\sigma(\mathscr{X}(t))$ is known, and so that $\boldsymbol{F}_{\mathscr{X}}^{+/-}(t) \subset \sigma(\mathscr{X}(t))$. Thus we have proved Theorem 3.3. (Q.E.D.)

§ 4. A Gaussian diffusion process

From Theorem 3.3 we find that a Gaussian process $(\mathscr{X}(t), P(\cdot\,|\,\mathscr{X}(0) = x)\,; t > 0, x \in \boldsymbol{R}^N)$ has the usual Markovian property. In this section we shall investigate several properties of such a Gaussian Markov process.

By (3.2) and Lemma 3.1 we have

https://doi.org/10.1017/S0027763000024673 Published online by Cambridge University Press

LEMMA 4.1. (i) $E_n(t) = \sqrt{2\pi}^{-1}(e^{tA}\boldsymbol{b})_n$ $(t > 0, 0 \leq n \leq N-1)$,

(ii) $\mathscr{X}(t) = e^{(t-s)A}\mathscr{X}(s) + \sqrt{2\pi}^{-1}\int_s^t e^{(t-u)A}\boldsymbol{b}dB(u)$ $(s < t)$.

We denote by $\mu(t,x)$ and $R(t,x)$ the mean vector and the covariance matrix, respectively, under the condition that $\mathscr{X}(0) = x$ $(t > 0, x \in \boldsymbol{R}^N)$:

$$\begin{cases} \mu(t,x) = E(\mathscr{X}(t) \mid \mathscr{X}(0) = x) , \\ R(t,x) = E(\mathscr{X}(t)\mathscr{X}(0)^* \mid \mathscr{X}(0) = x) . \end{cases}$$

It then follows from Lemma 4.1 that

$$\text{(4.1)} \qquad \begin{cases} \mu(t,x) = e^{tA}x , \\ R(t,x) = R(t) = \left((2\pi)^{-1}\int_0^t e^{sA}\boldsymbol{b}_m e^{sA}\boldsymbol{b}_n ds\right)_{0 \leq m,n \leq N-1} . \end{cases}$$

We shall prove

THEOREM 4.1. *$\{A^n\boldsymbol{b}; 0 \leq n \leq N-1\}$ is linearly independent.*

As an application of Theorem 4.1 we find that $R(t)$ is a positive definite matrix for each $t > 0$. Before the proof of Theorem 4.1, we shall prepare several lemmas.

LEMMA 4.2. *For any $n \in \{0, 1, \cdots, N-1\}$ we set*

$$G_n(t) = \begin{cases} \sum_{m=0}^{N-1}(-1)^m b_m F^{(n+m)}(t) & (t > 0) , \\ 0 & (t \leq 0) . \end{cases}$$

Then

$$\{G_n; 0 \leq n \leq N-1\} \qquad \textit{is linearly independent in } L^2(\boldsymbol{R}) .$$

Proof. Let α_n $(0 \leq n \leq N-1)$ be real constants such that $\sum_{n=0}^{N-1}\alpha_n G_n = 0$. We define a polynomial $S(z) = \sum_{n=0}^{N-1}\alpha_n(iz)^n$. Since $G_m(t) = G_0^{(m)}(t)$ for any $t \in \boldsymbol{R} - \{0\}$, we find that $S\left(\frac{1}{i}\frac{d}{dt}\right)G_0 = 0$ in $\boldsymbol{R} - \{0\}$ in the sense of distributions. Therefore, there exists a polynomial Q_1 such that $S\left(\frac{1}{i}\frac{d}{dt}\right)G_0 = Q_1\left(\frac{1}{i}\frac{d}{dt}\right)\delta$ in $\boldsymbol{R}$. Noting that $G_0 \in L^2(\boldsymbol{R})$ and taking the inverse Fourier transform of both sides, we find that $S(-\lambda)\tilde{G}_0(\lambda) = Q_1(-\lambda)$ $(\lambda \in \boldsymbol{R})$. On the other hand, we see that $\tilde{G}_0(\lambda) = Q(-\lambda)\tilde{F}(\lambda)$, since $G_0 =$

$Q\left(\frac{1}{i}\frac{d}{dt}\right)F$. Hence, it follows from (2.2) that $S(\lambda)Q(\lambda) = Q_1(\lambda)P(\lambda)$ $(\lambda \in \boldsymbol{R})$. Since S is a polynomial of at most degree $N-1$, this implies by (2.1) that $S = 0$ and so $\alpha_n = 0$ $(0 \leq n \leq N-1)$. Thus we have proved Lemma 4.2. (Q.E.D.)

LEMMA 4.3. *For any $m, n \in \{0, 1, \cdots, N-1\}$ we set*

$$\gamma_{mn} = \sum_{\ell=0}^{N-1} (-1)^{\ell} b_\ell F_n^{(m+\ell)}(0+) .$$

Then the $N \times N$-matrix $(\gamma_{mn})_{0 \leq m,n \leq N-1}$ is nonsingular.

Proof. Differentiating (i) in Lemma 2.3 $\ell + m$ times at $s = 0$, we have

$$F^{(\ell+m)}(t) = \sum_{n=0}^{N-1} (-1)^n F^{(n)}(t) F_n^{(\ell+m)}(0+) \qquad (t > 0, 0 \leq \ell, m \leq N-1) .$$

Multiplying it by $(-1)^\ell b_\ell$ and then summing up with respect to ℓ, we get

$$\sum_{\ell=0}^{N-1} (-1)^\ell b_\ell F^{(\ell+m)}(t) = \sum_{n=0}^{N-1} (-1)^n \gamma_{mn} F^{(n)}(t) \qquad (t > 0) .$$

Therefore, by Lemmas 2.1 (iii) and 4.2, we obtain the desired result. (Q.E.D.)

LEMMA 4.4. *The $N \times N$-matrix $(E_n^{(m)}(0+))_{0 \leq m,n \leq N-1}$ is nonsingular.*

Proof. Differentiating (ii) in Theorem 2.1 m times at $t = 0$ and then letting s tend to zero, we have

$$E_n^{(m)}(0+) = (2\pi)^{-1} \sum_{\ell=0}^{N-1} (-1)^\ell \left(\sum_{k=0}^{N-n-1} (-1)^{k+1} c_{k+n+1} F^{(m+k+\ell)}(0+) \right) E_\ell(0+) .$$

On the other hand, differentiating (i) in Lemma 2.3 m times and $k + \ell$ times at $t = 0$ and $s = 0$, respectively, we get

$$F^{(m+k+\ell)}(0+) = \sum_{j=0}^{N-1} (-1)^j F^{(m+j)}(0+) F_j^{(k+\ell)}(0+) .$$

Therefore it follows from Lemma 2.7 (i) that

$$E_n^{(m)}(0+) = (2\pi)^{-1} \sum_{j=0}^{N-1} \sum_{k=0}^{N-n-1} F^{(m+j)}(0+)(-1)^j \gamma_{kj} (-1)^{k+1} c_{k+n+1} .$$

https://doi.org/10.1017/S0027763000024673 Published online by Cambridge University Press

By Lemma 2.1 (i), the matrix $(F^{(m+j)}(0+))_{0\leq m,j\leq N-1}$ must be nonsingular. Therefore, we obtain the desired result noting that c_N is not zero and using Lemma 4.3. (Q.E.D.)

LEMMA 4.5. *The* $N\times N$*-matrix* $(E^{(m+n)}(0+))_{0\leq m,n\leq N-1}$ *is nonsingular.*

Proof. Differentiating (i) in Theorem 2.1 ℓ times and m times at $t=0$ and $s=0$, respectively, we have

$$E^{(\ell+m)}(0+)=\sum_{n=0}^{N-1}(-1)^n F^{(\ell+n)}(0+)E_n^{(m)}(0+)\ .$$

Therefore, by Lemma 4.4, we get the result. (Q.E.D.)

LEMMA 4.6. $\{A^n\boldsymbol{a}\,;\,0\leq n\leq N-1\}$ *is linearly independent, where* $\boldsymbol{a}=(a_0\cdots a_{N-1})^*$.

Proof. Since $A\boldsymbol{a}=-(0a_0\cdots a_{N-2})^*+a_{N-1}\boldsymbol{a}$, we have the result noting that a_0 is not zero. (Q.E.D.)

LEMMA 4.7. *For any* ℓ, m *and* $n\in\{0,1,\cdots,N-1\}$,

$$(A^n)_{\ell m}=(2\pi)^{-1}\sum_{k=0}^{N-\ell-1}c_{\ell+k+1}(-1)^{m+k+1}F^{(m+k+n)}(0+)\ .$$

Proof. Differentiating e^{tA} k times at $t=0$, we obtain the result from (3.7) and Lemma 3.1 (ii). (Q.E.D.)

LEMMA 4.8. $\sum_{n=0}^{N-1}(-1)^n b_n A^n$ *is nonsingular.*

Proof. We denote by $\boldsymbol{a}_\ell$ the $\ell+1$ row of the matrix $\sum_{n=0}^{N-1}(-1)^n b_n A^n$ and set $\boldsymbol{e}_\ell=(\cdots(-1)^{n+1}c_{\ell+n+1}\cdots)^*$ $(0\leq\ell\leq N-1)$, where $c_m=0$ for $m\geq N+1$. By (2.6) and Lemma 4.7 we have

$$\begin{aligned}\boldsymbol{a}_\ell&=(2\pi)^{-1}(-1)^\ell\sum_{n=0}^{N-1}\sum_{k=0}^{N-1}(-1)^k b_k F^{(\ell+k+n)}(0+)\boldsymbol{e}_n\\&=(2\pi)^{-1}(-1)^\ell\sum_{n=0}^{N-1}\left(Q\left(\frac{1}{i}\frac{d}{dt}\right)F(t)\right)^{(\ell+n)}\Bigg|_{t=0}\boldsymbol{e}_n\\&=(2\pi)^{-1}(-1)^\ell\sum_{n=0}^{N-1}E^{(\ell+n)}(0+)\boldsymbol{e}_n\ .\end{aligned}$$

Therefore, since $\det(\boldsymbol{e}_0\cdots\boldsymbol{e}_{N-1})=((-1)^N c_N)^N$ is not zero, we have the desired result from Lemma 4.5. (Q.E.D.)

After these preparations, we are in a position to prove Theorem 4.1.

Proof of Theorem 4.1.: Let α_n $(0\leq n\leq N-1)$ be real constants

https://doi.org/10.1017/S0027763000024673 Published online by Cambridge University Press

such that $\sum_{n=0}^{N-1} \alpha_n A^n \boldsymbol{b} = 0$. Since $A\boldsymbol{b} = -(0b_0 \cdots b_{N-2})^* + b_{N-1}\boldsymbol{a}$, we have

$$A^{N+n}\boldsymbol{b} = (-1)^{N-1} \sum_{m=0}^{N-1} (-1)^m b_m A^{m+n}\boldsymbol{a} \qquad (0 \leq n \leq N-1) .$$

Then operating the matrix A^N to both sides, we get

$$\left(\sum_{m=0}^{N-1} (-1)^m b_m A^m\right)\left(\sum_{n=0}^{N-1} \alpha_n A^n \boldsymbol{a}\right) = \sum_{n=0}^{N-1} \alpha_n A^{N+n}\boldsymbol{b} = 0 .$$

and so $\alpha_n = 0$ $(0 \leq n \leq N-1)$ by Lemmas 4.8 and 4.6. This completes the proof of Theorem 4.1. (Q.E.D.)

As an application of Lemma 4.4 we shall show the following

THEOREM 4.2. (i) *There exist N positive numbers $t_0 < t_1 < \cdots < t_{N-1}$ such that the matrix $(E^{(m)}(t_n))_{0 \leq n \leq N-1}$ is nonsingular.*

(ii) *In order that for any N positive numbers $t_0 < t_1 < \cdots < t_{N-1}$ the matrix $(E^{(m)}(t_n))_{0 \leq m, n \leq N-1}$ is nonsingular, it is a necessary and sufficient condition that the zero points of P are located in the positive imaginary axis.*

Proof. Differentiating (i) in Theorem 2.1 m times at $s = 0$, we have

$$E^{(m)}(t) = \sum_{n=0}^{N-1} (-1)^n F^{(n)}(t) E_n^{(m)}(0+) \qquad (t > 0) .$$

Therefore, combining Lemmas 2.5, 2.6 and 4.4, we obtain the result. (Q.E.D.)

Now we shall apply Theorem 4.1 to get several properties of the Gaussian Markov process $(\mathscr{X}(t), P(\cdot \mid \mathscr{X}(0) = x); t > 0, x \in \boldsymbol{R}^N)$. It is easy to see from (4.1) that the covariance matrices $R(t)$ $(t > 0)$ are positive definite. Therefore it follows from (4.1) that the Gaussian Markov process $(\mathscr{X}(t), P(\cdot \mid \mathscr{X}(0) = x); t > 0, x \in \boldsymbol{R}^N)$ has a transition probability density $P(t, x, y)$;

$$(4.2) \qquad \begin{cases} P(\mathscr{X}(t) \in dy \mid \mathscr{X}(0) = x) = P(t, x, y)dy , \\ P(t, x, y) = (2\pi)^{-N/2}(\det R(t))^{-1/2} e^{-1/2(y - e^{tA}x, R^{-1}(t)(y - e^{tA}x))} . \end{cases}$$

Since $\boldsymbol{b}$ is not zero, it follows from Theorem 3.1 that

$$(4.3) \quad \sigma(B(s) - B(t); s, t \in D) \subset \boldsymbol{F}_{\mathfrak{X}}(D) \qquad \text{for any open set } D \text{ in } \boldsymbol{R} .$$

Therefore, by (3.2), (3.8) and (4.3), we can apply K. Ito's formula to the stochastic differential equation in Theorem 3.1 and find that the Gaussian

Markov process $(\mathscr{X}(t), P(\cdot\,|\,\mathscr{X}(0) = x)\,;\, t > 0, x \in \boldsymbol{R}^N)$ becomes a diffusion process whose infinitesimal generator $\mathscr{G}_x$ is given by

(4.4) $$\mathscr{G}_x = \tfrac{1}{2}(\sqrt{2\pi}^{-1}\boldsymbol{b}\cdot\nabla)^2 + (Ax)\cdot\nabla\ .$$

From Theorem 4.1 we find that this differential operator $\mathscr{G}_x$ is hypoelliptic ([4]).

It is easy to see from (2.1) and (3.6) that the characteristic equation of the matrix A is equal to $(-1)^N c_N^{-1} P(i^{-1}\lambda)$:

(4.5) $$\det(\lambda - A) = (-1)^N c_N^{-1} P(i^{-1}\lambda) = (-1)^N \sum_{n=0}^{N} a_n(-\lambda)^n\ .$$

This particulary implies that the real part of all eigenvalues of A is negative. Noting this fact and applying Theorems 4.1, 6.1 and 7.1 in [2] to our Gaussian diffusion process, we have

THEOREM 4.3. *The Gaussian diffusion process* $(\mathscr{X}(t), P(\cdot\,|\,\mathscr{X}(0) = x)\,;$ $t > 0, x \in \boldsymbol{R}^N)$ *is recurrent and there uniquely exists an invariant measure* $\mu(dy)$:

(4.6) $$\begin{cases} \mu(dy) = \varphi(y)dy\ , \\ \varphi(y) = e^{-\frac{1}{2}(y, R^{-1}(\infty)y)}\ , \end{cases}$$

where $R^{-1}(\infty)$ *is the inverse matrix of a positive definite matrix* $R(\infty)$ $= \lim_{t\to\infty} R(t)$.

Remark 4.1. It follows from (4.1) that

(4.7) $$R(\infty) = \left((2\pi)^{-1}\int_0^\infty e^{tA}\boldsymbol{b}_m e^{tA}\boldsymbol{b}_n dt\right)_{0\le m,n\le N-1}\ .$$

§ 5. $F_X^{+/-}(t)$ (II)

We have constructed in § 3 an example $\mathscr{X}$ of N-dimensional stationary Gaussian processes $\mathscr{Y} = (\mathscr{Y}(t)\,;\, t \in \boldsymbol{R})$ satisfying the following conditions:

(5.1) $\mathscr{Y}(t)$ is continuous in the mean;

(5.2) For any $t \in \boldsymbol{R}$, each component of $\mathscr{Y}(t)$ belongs to $\boldsymbol{M}$ and {the n-th component of $\mathscr{Y}(t)\,;\, 1 \le n \le N$} is linearly independent;

(5.3) $F_X^{+/-}(t) = \sigma(\mathscr{Y}(t))$ for any $t \in \boldsymbol{R}$.

In this section we shall show the next theorem about the uniqueness of such a process.

https://doi.org/10.1017/S0027763000024673 Published online by Cambridge University Press

THEOREM 5.1. *For any N-dimensional stationary Gaussian process* $\mathscr{Y} = (\mathscr{Y}(t); t \in \boldsymbol{R})$ *satisfying* (5.1), (5.2) *and* (5.3), *there uniquely exists a constant nonsingular* $N \times N$*-matrix* T *such that* $\mathscr{Y}(t) = T\mathscr{X}(t)$ *for any* $t \in \boldsymbol{R}$.

Before proving this theorem, we shall prepare three lemmas. We define for any $t \in \boldsymbol{R}$ an $N \times N$-matrix $K_{\mathfrak{x}}(t)$ by

$$K_{\mathfrak{x}}(t) = E(\mathscr{X}(t)\mathscr{X}(0)^*) . \tag{5.4}$$

By Theorem 3.2 (ii) and Lemma 3.1 we have

LEMMA 5.1. (i) $K_{\mathfrak{x}}(0)$ *is symmetric and positive definite,*

(ii) $$K_{\mathfrak{x}}(t) = \begin{cases} e^{tA}K_{\mathfrak{x}}(0) & (t \geq 0) , \\ K_{\mathfrak{x}}(0)e^{-tA} & (t < 0) . \end{cases}$$

We define a symmetric $N \times N$-matrix B by

$$B = (b_m b_n)_{0 \leq m,n \leq N-1} . \tag{5.5}$$

Then we shall prove

LEMMA 5.2. $AK_{\mathfrak{x}}(0) + K_{\mathfrak{x}}(0)A^* = -(2\pi)^{-1}B.$

Proof. Since $\mathscr{X} = (\mathscr{X}(t); t \in \boldsymbol{R})$ is stationary, it follows from (3.2), (3.8) and Lemma 4.1 (ii) that

$$K_{\mathfrak{x}}(0) = e^{tA}K_{\mathfrak{x}}(0)e^{tA^*} + (2\pi)^{-1}\int_0^t e^{sA}Be^{sA^*}ds \qquad (t > 0) .$$

Differentiating it at $t = 0$, we obtain the result. (Q.E.D.)

Next we shall show the following general statement.

LEMMA 5.3. *Let* A, B *and* K *be real* $N \times N$*-matrices such that*

(i) $B = (b_m b_n)_{0 \leq m,n \leq N-1}$, $\boldsymbol{b} = (b_0 \cdots b_{N-1})^* \neq 0$,

(ii) K *is symmetric and positive definite,*

(iii) $AK + KA^* = -B$

and

(iv) $\{A^n \boldsymbol{b}; 0 \leq n \leq N-1\}$ *is linearly independent.*

If an $N \times N$*-matrix* $\tilde{A}$ *satisfies*

$$e^{t\tilde{A}}Ke^{t\tilde{A}^*} = e^{tA}Ke^{tA^*} \qquad \text{for any } t \in \boldsymbol{R} ,$$

then

$$\tilde{A} = A .$$

Proof. Since K has a symmetric and positive definite root $K^{\frac{1}{2}}$, we can define $A_1 = K^{-\frac{1}{2}}AK^{\frac{1}{2}}$, $\tilde{A}_1 = K^{-\frac{1}{2}}\tilde{A}K^{\frac{1}{2}}$ and $B_1 = K^{-\frac{1}{2}}BK^{-\frac{1}{2}}$. It then follows that

$$\begin{cases} A_1 + A_1^* = -B_1\,, \\ e^{t\tilde{A}_1}e^{t\tilde{A}_1^*} = e^{tA_1}e^{tA_1^*} \qquad \text{for any } t \in \boldsymbol{R}\,. \end{cases} \tag{5.6}$$

Since B_1 is a symmetric, nonnegative definite matrix of rank one, there exist an orthogonal matrix P_1 and a positive number ε such that $B_1 = P_1 \begin{bmatrix} \varepsilon_0 & & 0 \\ & \ddots & \\ 0 & & 0 \end{bmatrix} P_1^{-1}$ and so $\varepsilon^{-1}\sum_{n=0}^{N-1}(B_1)_{nn} = 1$. Therefore we can find another orthogonal matrix P_2 such that $(P_2)_{n0} = \sqrt{\varepsilon^{-1}}(K^{-\frac{1}{2}}\boldsymbol{b})_n$ $(0 \leq n \leq N-1)$, because $(B_1)_{nn} = (K^{-\frac{1}{2}}\boldsymbol{b})_n^2$. It is then easy to see that $P_2 \begin{bmatrix} \varepsilon_0 & & 0 \\ & \ddots & \\ 0 & & 0 \end{bmatrix} P_2^{-1} = B_1$. Hence, setting $A_2 = P_2^{-1}A_1P_2$, $\tilde{A}_2 = P_2^{-1}\tilde{A}_1P_2$ and $T = \begin{bmatrix} -\varepsilon_0 & & 0 \\ & \ddots & \\ 0 & & 0 \end{bmatrix}$, we see from (5.6) and Theorem 4.1 that

$$A_2 + A_2^* = T\,, \tag{5.7}$$

$$e^{t\tilde{A}_2}e^{t\tilde{A}_2^*} = e^{tA_2}e^{tA_2^*} \qquad \text{for any } t \in \boldsymbol{R} \tag{5.8}$$

and

$$\{((A_2^nT)_{00}, (A_2^nT)_{10}, \cdots, (A_2^nT)_{N-1\,0})^*\,;\, 0 \leq n \leq N-1\} \tag{5.9}$$

is linearly independent.

We define a sequence $(D_p)_{p=0}^{\infty}$ of $N \times N$-matrices by

$$D_p = A_2D_{p-1} + D_{p-1}A_2^* \qquad (p = 1, 2, \cdots),\ D_0 = I\,. \tag{5.10}$$

Since $D_1 = T$ by (5.7), we have

$$D_{p+1} = \sum_{k=0}^{p}\binom{p}{k}A_2^kTA_2^{*p-k} \qquad (p = 0, 1, 2, \cdots)\,. \tag{5.11}$$

Setting $L = \tilde{A}_2 - A_2$ and then differentiating (5.8) at $t = 0$, we get

$$LD_p + D_pL^* = 0 \qquad (p = 0, 1, 2, \cdots)\,. \tag{5.12}$$

Therefore, putting $S = [L, A_2]$ $(= LA_2 - A_2L)$, we see from (5.10) and (5.12) that

$$SD_p + D_pS^* = 0 \qquad (p = 0, 1, 2, \cdots) . \tag{5.13}$$

From (5.12) in the case of $p = 1$ we have

$$L + L^* = 0 . \tag{5.14}$$

Furthermore, applying (5.12) in the case of $p = 1$, we find that $[L, T]$ $= 0$. Therefore, since $T = \begin{bmatrix} -\varepsilon_0 & & 0 \\ & \ddots & \\ 0 & & 0 \end{bmatrix}$, we get

$$LT = TL = 0 . \tag{5.15}$$

Similarly it follows from (5.13) in the case of $p = 0$ and $p = 1$ that

$$S + S^* = 0 \tag{5.16}$$

and

$$ST = TS = 0 . \tag{5.17}$$

Fixing any $p_0 \in \{0, 1, 2, \cdots\}$ we shall assume that $SA_2^pT = TA_2^pS = 0$ for any $p \in \{0, \cdots, p_0\}$. By (5.7), (5.11), (5.13), (5.16) and (5.17), we find that $SA_2^{p_0+1}T = TA_2^{p_0+1}S$. Since $T = \begin{bmatrix} -\varepsilon_0 & & 0 \\ & \ddots & \\ 0 & & 0 \end{bmatrix}$, this implies that $(SA_2^{p_0+1})_{n0}$ $= 0$ for any $n \in \{1, 2, \cdots, N-1\}$. Moreover we see that $(SA_2^{p_0+1})_{00} = 0$ because S_{0n} for any $n \in \{0, 1, \cdots, N-1\}$ by (5.17). For this reason it follows that $SA_2^{p_0+1}T = TA_2^{p_0+1}S = 0$. By mathematical induction on p_0, we conclude that $SA_2^pT = 0$ for any $p \in \{0, 1, 2, \cdots\}$. Therefore, using (5.9), we find that $S = 0$. Since this conclusion implies that L commutes with A_2, it follows from (5.15) that $LA_2^pT = 0$ for any $p \in \{0, 1, \cdots\}$. Consequently, using (5.9) again, we see that $L = 0$ and so $\tilde{A} = A$. Now we complete the proof of Lemma 5.3. (Q.E.D.)

After these preparations, we are in a position to prove Theorem 5.1.

Proof of Theorem 5.1: Since the subspace of $\boldsymbol{M}$ whose elements are $F_X^{+/-}(t)$-measurable is equal to the space $\boldsymbol{M}^{+/-}(t)$ with the algebraic dimension N, it follows from (5.2) and (5.3) that there exists a non-singular $N \times N$-matrix $T(t)$ satisfying $\mathscr{Y}(t) = T(t)\mathscr{X}(t)$ $(t \in \boldsymbol{R})$. For any

https://doi.org/10.1017/S0027763000024673 Published online by Cambridge University Press

s and $t \in \boldsymbol{R}$, $s < t$, we define an $N \times N$-matrix $C(t, s)$ by

$$C(t, s) = T(t)e^{(t-s)A}T(s)^{-1} .$$

Then it follows from Lemma 3.1 and (5.2) that

$$(5.18) \qquad C(u, s) = C(u, t)C(t, s) \qquad (s < t < u)$$

and

$$(5.19) \qquad E(\mathscr{Y}(t) \mid \boldsymbol{F}_X^-(s)) = C(t, s)\mathscr{Y}(s) \qquad (s < t) .$$

Since $\mathscr{Y} = (\mathscr{Y}(t); t \in \boldsymbol{R})$ is stationary, we see from (5.2) and (5.19) that $C(t, s) = C(t - s, 0)$ $(s < t)$. Setting $C(t) = C(t, 0)$ $(t > 0)$, we can show from (5.1), (5.2) and (5.18) that $C(t)$ is continuous in $t \in [0, \infty)$, $C(0) = I$ and $C(s + t) = C(s)C(t)$ $(s, t \in [0, \infty))$. Therefore, there exists an $N \times N$-matrix $\tilde{A}$ such that $C(t) = e^{tT(0)\tilde{A}T(0)^{-1}}$ $(t \geq 0)$. Since it is easily seen that $T(t)$ is real analytic in $t \in \boldsymbol{R}$, we obtain

$$(5.20) \qquad T(t) = T(0)e^{t\tilde{A}}e^{-tA} \qquad \text{for any } t \in \boldsymbol{R} .$$

On the other hand, by Lemma 5.1 and (5.19), we have

$$C(t - s)T(0)K_{\mathfrak{X}}(0)T(0)^* = T(t)e^{(t-s)A}K_{\mathfrak{X}}(0)T(s)^* \qquad (s < t) .$$

Combining this with (5.20), we get

$$e^{t\tilde{A}}K_{\mathfrak{X}}(0)e^{t\tilde{A}^*} = e^{tA}K_{\mathfrak{X}}(0)e^{tA^*} \qquad (t \in \boldsymbol{R}) .$$

Therefore, by Theorem 4.1, Lemmas 5.1 (i) and 5.2, we can apply Lemma 5.3. to obtain the conclusion. (Q.E.D.)

EXAMPLE 6.1. Using N positive numbers t_n in Lemma 2.5, we define a nonsingular $N \times N$-matrix $T = ((-1)^n F^{(m)}(t_n))_{0 \leq m,n \leq N-1}$ and a stationary Gaussian process $\mathscr{Y} = (\mathscr{Y}(t); t \in \boldsymbol{R}) = (T\mathscr{X}(t); t \in \boldsymbol{R})$. It follows from Theorem 3.2 (i) that the $n + 1$-th component of $\mathscr{Y}(t)$ is equal to $E(X(t + t_n) \mid \boldsymbol{F}_X^-(t))$ $(t \in \boldsymbol{R}, 0 \leq n \leq N - 1)$.

§ 6. $\boldsymbol{F}_X^{+/-}(t)$ (III)

Using the L^2-function F in (2.2) and the Brownian motion B in (3.1), we define a real stationary Gaussian process $\boldsymbol{Y} = (Y(t); t \in \boldsymbol{R})$ such that

$$(6.1) \qquad Y(t) = \sqrt{2\pi}^{-1}\int_{-\infty}^{t} F(t - s)dB(s) \qquad (t \in \boldsymbol{R}) .$$

It is easy to see that this representation is canonical and $\boldsymbol{Y}$ has the N-ple Markovian property in the narrow sense. Since Q is a polynomial of at most degree $N-1$, we see from Lemma 2.1 (i), (2.6) and (3.1) that

$$X(t) = Q\left(\frac{1}{i}\frac{d}{dt}\right)Y(t) \qquad (t \in \boldsymbol{R}) . \tag{6.2}$$

Now we define an $N \times N$-matrix T by

$$T = (\boldsymbol{b}(-A)\boldsymbol{b} \cdots (-A)^{N-1}\boldsymbol{b}) , \tag{6.3}$$

which is nonsingular by virtue of Theorem 4.1. Since the characteristic polynomial of A is $(-1)^N c_N^{-1} P(i^{-1}\lambda)$, it follows from Caley-Hamilton's theorem that $\sum_{n=0}^N a_n(-A)^n = 0$ ((4.5)). Therefore we can easily see that

$$T^{-1}\boldsymbol{b} = (10 \cdots 0)^* \tag{6.4}$$

and

$$T^{-1}AT = A . \tag{6.5}$$

Using this matrix T we define an N-dimensional stationary Gaussian process $\mathscr{Y} = (\mathscr{Y}(t)\,;\, t \in \boldsymbol{R})$ satisfying (5.1), (5.2) and (5.3) as follows:

$$\mathscr{Y}(t) = T^{-1}\mathscr{X}(t) \qquad (t \in \boldsymbol{R}) . \tag{6.6}$$

We denote by $Y_n(t)$ the $n+1$-th component of $\mathscr{Y}(t)$ $(0 \le n \le N-1, t \in \boldsymbol{R})$. By (2.3), (3.3), (3.7), Lemma 3.1 (ii) and 4.1 (i), we can show that

$$\mathscr{Y}(t) = \sqrt{2\pi}^{-1} \int_{-\infty}^t e^{(t-s)A} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} dB(s) \qquad (t \in \boldsymbol{R}) \tag{6.7}$$

and

$$\left[e^{tA} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right]_n = F_n(t) \qquad (t > 0, 0 \le n \le N-1) . \tag{6.8}$$

By (2.4), we particularly find

https://doi.org/10.1017/S0027763000024673 Published online by Cambridge University Press

(6.9) $$Y_{N-1}(t) = (-2\pi)^{-1} c_N Y(t) \qquad (t \in \boldsymbol{R}) .$$

By (3.8) and (6.6) we note

(6.10) $$\boldsymbol{F}_X^-(t) = \boldsymbol{F}_{\mathscr{Y}}^-(t) .$$

Using Theorem 3.1, Lemmas 3.1 and 4.1 (ii), we see from (6.4) and (6.5) that

THEOREM 6.1. *For almost all* ω

(i) $\mathscr{Y}(t) - \mathscr{Y}(s) = \sqrt{2\pi}^{-1}(B(t) - B(s), 0, \cdots, 0)^* + \int_s^t A\mathscr{Y}(u)du \ (s < t),$

(ii) $\mathscr{Y}(t) = e^{(t-s)A}\mathscr{Y}(s) + \sqrt{2\pi}^{-1}\int_s^t e^{(t-s)A}\begin{bmatrix}1\\0\\ \vdots \\0\end{bmatrix} dB(u) \ (s < t),$

(iii) $E(\mathscr{Y}(t) \mid \boldsymbol{F}_X^-(s)) = e^{(t-s)A}\mathscr{Y}(s) \ (s < t).$

Noting (3.6) we can show from (6.6), (6.9) and Theorem 6.1 (i) that

(6.11) $\boldsymbol{F}_{\mathscr{X}}(D) = \boldsymbol{F}_{\mathscr{Y}}(D) = \boldsymbol{F}_Y(D)$ for any open set D in $\boldsymbol{R}$

and

(6.12) $$\boldsymbol{F}_Y^{+/-}(t) = \partial \boldsymbol{F}_Y(t) \qquad \text{for any } t \in \boldsymbol{R} .$$

Therefore, combining these with Theorem 3.3, we get

THEOREM 6.2.

$$\boldsymbol{F}_X^{+/-}(t) = \boldsymbol{F}_{\mathscr{Y}}^{+/-}(t) = \sigma(\mathscr{Y}(t)) = \boldsymbol{F}_Y^{+/-}(t) = \partial \boldsymbol{F}_Y(t) \qquad \textit{for any } t \in \boldsymbol{R} .$$

Finally we shall give an alternative expression of the linear predictor by using the process $\mathscr{Y}$.

THEOREM 6.3. (i) *For any* s *and* $t \in \boldsymbol{R}$, $s < t$,

$$E(X(t) \mid \boldsymbol{F}_X^-(s)) = \sum_{n=0}^{N-1} (-1)^n E^{(n)}(t-s) Y_n(s) .$$

(ii) $\{Y_n(t); 0 \leq n \leq N-1\}$ *is linearly independent in* $\boldsymbol{M}$ *for any* $t \in \boldsymbol{R}$.

Proof. By Theorem 3.2 (i) and (6.6) we have (ii). It follows from Theorem 2.1 (i) and Lemma 4.1 (i) that

$$E(t-s) = \sum_{\ell=0}^{N-1} (-1)^\ell F^{(\ell)}(t)(e^{-sA}\boldsymbol{b})_\ell \qquad (s < 0, t > 0) .$$

Differentiating both sides n times at $s=0$, we get

$$E(t)=\sum_{\ell=0}^{N-1}(-1)^{\ell}F^{(\ell)}(t)(A^{n}\boldsymbol{b})_{\ell}\qquad(0\leq n\leq N-1)\ .$$

Therefore, by Theorem 3.2 (i) and (6.6), we obtain (i). (Q.E.D.)

§ 7. Applications

7.1. Markovian property.

At first we shall characterize the Markovian property of stationary Gaussian processes from the point of view of representations. In [6] we have proved

THEOREM 7.1. ([6]) *In order that a real mean continuous, purely nondeterministic stationary Gaussian process* $\boldsymbol{X}$ *has the Markovian property:*

(7.1) $$\boldsymbol{F}_{\boldsymbol{X}}^{+/-}(t)=\partial\boldsymbol{F}_{\boldsymbol{X}}(t)\qquad \text{for any } t\in\boldsymbol{R}\ ,$$

it is a necessary and sufficient condition that there exists a canonical representation $(\sqrt{2\pi}^{-1}E(t),B(t))$ *possesing*

(7.2) $$\sigma(B(s)-B(t)\,;\,s,t\in D)\subset\boldsymbol{F}_{\boldsymbol{X}}(D)\qquad \text{for any open set } D \text{ in } \boldsymbol{R}\ .$$

We shall give another proof of Theorem 7.1 in case $\boldsymbol{X}$ has a rational spectral density Δ of the form (2.1). Now let's assume (7.2). It then follows from (3.5), (3.6) and Theorem 3.1 that $\mathscr{X}(t)$ is $\partial\boldsymbol{F}_{\boldsymbol{X}}(t)$-measurable for any $t\in\boldsymbol{R}$. Therefore, by Theorem 3.2 (i), we find that $E(X(u)\,|\,\boldsymbol{F}_{\boldsymbol{X}}^{-}(t))$ is $\partial\boldsymbol{F}_{\boldsymbol{X}}(t)$-measurable $(t<u)$ and so that (7.1) holds. Conversely let's assume (7.1). It then follows from Lemma 2.5 and Theorem 3.2 (i) that $\mathscr{X}(t)$ is $\partial\boldsymbol{F}_{\boldsymbol{X}}(t)$-measurable for any $t\in\boldsymbol{R}$. Therefore, by (3.6) and Theorem 3.1, we obtain (7.2) since $\boldsymbol{b}$ is not zero. (Q.E.D.)

Next we shall characterize the N-ple Markovian property in the sense of T. Hida ([3]). Immediately from Lemma 2.6 and Theorem 3.2 (i) we can show

THEOREM 7.2. *In order that a real mean continuous, purely nondeterministic stationary Gaussian process* $\boldsymbol{X}$ *has the N-ple Markovian property in the sense of T. Hida, it is a necessary and sufficient condition that* $\boldsymbol{X}$ *has a rational spectral density* Δ *of the form* (2.1) *with an additional property*

https://doi.org/10.1017/S0027763000024673 Published online by Cambridge University Press

(7.3) $$V_p^r \subset \{z \in \boldsymbol{C}^+ ; \operatorname{Re} z = 0\} .$$

7.2. **Initial value problem.**

We shall characterize the linear predictor using the past as a unique solution of an initial value problem. We define an $N \times N$-matrix $D = (D_{mn})_{0 \leq m,n \leq N-1}$ by

(7.4) $$D_{mn} = (-1)^n E^{(m+n)}(0+) ,$$

which is nonsingular by Lemma 4.5.

THEOREM 7.3. *We denote by $Z(t, \omega)$ the linear predictor of $X(t)$ using the whole past;*

$$Z(t, \omega) = E(X(t) \,|\, \boldsymbol{F}_X^-(0)) \qquad (t > 0) .$$

Then, for almost all $\omega \in \Omega$, $Z(t, \omega)$ $(t > 0)$ is a unique solution of the following initial value problem (7.5):

(7.5) $$\begin{cases} Z(\cdot, \omega) \in \mathscr{A}((0, \infty)) \cap L^2((0, \infty)) , \\ P\left(\dfrac{1}{i} \dfrac{d}{dt}\right) Z(t, \omega) = 0 \qquad \text{in } (0, \infty) , \\ Z^{(n)}(0+, \omega) = (D\mathscr{Y}(0))_n \qquad (0 \leq n \leq N-1) . \end{cases}$$

Proof. Since $F^{(n)} \in \mathscr{A}((0, \infty)) \cap L^2((0, \infty))$ $(n = 0, 1, 2, \cdots)$ and $P\left(\frac{1}{i} \frac{d}{dt}\right) F = 0$ in $(0, \infty)$, it follows from Theorem 2.1 (i) that $E^{(n)} \in \mathscr{A}((0, \infty)) \cap L^2((0, \infty))$ and $P\left(\frac{1}{i} \frac{d}{dt}\right) E^{(n)} = 0$ in $(0, \infty)$ $(n = 0, 1, 2, \cdots)$. Therefore, by Theorem 6.3 (i), we have (7.5). It is clear that $Z(\cdot, \omega)$ is a unique solution of (7.5), because P is a polynomial of degree N.

(Q.E.D.)

Remark 7.1. By Theorem 6.3 (ii) we note that $\{(D\mathscr{Y}(0))_n ; 0 \leq n \leq N-1\}$ is linearly independent.

7.3. **Nonlinear prediction.**

As the last application, we shall give an expression of nonlinear predictors of $X(t)$ using the past $\boldsymbol{F}_X^-(0)$ in terms of the transition probability density $P(t, x, y)$ of the Gaussian diffusion process $(\mathscr{X}(t), P(\cdot \,|\, \mathscr{X}(0) = x) ; t > 0, x \in \boldsymbol{R}^N)$. Immediately from (3.5), Theorem 3.3 and (4,2) we have

https://doi.org/10.1017/S0027763000024673 Published online by Cambridge University Press

THEOREM 7.4. *For any bounded measurable function f (or any polynomial) on $\boldsymbol{R}$ and any $t > 0$,*

$$E(f(X(t))\,|\,F_X^-(0)) = \int_{R^N} f(-2\pi c_N^{-1} y_{N-1}) P(t, \mathscr{X}(0), y) dy_0 \cdots dy_{N-1}\,.$$

REFERENCES

[1] J. L. Doob: The elementary Gaussian processes, Ann. Math. Statist. **15** (1944), 229–282.

[2] H. Dym: Stationary measures for the flow of a linear differential equation driven by white noise, Trans. Amer. Math. Soc. **123** (1966), 130–164.

[3] T. Hida: Canonical representations of Gaussian processes and their applications, Mem. Coll. Sci. Univ. Kyoto, Ser. A, Math. **34** (1960), 109–155.

[4] L. Hörmander: Hypoelliptic second order differential equations, Acta Math. **119** (1967), 141–171.

[5] N. Levinson and H. P. McKean, Jr.: Weighted trigonometrical approximation on $\boldsymbol{R}^1$ with application to the germ field of a stationary Gaussian noise, Acta Math. **112** (1964), 99–143.

[6] Y. Okabe: Stationary Gaussian processes with Markovian property and M. Sato's hyperfunctions, to appear in Japanese J. of Math., **41** (1973), 69–122.

Nagoya University

https://doi.org/10.1017/S0027763000024673 Published online by Cambridge University Press