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OSCILLATION THEOREMS FOR NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS OF EVEN ORDER

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Consider the differential equation

$$y^{(n)}+f(t, y)=0, \tag{1}$$

where n is even and $f(t, y)$ is subject to the following conditions:

(2)
(a) $f(t, y)$ is continuous on $[0, \infty)\times R$;
(b) $f(t, y)$ is nondecreasing in y for each fixed $t\in[0, \infty)$;
(c) $yf(t, y)>0$ for $y\neq 0$ and $t\in[0, \infty)$.

By a *proper solution* of equation (1) we mean a function $y:[T_y, \infty)\to R$ which satisfies (1) on $[T_y, \infty)$ and $\sup\{|y(t)| : t\geq T\}>0$ for any $T\geq T_y$. For second order equations the question of existence of proper solutions has been addressed by Coffman and Wong [5] and in references contained therein. The corresponding question for nth order equations does not seem to have been extensively studied (we know only of the work of Burton [3] and Kiguradze [10]). Accordingly the difficulties arising from the non-existence of proper solutions are not addressed here, but a standing hypothesis is that equation (1) does possess proper solutions. A proper solution of (1) is called *oscillatory* if it has arbitrarily large zeros, and it is called *nonoscillatory* otherwise. For brevity we say that *equation* (1) *is oscillatory* if every proper solution of (1) is oscillatory.

We are interested in the problem of characterizing the oscillation of equation (1), that is, obtaining necessary and sufficient conditions for (1) to be oscillatory in the sense defined above. This problem was first studied in the fundamental paper of Atkinson [1], which was followed by Belohorec [2]. Their results are summarized in the following theorem.

THEOREM 0. *Consider the equation*

$$y''+p(t)\,|y|^{\gamma}\operatorname{sgn} y=0, \tag{3}$$

where $\gamma>0$ *and* $p(t)$ *is positive and continuous on* $[0, \infty)$.

(i) (Atkinson) *Let* $\gamma>1$. *Then equation* (3) *is oscillatory if and only if*

$$\int^{\infty} tp(t)\,dt=\infty.$$

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(ii) (Belohorec) *Let* $0<\gamma<1$. *Then equation* (3) *is oscillatory if and only if*

$$\int^{\infty} t^{\gamma} p(t)\, dt = \infty.$$

Extensions of Theorem 0 to equation (1) have been undertaken by numerous authors; see, for example, Coffman and Wong [4, 5], Izjumova [6], Kiguradze [7, 8, 9], Ličko and Švec [12], Macki and Wong [13], Onose [14], and Ryder and Wend [15]. Such extensions are based on the introduction of an appropriate class of *superlinear* [resp. *sublinear*] functions $f(t, y)$ including $p(t)|y|^{\gamma} \operatorname{sgn} y$ with $\gamma>1$ [resp. $0<\gamma<1$] as a particular case.

The purpose of this paper is to give new definitions of superlinearity and sublinearity of $f(t, y)$ which unify the corresponding definitions found in the literature, and to obtain a characterization for oscillation of equation (1) with such super- and sublinearity so as to cover all the known extensions of Theorem 0 to even order equations of the form (1).

Definition 1. The function $f(t, y)$ is called *superlinear* if

$$\text{(4)} \qquad \int^{\infty} \frac{f(\varphi(u), c)}{f(\varphi(u), u)}\, du<\infty \quad \text{and} \quad \int^{\infty} \frac{f(\varphi(u), -c)}{f(\varphi(u), -u)}\, du<\infty$$

for some constant $c>0$ and every strictly increasing function $\varphi(u) \in C[(0, \infty), (0, \infty)]$ such that $\varphi(u) \uparrow \infty$ as $u \uparrow \infty$.

Definition 2. The function $f(t, y)$ is called *sublinear* if

$$\text{(5)} \qquad \int_{+0} \frac{f(\psi(u), c\chi(u))}{f(\psi(u), u\chi(u))}\, du<\infty \quad \text{and} \quad \int_{+0} \frac{f(\psi(u), -c\chi(u))}{f(\psi(u), -u\chi(u))}\, du<\infty$$

for some constant $c>0$ and every pair of strictly decreasing functions $\psi(u)$, $\chi(u) \in C[(0, \delta), (0, \infty)]$, $\delta>0$, such that $\psi(u) \uparrow \infty$ and $\chi(u) \uparrow \infty$ as $u \downarrow 0$.

We now state and prove our oscillation theorems.

Theorem 1. *Let $f(t, y)$ be superlinear. Then equation* (1) *is oscillatory if and only if*

$$\text{(6)} \qquad \int^{\infty} t^{n-1} |f(t, c)|\, dt = \infty \quad \textit{for every} \quad c \neq 0.$$

Proof. Let $y(t)$ be a nonoscillatory solution of (1). We may suppose that $y(t)$ is eventually positive, since a parallel argument holds if $y(t)$ is eventually negative. Then there exists a $T>0$ such that

$$\text{(7)} \qquad y(t) \geq \int_{T}^{t} \frac{(s-T)^{n-1}}{(n-1)!} f(s, y(s))\, ds+\frac{(t-T)^{n-1}}{(n-1)!} \int_{t}^{\infty} f(s, y(s))\, ds$$

for $t \geq T$. For the proof see, for example, Kusano and Onose [11]. In particular

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we have

$$y(t) \geq \int_T^t \frac{(s-T)^{n-1}}{(n-1)!} f(s, y(s)\, ds \stackrel{\text{def}}{=} \Phi(t) \tag{8}$$

for $t \geq T$. Let $u = \Phi(t)$. Since

$$\frac{du}{dt} = \frac{(t-T)^{n-1}}{(n-1)!} f(t, y(t)) > 0, \qquad t > T,$$

$u = \Phi(t)$ has the inverse function, which we denote by $t = \varphi(u)$. It is clear that $\varphi(u) \uparrow \infty$ as $u \uparrow \infty$. Noting that $f(t, y(t)) \geq f(t, \Phi(t))$ by 2(b) and (8), we have

$$\begin{aligned} \int_{t_1}^{t_2} \frac{(t-T)^{n-1}}{(n-1)!} f(t, c)\, dt &\leq \int_{t_1}^{t_2} \frac{(t-T)^{n-1}}{(n-1)!} f(t, y(t)) \frac{f(t, c)}{f(t, \Phi(t))}\, dt \\ &= \int_{\Phi(t_1)}^{\Phi(t_2)} \frac{f(\varphi(u), c)}{f(\varphi(u), u)}\, du \end{aligned} \tag{9}$$

for any $t_2 > t_1 > T$ and some $c > 0$. Letting $t_2 \to \infty$ in (9) and using (4), we have for some $c > 0$

$$\int_{t_1}^{\infty} \frac{(t-T)^{n-1}}{(n-1)!} f(t, c)\, dt \leq \int_{\Phi(t_1)}^{\infty} \frac{f(\varphi(u), c)}{f(\varphi(u), u)}\, du < \infty,$$

which contradicts (6).

Conversely, if (6) is not satisfied for some $c \neq 0$, then we can solve the integral equation

$$y(t) = \frac{c}{2} + \int_t^{\infty} \frac{(s-t)^{n-1}}{(n-1)!} f(s, y(s))\, ds \tag{10}$$

by the method of successive approximations or via the Schauder–Tychonoff fixed point theorem. The solution of (10) is clearly a nonoscillatory solution of equation (1). This completes the proof of Theorem 1.

THEOREM 2. *Let $f(t, y)$ be sublinear. Then equation* (1) *is oscillatory if and only if*

$$\int^{\infty} |f(t, ct^{n-1})|\, dt = \infty \quad \textit{for every} \quad c \neq 0. \tag{11}$$

Proof. Let $y(t)$ be a nonoscillatory solution of (1), which may be supposed positive without loss of generality. Then there exists a $T > 0$ such that (8) holds for $t \geq T$. Hence we have

$$y(t) \geq \frac{(t-T)^{n-1}}{(n-1)!} \int_t^{\infty} f(s, y(s))\, ds \geq \frac{t^{n-1}}{n!} \int_t^{\infty} f(s, y(s))\, ds \tag{12}$$

for $t \geq T'$, provided $T' > T$ is sufficiently large. Put

$$u = \Psi(t) = \frac{1}{n!} \int_t^\infty f(s, y(s))\, ds$$

and denote its inverse function by $t = \Psi(u)$. Since we must have $y^{(n-1)}(t) \downarrow 0$ as $t \uparrow \infty$, it is obvious that $\psi(u) \uparrow \infty$ as $u \downarrow 0$ and $du/dt = -f(t, y(t))/n!$. In view of 2(b) and (12), we have $f(t, y(t)) \geq f(t, t^{n-1}\psi(t))$ for $t \geq T'$, and

$$\begin{aligned}
\frac{1}{n!}\int_{t_1}^{t_2} f(t, ct^{n-1})\, dt &\leq \int_{t_1}^{t_2} \frac{f(t, y(t))}{n!} \frac{f(t, ct^{n-1})}{f(t, t^{n-1}\Psi(t))}\, dt \\
&= \int_{\Psi(t_2)}^{\Psi(t_1)} \frac{f(\psi(u), c\psi^{n-1}(u))}{f(\psi(u), u\psi^{n-1}(u))}\, du
\end{aligned} \tag{13}$$

for any $t_2 > t_1 > T'$ and some $c > 0$. Passing to the limit as $t_2 \to \infty$ and using (5), we obtain from (13) that for some $c > 0$

$$\int_{t_1}^\infty f(t, ct^{n-1})\, dt \leq n! \int_{+0}^{\Psi(t_1)} \frac{f(\psi(u), c\psi^{n-1}(u))}{f(\psi(u), u\psi^{n-1}(u))}\, du < \infty,$$

which contradicts (11).

If (11) is violated for some $c > 0$ [resp. $c < 0$], then equation (1) has a nonoscillatory solution $y(t)$ satisfying $\lim_{t\to\infty}[y(t)/t^{n-1}] = \text{const} > 0$ [resp. < 0] which is obtained as a solution to the integral equation

$$y(t) = \frac{c}{2} t^{n-1} + \int_T^t \frac{(t-s)^{n-2}}{(n-2)!} \int_s^\infty f(u, y(u))\, du\, ds,$$

where $T > 0$ is chosen suitably large. See also Kiguradze [6]. Thus the proof of Theorem 2 is complete.

By way of examples, we note that $f(t, y) = p(t, y)y(\log y^2)^{1+\varepsilon}$ satisfies our definition of superlinearity whenever $\varepsilon > 0$ and $p(t, y)$ is continuous on $[0, \infty] \times R$, nondecreasing in y, and satisfying $yp(t, y) > 0$ for $y \neq 0$. For then $f(t, y)$ satisfies 2(a)–(c) and there exists a constant k such that

$$\frac{f(\varphi(u), c)}{f(\varphi(u), u)} \leq \frac{k}{u} (\log u^2)^{-1-\varepsilon}$$

for $u \geq c$ and

$$\frac{f(\varphi(u), -c)}{f(\varphi(u), -u)} \leq \frac{k}{u} (\log u^2)^{-1-\varepsilon}$$

for $-u \leq -c$. In this case our generalization of Theorem 0 asserts that (1) is oscillatory if and only if $\int^\infty t^{n-1}p(t, c)\, dt = \infty$ for every $c \neq 0$. Our definition of sublinearity also contains $f(t, y) = p(t)\, |y|^\gamma \operatorname{sgn} y$ whenever $0 < \gamma < 1$ and $p(t)$ is continuous and positive in R. For then we have

$$\frac{f(\varphi(u), c\chi(u))}{f(\varphi(u), u\chi(u))} = \frac{f(\varphi(u), -c\chi(u))}{f(\varphi(u), -u\chi(u))} = \left(\frac{c}{u}\right)^{\gamma}$$

so that (5) is satisfied.

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