Modular subgroups, *dessins d'enfants* and elliptic K3 surfaces

Yang-Hui He, John McKay and James Read

ABSTRACT

We consider the 33 conjugacy classes of genus zero, torsion-free modular subgroups, computing ramification data and Grothendieck's *dessins d'enfants*. In the particular case of the index 36 subgroups, the corresponding Calabi–Yau threefolds are identified, in analogy with the index 24 cases being associated to K3 surfaces. In a parallel vein, we study the 112 semi-stable elliptic fibrations over $\mathbb{P}^1$ as extremal K3 surfaces with six singular fibres. In each case, a representative of the corresponding class of subgroups is identified by specifying a generating set for that representative.

1. *Introduction*

The congruence subgroups of the modular group $\Gamma = \mathrm{PSL}(2;\mathbb{Z})$ play a key role in many branches of modern mathematics. Whereas the upper half of the complex plane $\mathcal{H}$, when adjoining cusps on the real line, can quotient Γ to give a Riemann sphere, it is an interesting question as for which subgroups a similar sphere arises. Such a subgroup is called genus zero. Inspired by Moonshine, the authors of [11] looked into the classification problem of congruence, genus zero, torsion-free subgroups of the modular group and found these to be rather rare: there are only 33 conjugacy classes of these subgroups, of finite index $\mu \in \{6, 12, 24, 36, 48, 60\}$.

Therefrom emerge many remarkable connections to geometry, graph theory, number theory and physics [8, 17, 20]. First, we naturally extend the modular group action to a so-called modular surface for each of the groups whose Euler number is precisely μ. These turn out to be elliptically fibred over the genus zero $\mathbb{P}^1$ formed by quotienting $\mathcal{H}$ by the group and are, in fact, semi-stable in the sense that all singular fibres are of Kodaira type I_n. Moreover, there are exactly $h = \mu/6 + 2$ such singular fibres, and the tuple $\{n_1, \ldots, n_h\}$ is called the cusp widths.

Next, one can construct the Schreier coset graphs for these groups because they have finite index. They turn out to be finite trivalent graphs of non-trivial topology. We can interpret them as clean cubic *dessin d'enfants* in the sense of Grothendieck. Focus has been made on the index $\mu = 24$ cases because they are K3 surfaces [8] and on the $\mu = 12$ cases because they have interesting associated modular forms [17]. One can check that the Belyi rational maps for the *dessins* are precisely the elliptic j-invariants of the Weierstraß models for the modular surfaces with base $\mathbb{P}^1$ and we will complete this story in the present paper.

In a slightly different but obviously related direction, the classification of extremal, semi-elliptic K3 surfaces with six singular fibres was wonderfully performed by Miranda and Persson in [12], and the associated *dessins* nicely compiled in [2, 3]. It is therefore a natural question whether they can be affiliated with subgroups of the modular group, not necessarily congruence. Looking at these *dessins*, one can see them to be trivalent and clean, and since these can be

interpreted as the Schreier coset graphs of a subgroup of the modular group, the answer must be affirmative. We shall undertake this task, using the *dessins* to find the generating set of a representative of the corresponding conjugacy class of subgroups in each case.

The organization of the paper is as follows. In §2, we introduce the 33 conjugacy classes of congruence, torsion-free, genus zero subgroups of the modular group, especially their j-invariants, *dessins* and Schreier coset graphs. We will see that as the index 24 cases are attached to K3 surfaces, so too can the index 36 cases be associated to Calabi–Yau threefolds. Then, in §3, we turn to the 112 extremal semi-stable K3 surfaces, studying their *dessins* and interpreting them as coset graphs of some conjugacy class of index 24 modular subgroups. Via a discussion of the elliptic j-invariants as Belyi maps and the so-called cartographic groups, we find the generating set for a representative of the class of subgroups in each case. Finally, we conclude with some outlook and prospects.

2. Torsion-free, genus zero congruence subgroups

In this section, we first recall some standard notation regarding the modular group Γ and its congruence subgroups. We focus specifically on the conjugacy classes of the *torsion-free, genus zero, congruence subgroups*; there are 33 such classes [11]. In each case, we will compute the ramification data, and draw the dessin d'enfant which encodes this data. The cases of index 24 and 36 are particularly interesting: the former can be associated to K3 surfaces and we will demonstrate, using the j-invariants, that each of the index 36 subgroups can be identified with a specific Calabi–Yau threefold.

2.1. The modular group and congruence subgroups

First, the central object of our study is the *modular group*

$$\Gamma \equiv \Gamma(1) = \mathrm{PSL}(2; \mathbb{Z}) = \mathrm{SL}(2; \mathbb{Z})/\{\pm I\}. \tag{2.1}$$

This is the group of linear fractional transformations $z \to (az + b)/(cz + d)$, with $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$. It is generated by the transformations T and S defined by

$$T\tau = \tau + 1, \quad S\tau = -1/\tau. \tag{2.2}$$

One presentation of Γ is $\langle S, T | S^2 = (ST)^3 = I \rangle$, and we will later discuss the presentations of certain modular subgroups. The 2×2 matrices for S and T are as follows:

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{2.3}$$

Letting $x = S$ and $y = ST$ denote the elements of order 2 and 3 respectively, we see that Γ is the free product of the cyclic groups $C_2 = \langle x | x^2 = I \rangle$ and $C_3 = \langle y | y^3 = I \rangle$. It follows that 2×2 matrices for x and y are

$$x = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}. \tag{2.4}$$

Cayley graphs. Given the free product structure of Γ, we see that its Cayley graph is an infinite free trivalent tree, but with each node replaced by an oriented triangle. Now, for any *finite index* subgroup H of Γ, we can quotient the Cayley graph to arrive at a finite cubic graph by associating nodes to right cosets and edges between cosets which are related by action of a group element. In other words, this graph encodes the permutation representation of Γ acting on the right cosets of H. This is called a *Schreier coset graph* and will occupy a central role in our ensuing discussions.

Congruence subgroups. The most important subgroups of Γ are the so-called congruence subgroups, defined by having the entries in the generating matrices S and T obeying some modular arithmetic. Some conjugacy classes of congruence subgroups of particular note are as follows.

– Principal congruence subgroups:

$$\Gamma(m) := \{A \in \mathrm{SL}(2;\mathbb{Z}), A \equiv \pm I \bmod m\}/\{\pm I\}.$$

– Congruence subgroups of level m: subgroups of Γ containing $\Gamma(m)$ but not any $\Gamma(n)$ for $n < m$.

– Unipotent matrices:

$$\Gamma_1(m) := \left\{A \in \mathrm{SL}(2;\mathbb{Z}), A \equiv \pm \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \bmod m\right\}/\{\pm I\}.$$

– Upper triangular matrices:

$$\Gamma_0(m) := \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma, c \equiv 0 \bmod m\right\}/\{\pm I\}.$$

In [**8**], our attention is drawn to the conjugacy classes of a particular family of subgroups of Γ: the so-called *genus zero, torsion-free* congruence subgroups. By *torsion-free* we mean that the subgroup contains no element of finite order other than the identity. To explain *genus zero*, first recall that the modular group acts on the upper half-plane $\mathcal{H} := \{\tau \in \mathbb{C}, \mathrm{Im}(\tau) > 0\}$ by linear fractional transformations $z \to (az + b)/(cz + d)$. Here $\mathcal{H}$ gives rise to a compactification $\mathcal{H}^*$ when adjoining *cusps*, which are points on $\mathbb{R} \cup \infty$ fixed under some parabolic element (i.e. an element $A \in \Gamma$ not equal to the identity and for which $\mathrm{Tr}(A) = 2$). The quotient $\mathcal{H}^*/\Gamma$ is a compact Riemann surface of genus zero, i.e. a sphere. It turns out that with the addition of appropriate cusp points, the extended upper half plane $\mathcal{H}^*$ factored by various congruence subgroups will also be compact Riemann surfaces, possibly of higher genus. Such a Riemann surface, as a complex algebraic variety, is called a *modular curve*. The genus of a subgroup of the modular group is the genus of the modular curve produced in this way.

Using the Riemann–Hurwitz theorem, the genus of the modular curve can be shown to be

$$g_C = 1 + \frac{\mu}{12} - \frac{\nu_2}{4} - \frac{\nu_3}{3} - \frac{\nu_\infty}{2}, \tag{2.5}$$

where μ is the index of the subgroup, ν_2, ν_3 are the number of inequivalent elliptic points of order 2 and 3, respectively, and ν_∞ is the number of inequivalent cusp points.

Our focus will be on genus zero, torsion-free subgroups, thus we have that

$$g_C = \nu_2 = \nu_3 = 0, \tag{2.6}$$

whence, by (2.5), we have a further constraint that $\mu = 6\nu_\infty - 12$, so that the index of all our subgroups of concern is a multiple of 6. This means that the index of the subgroup in question also constrains ν_∞, so that for a subgroup of index $6n$, we have $\nu_\infty = n + 2$.

2.1.1. *The 33 conjugacy classes of subgroups.* As classified in [**11**], the conjugacy classes of the genus zero, torsion-free congruence subgroups of the modular group are very rare. There are only 33 of them:

– $\Gamma(n)$ with $n = 2, 3, 4, 5$;
– $\Gamma_0(n)$ with $n = 4, 6, 8, 9, 12, 16, 18$;
– $\Gamma_1(n)$ with $n = 5, 7, 8, 9, 10, 12$;
– the intersections $\Gamma_0(a) \cap \Gamma(b)$ for $\{a, b\} = \{4, 2\}, \{3, 2\}, \{8, 2\}, \{2, 3\}, \{25, 5\}$; $\Gamma_1(8) \cap \Gamma(2)$ and $\Gamma_0(16) \cap \Gamma_1(8)$;

– the congruence subgroups

$$\Gamma\left(m;\frac{m}{d},\epsilon,\chi\right) := \left\{\pm\begin{pmatrix} 1+\dfrac{m}{\epsilon\chi}\alpha & d\beta \\ \dfrac{m}{\chi}\gamma & 1+\dfrac{m}{\epsilon\chi}\delta \end{pmatrix}, \gamma \equiv \alpha \bmod \chi\right\}$$

with $(m, d, \epsilon, \chi) = (8, 2, 1, 2)$, $(12, 2, 1, 2)$, $(16, 1, 2, 2)$, $(27, 1, 3, 3)$, $(8, 4, 1, 2)$, $(9, 3, 1, 3)$, $(16, 2, 2, 2)$, $(24, 1, 2, 2)$, $(32, 1, 4, 2)$.

For reference, in Table 1 we organize these 33 conjugacy classes of subgroups by index, and we will briefly explain the notation of the ramification data.

2.2. *Modular surfaces, ramification data and j-functions*

As detailed in [8], to each torsion-free, genus zero modular subgroup corresponds a distinct *Shioda modular surface* [18]. We briefly recall how this is constructed [20]. First, we extend the action of any subgroup G of Γ on $\mathcal{H}$ to an action

$$\mathcal{H} \times \mathbb{C} \ni (\tau, z) \longrightarrow \left(\gamma\tau, \frac{z + m\tau + n}{c\tau + d}\right), \tag{2.7}$$

for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ and $(m, n) \in \mathbb{Z}^2$. Next, the quotient of $\mathcal{H} \times \mathbb{C}$ by the above automorphism defines a surface equipped with a morphism to the modular curve C arising from the quotient of $\mathcal{H}$ by $\tau \to \gamma\tau$. The fiber over the image of this morphism to the modular curve is generically an elliptic curve corresponding to the lattice $\mathbb{Z} \oplus \mathbb{Z}\tau$. What we have therefore is a complex surface which is an elliptic fibration over the modular curve associated to G. This is the modular surface.

In our present case, because the base modular curves are genus zero (being the Riemann sphere $\mathbb{P}^1_C$), our modular surfaces will be elliptic fibrations over $\mathbb{P}^1_C$. It turns out the Euler number of this surface is the index μ of the group H. Hence, the astute reader will recognize that the index 24 groups should give us K3 surfaces, in particular elliptic K3 surfaces (we remark that $\Gamma_1(7)$, in particular, was realized by Tate to be related to a K3 surface). We will return to this point later, especially to the observation that index 36 groups, by analogy, give Calabi–Yau threefolds.

Furthermore, the elliptic modular surface is *semi-stable* in the sense that all of the singular fibres are of Kodaira type I_n; the set of integers $(n_1, n_2, \ldots, n_k)$ denoting the indices of the singular fibres are the *cusp widths*. For simplicity, we arrange this set in increasing order and group repeated entries by the short-hand $n_i^{a_i}$ with a_i being the multiplicity. Now, we are ready to introduce the concept of *ramification data* associated with a subgroup (and its modular surface); this is the following triplet of integer vectors:

$$\left\{\begin{matrix} 3^V \\ 2^E \\ n_1^{a_1}, n_2^{a_2}, \ldots, n_k^{a_k} \end{matrix}\right\}, \qquad \begin{matrix} V = \text{number of nodes in the Schreier coset graph} \\ E = \text{number of edges} \\ \{n_i^{a_i}\} = \text{cusp widths of the elliptic modular surface.} \end{matrix} \tag{2.8}$$

Note that the '3' reminds us that all nodes are trivalent and the '2' that each edge links two nodes. We have marked this ramification data in Table 1.

Indeed, given an elliptic fibration, we can write down a Weierstraß equation, from which we can then extract a (Klein) j-function. Owing to the fibration over $\mathbb{P}^1_C$, the j-function will explicitly depend on the projective coordinate of this base Riemann sphere and can thus be treated as a rational map from the base $\mathbb{P}^1_C$ to its range $\mathbb{P}^1$. All of these j-functions are given in [11] and certain simplifications are given in [8, 20]. Generalizing [8, 12], we find the nice fact in the following proposition.

PROPOSITION 2.1. *The j-function $j(t \in \mathbb{P}^1_C) \mapsto \mathbb{P}^1$ is ramified at only $(0, 1, \infty)$.*

In other words, the only points $t \in \mathbb{P}_C^1$ such that $(d/dt)j(t)|_{\tilde{t}} = 0$ are such that $j(\tilde{t}) = 0$, 1 or ∞. Moreover, the order of vanishing of the Taylor series at t is the ramification index

TABLE 1. *The 33 conjugacy classes of genus zero, torsion-free, congruence subgroups of the modular group, organized by index (which can be* 6, 12, 24, 36, 48 *or* 60). *The ramification data for each group are defined in* (2.8).

Index	Group	Ramification	Index	Group	Ramification
6	$\Gamma(2)$	$\left\{\begin{matrix} 3^2 \\ 2^3 \\ 2^3 \end{matrix}\right\}$	36	$\Gamma_0(2) \cap \Gamma(3)$	$\left\{\begin{matrix} 3^{12} \\ 2^{18} \\ 3^4, 6^4 \end{matrix}\right\}$
6	$\Gamma_0(4)$	$\left\{\begin{matrix} 3^2 \\ 2^3 \\ 1^2, 4 \end{matrix}\right\}$	36	$\Gamma_1(9)$	$\left\{\begin{matrix} 3^{12} \\ 2^{18} \\ 1^3, 3^2, 9^3 \end{matrix}\right\}$
12	$\Gamma(3)$	$\left\{\begin{matrix} 3^4 \\ 2^6 \\ 3^4 \end{matrix}\right\}$	36	$\Gamma(9; 3, 1, 3)$	$\left\{\begin{matrix} 3^{12} \\ 2^{18} \\ 3^6, 9^2 \end{matrix}\right\}$
12	$\Gamma_0(4) \cap \Gamma(2)$	$\left\{\begin{matrix} 3^4 \\ 2^6 \\ 2^2, 4^2 \end{matrix}\right\}$	36	$\Gamma_1(10)$	$\left\{\begin{matrix} 3^{12} \\ 2^{18} \\ 1^2, 2^2, 5^2, 10^2 \end{matrix}\right\}$
12	$\Gamma_1(5)$	$\left\{\begin{matrix} 3^4 \\ 2^6 \\ 1^2, 5^2 \end{matrix}\right\}$	36	$\Gamma_0(18)$	$\left\{\begin{matrix} 3^{12} \\ 2^{18} \\ 1^3, 2^3, 9, 18 \end{matrix}\right\}$
12	$\Gamma_0(6)$	$\left\{\begin{matrix} 3^4 \\ 2^6 \\ 1, 2, 3, 6 \end{matrix}\right\}$	36	$\Gamma(27; 27, 3, 3)$	$\left\{\begin{matrix} 3^{12} \\ 2^{18} \\ 1^6, 3, 27 \end{matrix}\right\}$
12	$\Gamma_0(8)$	$\left\{\begin{matrix} 3^4 \\ 2^6 \\ 1^2, 2, 8 \end{matrix}\right\}$	48	$\Gamma_1(8) \cap \Gamma(2)$	$\left\{\begin{matrix} 3^{16} \\ 2^{24} \\ 2^4, 4^2, 8^4 \end{matrix}\right\}$
12	$\Gamma_0(9)$	$\left\{\begin{matrix} 3^4 \\ 2^6 \\ 1^3, 9 \end{matrix}\right\}$	48	$\Gamma(8; 2, 1, 2)$	$\left\{\begin{matrix} 3^{16} \\ 2^{24} \\ 4^8, 8^2 \end{matrix}\right\}$
24	$\Gamma(4)$	$\left\{\begin{matrix} 3^8 \\ 2^{12} \\ 4^6 \end{matrix}\right\}$	48	$\Gamma_1(12)$	$\left\{\begin{matrix} 3^{16} \\ 2^{24} \\ 1^2, 2, 3^2, 4^2, 6, 12^2 \end{matrix}\right\}$
24	$\Gamma_0(3) \cap \Gamma(2)$	$\left\{\begin{matrix} 3^8 \\ 2^{12} \\ 2^3, 6^3 \end{matrix}\right\}$	48	$\Gamma(12; 6, 1, 2)$	$\left\{\begin{matrix} 3^{16} \\ 2^{24} \\ 2^4, 4, 6^4, 12 \end{matrix}\right\}$
24	$\Gamma_1(7)$	$\left\{\begin{matrix} 3^8 \\ 2^{12} \\ 1^3, 7^3 \end{matrix}\right\}$	48	$\Gamma_0(16) \cap \Gamma_1(8)$	$\left\{\begin{matrix} 3^{16} \\ 2^{24} \\ 1^4, 2^2, 4^2, 16^2 \end{matrix}\right\}$
24	$\Gamma_1(8)$	$\left\{\begin{matrix} 3^8 \\ 2^{12} \\ 1^2, 2, 4, 8^2 \end{matrix}\right\}$	48	$\Gamma(16; 8, 2, 2)$	$\left\{\begin{matrix} 3^{16} \\ 2^{24} \\ 2^8, 16^2 \end{matrix}\right\}$

TABLE 1. *Continued.*

Index	Group	Ramification	Index	Group	Ramification
24	$\Gamma_0(8) \cap \Gamma(2)$	$\left\{ \begin{matrix} 3^8 \\ 2^{12} \\ 2^4, 8^2 \end{matrix} \right\}$	48	$\Gamma(24; 24, 2, 2)$	$\left\{ \begin{matrix} 3^{16} \\ 2^{24} \\ 1^4, 3^4, 8, 24 \end{matrix} \right\}$
24	$\Gamma(8; 4, 1, 2)$	$\left\{ \begin{matrix} 3^8 \\ 2^{12} \\ 2^2, 4^3, 8 \end{matrix} \right\}$	48	$\Gamma(32; 32, 4, 2)$	$\left\{ \begin{matrix} 3^{16} \\ 2^{24} \\ 1^8, 8, 32 \end{matrix} \right\}$
24	$\Gamma_0(12)$	$\left\{ \begin{matrix} 3^8 \\ 2^{12} \\ 1^2, 3^2, 4, 12 \end{matrix} \right\}$	60	$\Gamma(5)$	$\left\{ \begin{matrix} 3^{20} \\ 2^{30} \\ 5^{12} \end{matrix} \right\}$
24	$\Gamma_0(16)$	$\left\{ \begin{matrix} 3^8 \\ 2^{12} \\ 1^4, 4, 16 \end{matrix} \right\}$	60	$\Gamma_0(25) \cap \Gamma_1(5)$	$\left\{ \begin{matrix} 3^{20} \\ 2^{30} \\ 1^{10}, 25^2 \end{matrix} \right\}$
24	$\Gamma(16; 16, 2, 2)$	$\left\{ \begin{matrix} 3^8 \\ 2^{12} \\ 1^2, 2^3, 16 \end{matrix} \right\}$			

(we remark that depending on convention, we may need to appropriately normalize to the *j*-function by a factor of 1728).

Specifically, for our 33 conjugacy classes of subgroups, we find that for a group of index $6n$, there are $2n$ pre-images of 0 and the ramification indices of all of these marked points is 3; there are $3n$ pre-images of 1 and ramification indices of all of these marked points is 2; there are $n + 2$ pre-images of ∞, the ramification indices of which are given as the cusp widths in Table 1.

2.3. *Dessins d'enfants: j-functions as Belyi maps*

Maps to $\mathbb{P}^1$ ramified only at $(0, 1, \infty)$ are called *Belyi maps* and can be represented graphically as Grothendieck's *dessins d'enfants*. To draw such a dessin is simple: given ramification data $\left\{ \begin{matrix} \{r_0(i)\} \\ \{r_1(i)\} \\ \{r_\infty(i)\} \end{matrix} \right\}$ specifying the ramification indices at the various pre-images of 0, 1 and ∞, one marks one white node for the *i*th pre-image of 0, with $r_0(i)$ edges emanating therefrom; similarly, one marks one black node for the *j*th pre-image of 1, with $r_1(j)$ edges. Thus, we have a *bipartite graph* embedded on a Riemann sphere, with W white nodes and B black nodes. Now we connect the nodes with the edges, joining only black with white, such that we have I faces, each being a polygon with $2r_\infty(k)$ sides. *Dessins* for all the 33 torsion-free, genus zero modular subgroups are presented in § 2.3.1.

The Schreier coset graphs for the 33 conjugacy classes of subgroups are discussed and drawn explicitly in [**11**]. In our case, let Γ^* be the torsion-free, genus zero modular subgroup in question. The coset graphs display the permutation action of generators x and y with the relations $x^2 = y^3 = I$ on the coset space of $\mathrm{PSL}(2, \mathbb{Z}) = \bigcup_{i=1}^{\mu} \Gamma^* x_i$ where μ is the index of Γ^* in $\mathrm{PSL}(2, \mathbb{Z})$. These graphs have μ nodes, where $\mu \in \{6, 12, 24, 36, 48, 60\}$ is the index of the modular subgroup in question. The graphs are built from simple edges (x) and triangles (y) which are assumed to be positively oriented. Reversal of orientation corresponds to applying the automorphism of the modular group that inverts each of the two generators x and y.

Interestingly, there is a direct connection between the Schreier coset graph and the *dessins d'enfant* for each modular subgroup [**19**]. This connection can be stated as follows.

PROPOSITION 2.2. *The dessins can be constructed from the Schreier coset graphs by replacing each positively oriented triangle with a white node, and inserting a black node into every edge. Equivalently, the Schreier coset graphs can be constructed from the dessins by replacing each white node with a positively oriented triangle, and removing the black node from every edge.*

The process of inserting into an edge a bivalent node of a different colouring is standard in the study of *dessins d'enfants*; this gives rise to so-called *clean dessins*. The contraction of triangles into a node is a less obvious one and should correspond to taking a cube root of the polynomial factors in the Belyi map.

2.3.1. *Dessins for the* 33 *conjugacy classes of subgroups.* In this section, we draw the *dessins d'enfants* for all 33 conjugacy classes of genus zero, torsion-free congruence subgroups, using the ramification data in Table 1. Each dessin is labelled by its corresponding conjugacy class of modular subgroups, and the unique cusp widths for that class. The reader can visually confirm the aforementioned correspondence between these *dessins* and the Schreier coset graphs constructed in [**11**].

$\Gamma(2)$
$2, 2, 2$

$\Gamma_0(4)$
$4, 1, 1$

$\Gamma(3)$
$3, 3, 3, 3$

$\Gamma_0(4) \cap \Gamma(2)$
$4, 4, 2, 2$

$\Gamma_1(5)$
$5, 5, 1, 1$

$\Gamma_0(6)$
$6, 3, 2, 1$

$\Gamma_0(8)$
$8, 2, 1, 1$

$\Gamma_0(9)$
$9, 1, 1, 1$

$\Gamma(4)$
$4, 4, 4, 4, 4, 4$

$\Gamma(8; 4, 1, 2)$
$8, 4, 4, 4, 2, 2$

$\Gamma_0(3) \cap \Gamma(2)$
$6, 6, 6, 2, 2, 2$
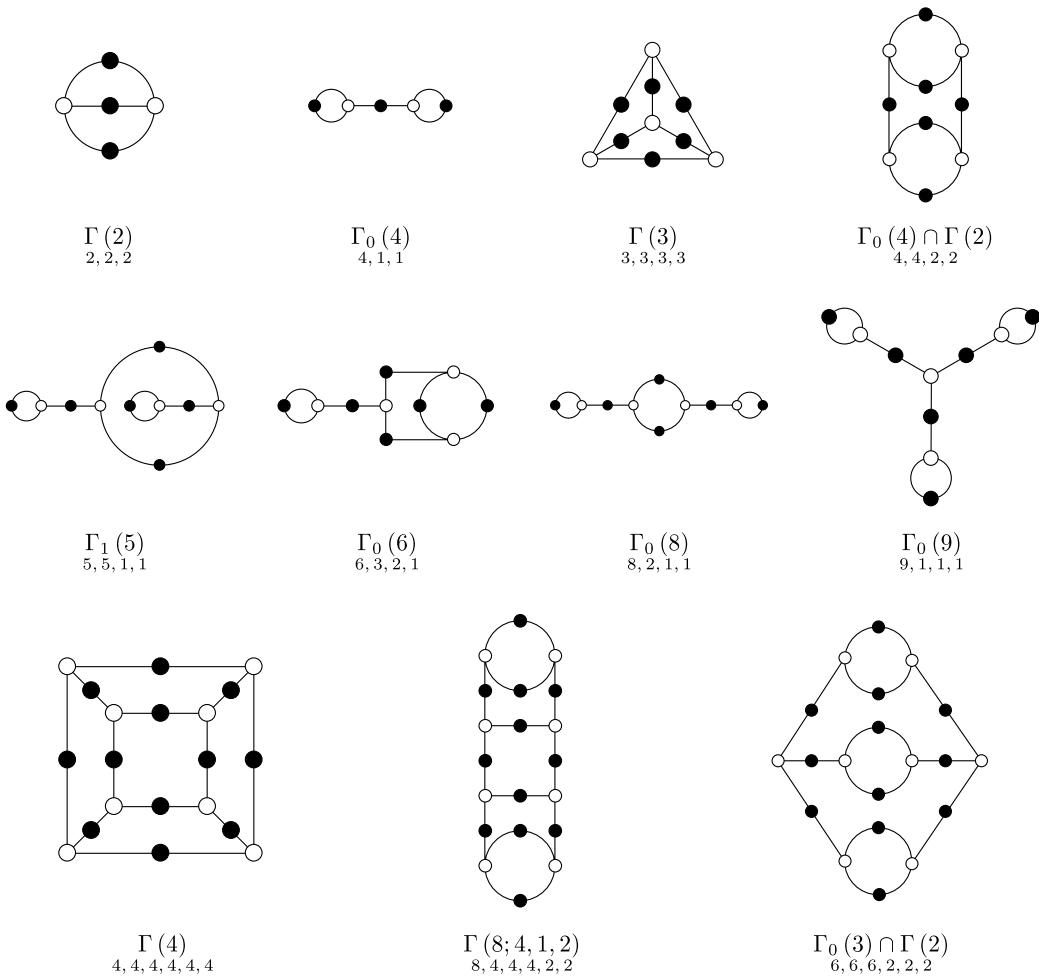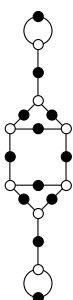
$\Gamma_0\left(12\right)$
12, 4, 3, 3, 1, 1
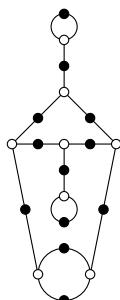
$\Gamma_1\left(8\right)$
8, 8, 4, 2, 1, 1
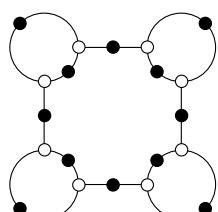
$\Gamma_0\left(8\right)\cap\Gamma\left(2\right)$
8, 8, 2, 2, 2, 2
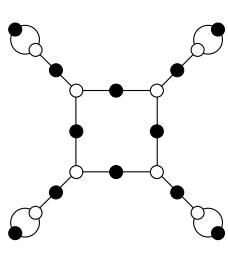
$\Gamma_0\left(16\right)$
16, 4, 1, 1, 1, 1
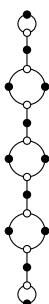
$\Gamma\left(16;16,2,2\right)$
16, 2, 2, 2, 1, 1
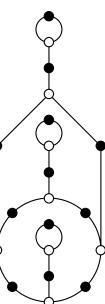
$\Gamma_1\left(7\right)$
7, 7, 7, 1, 1, 1
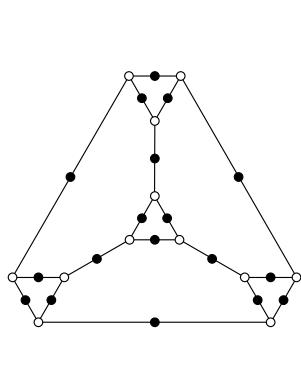
$\Gamma_0\left(2\right)\cap\Gamma\left(3\right)$
6, 6, 6, 6, 3, 3, 3, 3
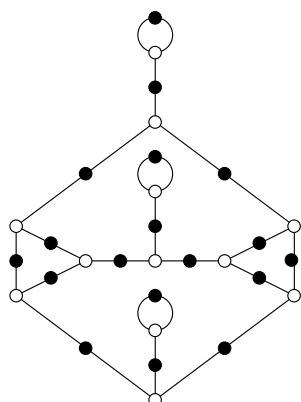
$\Gamma_1\left(9\right)$
9, 9, 9, 3, 3, 1, 1, 1
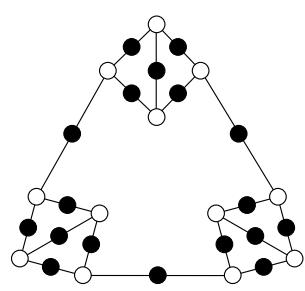
$\Gamma\left(9;3,1,3\right)$
9, 9, 3, 3, 3, 3, 3, 3
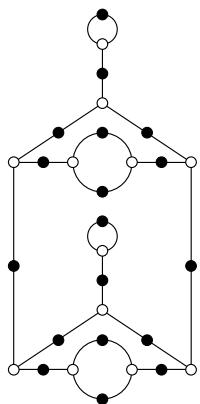
$\Gamma_1(10)$
$10, 10, 5, 5, 2, 2, 1, 1$
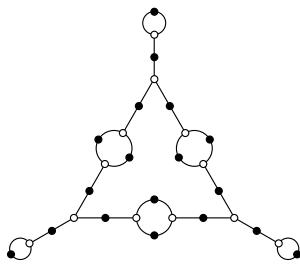
$\Gamma_0(18)$
$18, 9, 2, 2, 2, 1, 1, 1$
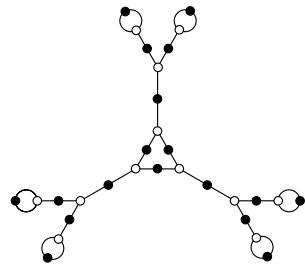
$\Gamma(27; 27, 3, 3)$
$27, 3, 1, 1, 1, 1, 1, 1$
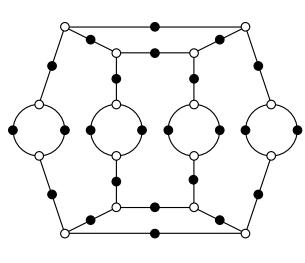
$\Gamma_1(8) \cap \Gamma(2)$
$8, 8, 8, 8, 4, 4, 2, 2, 2, 2$
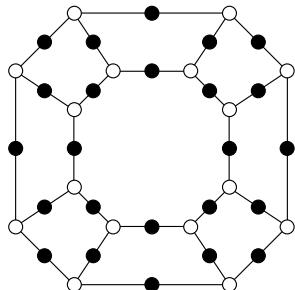
$\Gamma(8; 2, 1, 2)$
$8, 8, 4, 4, 4, 4, 4, 4, 4, 4$
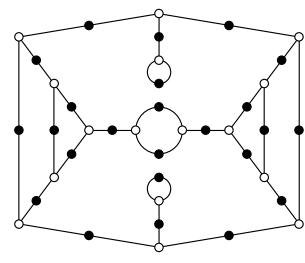
$\Gamma_1(12)$
$12, 12, 6, 4, 4, 3, 3, 2, 1, 1$
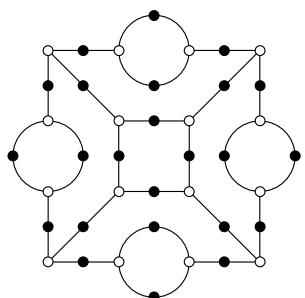
$\Gamma(12; 6, 1, 2)$
$12, 6, 6, 6, 6, 4, 2, 2, 2, 2$
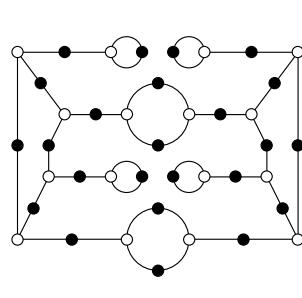
$\Gamma_0(16) \cap \Gamma_1(8)$
$16, 16, 4, 4, 2, 2, 1, 1, 1, 1$

$\Gamma(16; 8, 2, 2)$
$16, 16, 2, 2, 2, 2, 2, 2, 2, 2$

$$\Gamma\,(24; 24, 2, 2)$$
24, 8, 3, 3, 3, 3, 1, 1, 1, 1

$$\Gamma\,(32; 32, 4, 2)$$
32, 8, 1, 1, 1, 1, 1, 1, 1, 1

$$\Gamma\,(5)$$
5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5

$$\Gamma_0\,(25) \cap \Gamma_1\,(5)$$
25, 25, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1
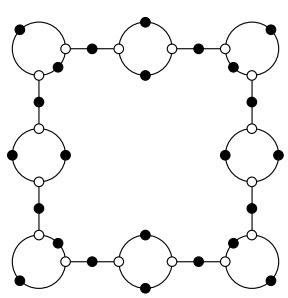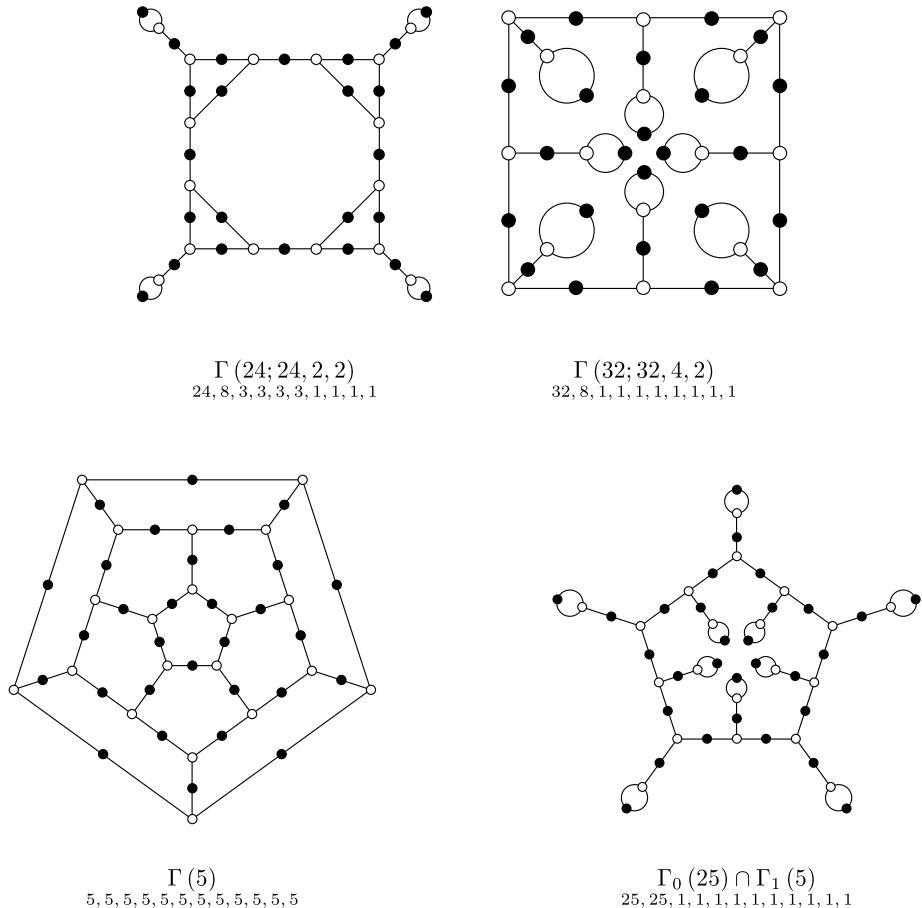
2.4. *Index 36 subgroups and Calabi–Yau threefolds*

Since all of the index 24 torsion-free, genus zero congruence subgroups have been associated in [**2, 3, 11, 20**] to K3 surfaces, in particular elliptically fibred K3 surfaces over $\mathbb{P}^1$, it is natural to enquire whether the index 36 groups can be associated to Calabi–Yau threefolds. We shall see that this is indeed the case, and the threefolds are a simple generalization of the K3 case, being now elliptic fibrations over $\mathbb{P}^2$.

First, let us recall some standard facts about elliptic Calabi–Yau threefolds. The classification of base surfaces B over which an elliptic fibration is a smooth Calabi–Yau threefold was performed in [**13, 14**]: they can be: Hirzebruch surfaces and their blowups; del Pezzo surfaces; or the Enriques surface. In particular, the local Weierstraß model looks as follows:

$$y^2 = x^3 + f(z_i)x + g(z_i) \subset \mathbb{P}^2_{[x,y]}, \tag{2.9}$$

with

$$f(z_i) \in \Gamma((K_B^{-1})^{\otimes 4}), \quad g(z_i) \in \Gamma((K_B^{-1})^{\otimes 6}). \tag{2.10}$$

This is an elliptic curve in a $\mathbb{P}^2$ bundle over B, whose coordinates are z_i; the (x, y) are affine coordinates on the $\mathbb{P}^2$. The functions f and g are homogeneous polynomials in the base coordinates. Specifically, they should be considered as sections Γ of appropriate tensor powers of the anti-canonical bundle K_B^{-1} of B: for f, it is the fourth power and for g, the sixth. A nice summary discussion of this can be found, for example, in [**13**, §3].

The j-invariant of a Weierstraß model is given by

$$j(z_i) = 1728 \cdot \frac{4f(z_i)^3}{4f(z_i)^3 - 27g(z_i)^2}. \tag{2.11}$$

Here, we have marked explicitly the dependence on the base coordinates z_i. In particular, we can see that both the numerator and denominator are sections of $(K_B^{-1})^{\otimes 12}$.

Let us return to the index 36 torsion-free, genus zero congruence subgroups. As mentioned, the j-invariants for these are computed and tabulated in [**11**]; these are rational functions in terms of the principal modulus t and we interpreted these as Belyi maps in our above discussions to construct the *dessins*. From j we can extract the corresponding polynomials $f(t)$ and $g(t)$ for each subgroup. We find that the $f(t)$ will be a degree 12 and $g(t)$, degree 18, polynomial in t. In general, for an index μ subgroup (recall again that $\mu \in \{6, 12, 24, 36, 48, 60\}$ for genus zero, torsion-free congruence subgroups), the degrees of f and g are $\mu/3$ and $\mu/2$.

Finally, let us recall a standard fact about $\mathbb{P}^2$ to complete the story. The anti-canonical bundle of the projective plane is $K_{\mathbb{P}^2}^{-1} = \mathcal{O}_{\mathbb{P}^2}(3)$, therefore sections of this would be degree 3 homogeneous polynomials in the coordinates. Therefore, if we choose our base B to be $\mathbb{P}^2$ with affine coordinates z_1, z_2, then $f(z_i)$ will be a degree $3 \times 4 = 12$ and $g(z_i)$ a degree $3 \times 6 = 18$ homogeneous polynomial in z_1, z_2. These are the right degrees to match those of $f(t)$ and $g(t)$ for the index 36 case. Therefore, setting a linear change of basis (normalizing the leading coefficient)

$$z_1 = t + a, \quad z_2 = t + b \quad a, b \in \mathbb{C} \tag{2.12}$$

and substituting into appropriately chosen $f(z_i)$ and $g(z_i)$ should give the required $f(t)$ and $g(t)$. Geometrically, this amounts to finding a hyper-plane $\mathbb{P}^1$ defined by (2.12) inside the base $B = \mathbb{P}^2$ on which our elliptically fibred Calabi–Yau threefold restricts to the required modular surface for index 36.

The analogous situation for the index 24 case is clear. There, we have K3 surfaces which are elliptic fibrations over a base $\mathbb{P}^1$: we know that the anti-canonical bundle is $K_{\mathbb{P}^2}^{-1} = \mathcal{O}_{\mathbb{P}^2}(2)$ whose sections are thus quadratics in the projective coordinate t. Hence, $f(t)$ and $g(t)$ should be of degree $2 \times 4 = 8$ and $2 \times 6 = 12$, which are indeed in accord with the aforementioned fact that the degrees should be $\mu/3$ and $\mu/2$, respectively, with $\mu = 24$ here.

2.4.1. *Example:* $\Gamma_0(2) \cap \Gamma(3)$. Let us see how the above setup works for the specific groups in detail. The j-invariant for the group $\Gamma_0(2) \cap \Gamma(3)$ is [**11**]:

$$j(t) = \frac{((t^3 + 4)(t^3 + 6t^2 + 4)(t^6 - 6t^5 + 36t^4 + 8t^3 - 24t^2 + 16))^3}{t^6(t + 1)^3(t^2 - t + 1)^3(t - 2)^6(t^2 + 2t + 4)^6}. \tag{2.13}$$

Therefore, solving for

$$((t^3 + 4)(t^3 + 6t^2 + 4)(t^6 - 6t^5 + 36t^4 + 8t^3 - 24t^2 + 16))^3 = 1728 \cdot 4 \cdot f(t)^3$$

and

$$t^6(t + 1)^3(t^2 - t + 1)^3(t - 2)^6(t^2 + 2t + 4)^6 = 4f(t)^3 - 27g(t)^2$$

gives

$$f(t) = \frac{1}{12 \cdot 2^{2/3}}(t^{12} + 232t^9 + 960t^6 + 256t^3 + 256),$$

$$g(t) = \frac{1}{216}(t^{18} - 516t^{15} - 12\,072t^{12} - 24\,640t^9 - 30\,720t^6 + 6144t^3 + 4096) \tag{2.14}$$

which we see to be polynomials of the correct degree. Note here that while $f = 0$ is solvable and has relatively simple roots, the corresponding equation for g is not solvable.

Now, the most general form of degree 12 and 18 polynomials in two variables are

$$f(z_1, z_2) = \sum_{i=0}^{12} c_i z_1^i z_2^{12-i}, \quad g(z_1, z_2) = \sum_{i=0}^{18} c_i z_1^i z_2^{18-i}, \tag{2.15}$$

with complex coefficients c_i and d_i. We can readily find the coefficients c_i and d_i as well as constants a, b which allow (2.15) to be transformed into the form of (2.14). Setting, for example, $c_1 = d_1 = 0$, we find that $a = 1$, $b = -2$, and that

$$f(z_1, z_2) = \frac{1}{8748 \cdot 2^{2/3}} (256 z_1^{12} - 256 z_2^3 z_1^9 + 960 z_2^6 z_1^6 - 232 z_2^9 z_1^3 + z_2^{12});$$

$$g(z_1, z_2) = \frac{1}{4\,251\,528} (-4096 z_1^{18} + 6144 z_2^3 z_1^{15} + 30\,720 z_2^6 z_1^{12} - 24\,640 z_2^9 z_1^9$$
$$+ 12072 z_2^{12} z_1^6 - 516 z_2^{15} z_1^3 - z_2^{18}). \tag{2.16}$$

We can readily find, using the same technique as above, the explicit Weierstraß equations, albeit with very complicated *algebraic* coefficients, of the remaining index 36 groups, as we had done for the index 24 cases in [8]. Although the complexity of the numerical expressions makes the presentation unworthwhile, it should be emphasized that all of the coefficients we find for the Weierstraß models must be algebraic numbers. This is in accord with the deep theorem of Belyi that a rational map from a Riemann surface to $\mathbb{P}^1$ is Belyi (i.e. ramified only on $0, 1, \infty$) if and only if there exists an algebraic equation for the Riemann surface over $\overline{\mathbb{Q}}$.

3. Semi-stable elliptic K3 surfaces

We have mentioned above that the nine conjugacy classes of index 24 subgroups are special in the sense that they correspond to semi-stable elliptic K3 surfaces. Exploring the physics of these was the subject of [8]. In this section, we will see how they belong to another important classification scheme which intersects nicely with our above discussion of congruence subgroups.

In [12], the authors study semi-stable elliptic fibrations over $\mathbb{P}^1$ of K3 surfaces with six singular fibres. From the cusp widths (or, equivalently, the ramification data at ∞) presented in Table 1, we see that our subgroups all have 6 cusps and hence the index 24 cases belong to this class. In their paper, the authors give a complete list of possible fibre types for such fibrations; there are 112 such cases. Our 9 conjugacy classes of index 24 subgroups of Γ discussed in the previous section correspond to 9 of these 112 surfaces.

Remarkably, the elliptic j-invariants for all the 112 surfaces also have the property that, as rational maps from $\mathbb{P}^1$ to $\mathbb{P}^1$, they are Belyi. Furthermore, all of the ramification indices at 0 are 3, and all of the ramification indices at 1 are 2; that is, we have trivalent *dessins*. In [2, 3], the *dessins d'enfants* for all 112 cases are constructed, along with explicit j-functions with coefficients in $\mathbb{Q}$ (for reference, we include the full list, emphasizing their clean, trivalent nature, in the appendix). Those with coefficients in $\overline{\mathbb{Q}} \setminus \mathbb{Q}$ have yet to be computed, due to the complexity of finding exact roots to polynomials in high degree. This is thus an interesting open challenge. Now, because any finite trivalent graph with nodes replaced by oriented triangles is the Schreier coset graph of a finite index subgroup of the modular group, one naturally wonders whether we could attach a conjugacy class of subgroups to each of these *dessins*. Of course, only nine of these will be congruence (they are all genus zero because the K3 surfaces are all fibred over $\mathbb{P}^1$), so what general subgroups can one encounter?

We will see that just as those 9 surfaces correspond to these congruence subgroups of Γ (up to modular conjugacy), it turns out that *all* of the 112 surfaces will correspond to some (not necessarily congruence) subgroup of Γ. In this section we shall elaborate on the connections between these surfaces and the subgroups of Γ. The structure of this section is as follows: first, we discuss the permutation representation of Γ on the right cosets of each modular subgroup,

and show how this defines a transitive permutation group known as a *cartographic group*. We then compute a generating set for a representative of the conjugacy class of modular subgroups corresponding to each of the 112 semi-stable elliptic fibrations over $\mathbb{P}^1$ of K3 surfaces with six singular fibres.

3.1. *Permutation representations and cartographic groups*

Returning to our original discussion of the modular group in §2.1, let σ_0 and σ_1 denote the permutations induced by the respective actions of S and ST on the cosets of each subgroup. We can find a third permutation σ_∞ by imposing the following condition, thereby constructing a *permutation triple*:

$$\sigma_0 \cdot \sigma_1 \cdot \sigma_\infty = 1. \tag{3.1}$$

The permutations σ_0, σ_1 and σ_∞ give the permutation representations of Γ on the right cosets of each modular subgroup in question. From the point of view of the *dessins*, there is an equivalent representation in terms of permutation triples [**10**, **16**, **21**]. Let d be the number of edges, labelled from 1 to d, and write elements of the symmetric group on d elements in the standard cycle notation. Now, at the ith black node write the labels for the incident edges clockwise, forming the cycle B_i; likewise, at the jth white node write the labels for the incident edges counter-clockwise, forming the cycle W_j. Then, σ_0 is simply $\prod_i B_i$ while $\sigma_1 = \prod_j W_j$. Remarkably, σ_∞ as determined by (3.1) will have as many cycle products as there are faces, such that twice the length of each cycle is the number of edges of the corresponding face. Conversely, a *unique dessin* and Belyi map of degree d is thus determined.

As elements of the symmetric group, σ_0 and σ_1 can be easily computed from the Schreier coset graphs in [**11**] by following the procedure elaborated in [**17**], i.e. by noting that the doubly directed edges represent an element x of order 2, while the positively oriented triangles represent an element y of order 3. Since the graphs are connected, the group generated by x and y is transitive on the vertices. Clearly, σ_0 and σ_1 tell us which vertex of the coset graph is sent to which, i.e. which coset of the modular subgroup in question is sent to which by the action of Γ on the right cosets of this subgroup.

To express the cycle decomposition of the permutation σ_∞, we choose a vertex and apply x, then y, i.e. we take an edge (doubly oriented) followed by a side of a triangle (always in the positive orientation), until we return back to the original vertex. The length of this circuit gives the length of a cycle in σ_∞. In this way, we can construct the permutation decompositions σ_0, σ_1 and σ_∞ for every *dessin* under consideration. In turn, any two of these three permutation decompositions define a transitive permutation group. Of course, we only need two of these three permutations since the third will simply be the inverse of the product of the other two by (3.1), and we will henceforth focus on σ_0 and σ_1. These permutation groups are known as *cartographic groups*, and are discussed in e.g. [**21**].

TABLE 2. *The permutation groups $L(n)$.*

Name	Size	Type
$L(6)$	60	$\text{PSL}(2,5) = A_5(6)$
$L(7)$	168	$\text{PSL}(3,2)$
$L(8)$	168	$\text{PSL}(2,7)$
$L(9)$	504	$\text{PSL}(2,8)$
$L(10)$	360	$\text{PSL}(2,9) = A_6(10)$
$L(11)$	660	$\text{PSL}(2,11)(11)$
$L(12)$	660	$\text{PSL}(2,11)$
$L(13)$	5616	$\text{PSL}(3,3)$
$L(14)$	1092	$\text{PSL}(2,13)$
$L(15)$	20 160	$\text{PSL}(4,2) = A_8(15)$

3.1.1. *Cartographic groups for the* 112 *K3 surfaces.* A priori, one would imagine that one could obtain the semi-stable elliptic K3 surfaces with 6 singular (I-type) singularities by partitioning 24 into 6 parts. There are 199 possibilities and [**12**] showed that only 112 are allowed. In general, one would try to partition 24 into at least 6 parts; the fact that there are at least 6 singular fibres is a restriction coming from the geometry of K3 surfaces. In our case, there are exactly 6 singular fibres, making these surfaces *extremal*. These surfaces are listed in [**12**].

We will thus distinguish our 112 extremal K3 surfaces by a 6-tuple partition $(I_1, \ldots, I_6)$ of 24; these will correspond to the I-type singular fibres as well as the ramification data (i.e. cusp widths) at infinity. A table of the cartographic groups for the partitions in [**12**] is given below. Now, *dessins* have intriguing number theoretic properties and the field of definition of (the coefficients in) the corresponding *j*-function as a Belyi map is an important quantity. For reference, we include the number field in brackets next to the partition. For simplicity, where we write $(\sqrt{-3})$, for example, we mean the field is $\mathbb{Q}(\sqrt{-3})$. Where it says e.g. 'cubic', we mean it is $\mathbb{Q}$ extended by the root of some cubic polynomial.

We note that the partitions do not necessarily correspond uniquely to a *dessin*; when they do, the *j*-function is always defined over $\mathbb{Q}$. Where a single partition has multiple *dessins* defined over different number fields, we have for completeness computed the cartographic groups for all such cases, indicating explicitly the field as in [**2, 3**]. We also keep the order of the entries the same as the order of the *dessins* in [**2, 3**] and in the appendix.

To identify these cartographic groups, we use the corresponding *dessins* given in [**2, 3**] to construct the Schreier coset graph in each case, following the procedure detailed in § 2.3. We then read the permutations σ_0 and σ_1 directly off these graphs. As mentioned, together these permutations define a transitive permutation group, which is the cartographic group in question. Extensive use will be made of the computer software GAP [**7**]. The decomposition of each group in terms of better known groups can be identified using commands such as `StructureDescription()` and `TransitiveIdentification()` and it is these decompositions which are given in the ensuing table[†]. We follow [**6**] and use the following group notation: A (alternating), AL (affine linear), C (cyclic), E (elementary), S (symmetric). Direct products are denoted by '×', split extensions by ':', and wreath products by '≀'. In addition, we use $L(n)$ to denote groups derived from linear groups as defined in Table 2 (see also [**6**]). A group name followed by a number n in round brackets '(n)' denotes that the group operates transitively on n points, or respectively that a point stabilizer has index n in the group [**6**].

Where no easy decomposition is available, we have simply given the cartographic group's index in the GAP library of transitive permutation groups (here a number of the form t24n...). The number after 't' in this index gives the degree of the transitive permutation group, while the number after the 'n' gives the location that group in the GAP library of transitive permutation groups for that particular group degree.

Partition (K3 surface)	Cartographic group	Cartographic group order
$19, 1, 1, 1, 1, 1$ ($\mathbb{Q}$)	A_{24}	$24!/2$
$18, 2, 1, 1, 1, 1$ ($\mathbb{Q}$)	t24n18368 = t24n16559 × C_2	$2^{11} \cdot 3^4$
$18, 2, 1, 1, 1, 1$ ($\sqrt{-3}$)	A_{24}	$24!/2$
$18, 2, 1, 1, 1, 1$ ($\sqrt{-3}$)	A_{24}	$24!/2$
$17, 3, 1, 1, 1, 1$ ($\mathbb{Q}$)	A_{24}	$24!/2$
$17, 2, 2, 1, 1, 1$ ($\sqrt{17}$)	A_{24}	$24!/2$
$17, 2, 2, 1, 1, 1$ ($\sqrt{17}$)	A_{24}	$24!/2$
$16, 4, 1, 1, 1, 1$ ($\mathbb{Q}$)	$(((C_8 \times C_2) : C_8) : C_3) : C_2$	$2^8 \cdot 3$
$16, 3, 2, 1, 1, 1$ ($\mathbb{Q}$)	t24n24710 = 2^{12} : t18n762	$2^{17} \cdot 3^7$
$16, 3, 2, 1, 1, 1$ ($\sqrt{-2}$)	A_{24}	$24!/2$

[†]We thank Alexander Hulpke for helping to identify many of the decompositions.

Partition (K3 surface)	Cartographic group	Cartographic group order
$16, 3, 2, 1, 1, 1\ (\sqrt{-2})$	A_{24}	$24!/2$
$16, 2, 2, 2, 1, 1\ (\mathbb{Q})$	$((C_8 \times C_8) : C_3) : C_2$	$2^7 \cdot 3$
$15, 5, 1, 1, 1, 1\ (\sqrt{-15})$	A_{24}	$24!/2$
$15, 5, 1, 1, 1, 1\ (\sqrt{-15})$	A_{24}	$24!/2$
$15, 4, 2, 1, 1, 1\ (\sqrt{-15})$	A_{24}	$24!/2$
$15, 4, 2, 1, 1, 1\ (\sqrt{-15})$	A_{24}	$24!/2$
$15, 3, 3, 1, 1, 1\ (\mathbb{Q})$	$A_6 \wr A_4$	$2^{14} \cdot 3^9 \cdot 5^4$
$15, 3, 2, 2, 1, 1\ (\mathbb{Q})$	A_{24}	$24!/2$
$15, 3, 2, 2, 1, 1\ (\sqrt{-15})$	A_{24}	$24!/2$
$15, 3, 2, 2, 1, 1\ (\sqrt{-15})$	A_{24}	$24!/2$
$14, 6, 1, 1, 1, 1\ (\sqrt{-3})$	A_{24}	$24!/2$
$14, 6, 1, 1, 1, 1\ (\sqrt{-3})$	A_{24}	$24!/2$
$14, 5, 2, 1, 1, 1\ (\mathbb{Q})$	$A_8 \wr S_3$	$2^{19} \cdot 3^7 \cdot 5^3 \cdot 7^3$
$14, 5, 2, 1, 1, 1\ (\text{cubic})$	A_{24}	$24!/2$
$14, 5, 2, 1, 1, 1\ (\text{cubic})$	A_{24}	$24!/2$
$14, 5, 2, 1, 1, 1\ (\text{cubic})$	A_{24}	$24!/2$
$14, 4, 3, 1, 1, 1\ (\mathbb{Q})$	A_{24}	$24!/2$
$14, 4, 2, 2, 1, 1\ (\sqrt{-7})$	$(E_8 : L(7)) \wr S_3 = \mathrm{ASL}(3, 2) \wr S_3$	$2^{19} \cdot 3^4 \cdot 7^3$
$14, 4, 2, 2, 1, 1\ (\sqrt{-7})$	$(E_8 : L(7)) \wr S_3 = \mathrm{ASL}(3, 2) \wr S_3$	$2^{19} \cdot 3^4 \cdot 7^3$
$14, 3, 3, 2, 1, 1\ (\mathbb{Q})$	$L(8) \wr S_3 = \mathrm{PSL}(2, 7) \wr S_3$	$2^{10} \cdot 3^4 \cdot 7^3$
$14, 3, 3, 2, 1, 1\ (\mathbb{Q})$	A_{24}	$24!/2$
$14, 3, 2, 2, 2, 1\ (\mathbb{Q})$	$A_8 \wr S_3$	$2^{19} \cdot 3^7 \cdot 5^3 \cdot 7^3$
$13, 7, 1, 1, 1, 1\ (\mathbb{Q})$	A_{24}	$24!/2$
$13, 6, 2, 1, 1, 1\ (\sqrt{-3})$	A_{24}	$24!/2$
$13, 6, 2, 1, 1, 1\ (\sqrt{-3})$	A_{24}	$24!/2$
$13, 5, 3, 1, 1, 1\ (\mathbb{Q})$	A_{24}	$24!/2$
$13, 5, 2, 2, 1, 1\ (\sqrt{65})$	A_{24}	$24!/2$
$13, 5, 2, 2, 1, 1\ (\sqrt{65})$	A_{24}	$24!/2$
$13, 4, 3, 2, 1, 1\ (\mathbb{Q})$	A_{24}	$24!/2$
$13, 3, 3, 2, 2, 1\ (\mathbb{Q})$	A_{24}	$24!/2$
$12, 7, 2, 1, 1, 1\ (\sqrt{-3})$	A_{24}	$24!/2$
$12, 7, 2, 1, 1, 1\ (\sqrt{-3})$	A_{24}	$24!/2$
$12, 6, 3, 1, 1, 1\ (\mathbb{Q})$	$A_6 \wr A_4$	$2^{14} \cdot 3^9 \cdot 5^4$
$12, 6, 2, 2, 1, 1\ (\mathbb{Q})$	t24n12260	$2^{11} \cdot 3^2$
$12, 6, 2, 2, 1, 1\ (\mathbb{Q})$	$(L(8) \wr S_3) \cap A_{24}$	$2^{12} \cdot 3^4 \cdot 7^3$
$12, 5, 3, 2, 1, 1\ (\text{quartic})$	A_{24}	$24!/2$
$12, 5, 3, 2, 1, 1\ (\text{quartic})$	A_{24}	$24!/2$
$12, 5, 3, 2, 1, 1\ (\text{quartic})$	A_{24}	$24!/2$
$12, 5, 3, 2, 1, 1\ (\text{quartic})$	A_{24}	$24!/2$
$12, 5, 2, 2, 2, 1\ (\mathbb{Q})$	$(S_8 \wr S_3) \cap A_{24}$	$2^{21} \cdot 3^7 \cdot 5^3 \cdot 7^3$
$12, 4, 4, 2, 1, 1\ (\mathbb{Q})$	t24n21388 $= 2^{12} \wr$ t18n97	$2^{15} \cdot 3^3$
$12, 4, 3, 3, 1, 1\ (\mathbb{Q})$	$S_4 \times A_4$	$4! \cdot 4!/2$
$12, 4, 3, 3, 1, 1\ (\mathbb{Q})$	$(\mathrm{PSL}(2, 5) \wr A_4) : C_2$	$2^{11} \cdot 3^5 \cdot 5^4$
$12, 4, 3, 2, 2, 1\ (\mathbb{Q})$	t24n23732 $= 2^{12} \wr$ t18n346	$2^{15} \cdot 3^5$
$12, 4, 3, 2, 2, 1\ (\sqrt{-3})$	$(S_8 \wr S_3) \cap A_{24}$	$2^{21} \cdot 3^7 \cdot 5^3 \cdot 7^3$
$12, 4, 3, 2, 2, 1\ (\sqrt{-3})$	$(S_8 \wr S_3) \cap A_{24}$	$2^{21} \cdot 3^7 \cdot 5^3 \cdot 7^3$
$12, 3, 3, 3, 2, 1\ (\mathbb{Q})$	$(A_6 \wr A_4) : C_2$	$2^{15} \cdot 3^9 \cdot 5^4$
$12, 3, 3, 2, 2, 2\ (\mathbb{Q})$	t24n12261	$2^{11} \cdot 3^2$
$11, 9, 1, 1, 1, 1\ (\mathbb{Q})$	A_{24}	$24!/2$
$11, 8, 2, 1, 1, 1\ (\text{cubic})$	A_{24}	$24!/2$
$11, 8, 2, 1, 1, 1\ (\text{cubic})$	A_{24}	$24!/2$
$11, 8, 2, 1, 1, 1\ (\text{cubic})$	A_{24}	$24!/2$
$11, 7, 3, 1, 1, 1\ (\text{cubic})$	A_{24}	$24!/2$

Partition (K3 surface)	Cartographic group	Cartographic group order
$11, 7, 3, 1, 1, 1$ (cubic)	A_{24}	$24!/2$
$11, 7, 3, 1, 1, 1$ (cubic)	A_{24}	$24!/2$
$11, 7, 2, 2, 1, 1$ ($\sqrt{-7}$)	A_{24}	$24!/2$
$11, 7, 2, 2, 1, 1$ ($\sqrt{-7}$)	A_{24}	$24!/2$
$11, 6, 4, 1, 1, 1$ ($\sqrt{33}$)	A_{24}	$24!/2$
$11, 6, 4, 1, 1, 1$ ($\sqrt{33}$)	A_{24}	$24!/2$
$11, 6, 3, 2, 1, 1$ (cubic)	A_{24}	$24!/2$
$11, 6, 3, 2, 1, 1$ (cubic)	A_{24}	$24!/2$
$11, 6, 3, 2, 1, 1$ (cubic)	A_{24}	$24!/2$
$11, 5, 5, 1, 1, 1$ ($\mathbb{Q}$)	A_{24}	$24!/2$
$11, 5, 4, 2, 1, 1$ (cubic)	A_{24}	$24!/2$
$11, 5, 4, 2, 1, 1$ (cubic)	A_{24}	$24!/2$
$11, 5, 4, 2, 1, 1$ (cubic)	A_{24}	$24!/2$
$11, 5, 3, 3, 1, 1$ ($\sqrt{5}$)	A_{24}	$24!/2$
$11, 5, 3, 3, 1, 1$ ($\sqrt{5}$)	A_{24}	$24!/2$
$11, 5, 3, 2, 2, 1$ ($\mathbb{Q}$)	A_{24}	$24!/2$
$11, 4, 4, 3, 1, 1$ ($\mathbb{Q}$)	A_{24}	$24!/2$
$11, 4, 3, 3, 2, 1$ ($\mathbb{Q}$)	A_{24}	$24!/2$
$10, 10, 1, 1, 1, 1$ ($\mathbb{Q}$)	$((C_2 \times C_2 \times C_2 \times C_2 \times C_2) : A_5) : C_2$	$2^8 \cdot 3 \cdot 5$
$10, 10, 1, 1, 1, 1$ ($\sqrt{5}$)	t24n22755	$2^{14} \cdot 3 \cdot 5 \cdot 11$
$10, 10, 1, 1, 1, 1$ ($\sqrt{5}$)	t24n22755	$2^{14} \cdot 3 \cdot 5 \cdot 11$
$10, 9, 2, 1, 1, 1$ ($\sqrt{5}$)	A_{24}	$24!/2$
$10, 9, 2, 1, 1, 1$ ($\sqrt{5}$)	A_{24}	$24!/2$
$10, 8, 3, 1, 1, 1$ ($\mathbb{Q}$)	A_{24}	$24!/2$
$10, 7, 4, 1, 1, 1$ ($\mathbb{Q}$)	A_{24}	$24!/2$
$10, 7, 3, 2, 1, 1$ ($\sqrt{21}$)	A_{24}	$24!/2$
$10, 7, 3, 2, 1, 1$ ($\sqrt{21}$)	A_{24}	$24!/2$
$10, 7, 2, 2, 2, 1$ ($\mathbb{Q}$)	$A_8 \wr S_3$	$2^{19} \cdot 3^7 \cdot 5^3 \cdot 7^3$
$10, 6, 5, 1, 1, 1$ ($\mathbb{Q}$)	$A_8 \wr S_3$	$2^{19} \cdot 3^7 \cdot 5^3 \cdot 7^3$
$10, 6, 5, 1, 1, 1$ (cubic)	A_{24}	$24!/2$
$10, 6, 5, 1, 1, 1$ (cubic)	A_{24}	$24!/2$
$10, 6, 5, 1, 1, 1$ (cubic)	A_{24}	$24!/2$
$10, 6, 4, 2, 1, 1$ ($\mathbb{Q}$)	$(S_8 \wr S_3) \cap A_{24}$	$2^{21} \cdot 3^7 \cdot 5^3 \cdot 7^3$
$10, 6, 4, 2, 1, 1$ ($\mathbb{Q}$)	$A_8 \wr S_3$	$2^{19} \cdot 3^7 \cdot 5^3 \cdot 7^3$
$10, 6, 3, 2, 2, 1$ ($\mathbb{Q}$)	$A_8 \wr S_3$	$2^{19} \cdot 3^7 \cdot 5^3 \cdot 7^3$
$10, 5, 5, 2, 1, 1$ ($\sqrt{5}$)	$((C_2 \times C_2 \times C_2 \times C_2 \times C_2) : A_5) : C_2$	$2^8 \cdot 3 \cdot 5$
$10, 5, 5, 2, 1, 1$ ($\sqrt{5}$)	$((C_2 \times C_2 \times C_2 \times C_2 \times C_2) : A_5) : C_2$	$2^8 \cdot 3 \cdot 5$
$10, 5, 4, 2, 2, 1$ (cubic)	$(S_8 \wr S_3) \cap A_{24}$	$2^{21} \cdot 3^7 \cdot 5^3 \cdot 7^3$
$10, 5, 4, 2, 2, 1$ (cubic)	$(S_8 \wr S_3) \cap A_{24}$	$2^{21} \cdot 3^7 \cdot 5^3 \cdot 7^3$
$10, 5, 4, 2, 2, 1$ (cubic)	$(S_8 \wr S_3) \cap A_{24}$	$2^{21} \cdot 3^7 \cdot 5^3 \cdot 7^3$
$10, 5, 4, 2, 2, 1$ (cubic)	$(S_8 \wr S_3) \cap A_{24}$	$2^{19} \cdot 3^7 \cdot 5^3 \cdot 7^3$
$10, 5, 3, 2, 2, 2$ ($\mathbb{Q}$)	$A_8 \wr S_3$	$2^{19} \cdot 3^7 \cdot 5^3 \cdot 7^3$
$10, 4, 4, 3, 2, 1$ ($\mathbb{Q}$)	$(S_8 \wr S_3) \cap A_{24}$	$2^{21} \cdot 3^7 \cdot 5^3 \cdot 7^3$
$10, 4, 3, 3, 2, 2$ ($\mathbb{Q}$)	$(S_8 \wr S_3) \cap A_{24}$	$2^{21} \cdot 3^7 \cdot 5^3 \cdot 7^3$
$9, 9, 2, 2, 1, 1$ ($\mathbb{Q}$)	t24n16559	$2^{10} \cdot 3^4$
$9, 9, 2, 2, 1, 1$ (cubic)	t24n24970	$2^{21} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11$
$9, 9, 2, 2, 1, 1$ (cubic)	t24n24970	$2^{21} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11$
$9, 9, 2, 2, 1, 1$ (cubic)	t24n24970	$2^{21} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11$
$9, 8, 3, 2, 1, 1$ (cubic)	A_{24}	$24!/2$
$9, 8, 3, 2, 1, 1$ (cubic)	A_{24}	$24!/2$
$9, 8, 3, 2, 1, 1$ (cubic)	A_{24}	$24!/2$
$9, 7, 5, 1, 1, 1$ ($\mathbb{Q}$)	A_{24}	$24!/2$
$9, 7, 4, 2, 1, 1$ ($\sqrt{-7}$)	A_{24}	$24!/2$
$9, 7, 4, 2, 1, 1$ ($\sqrt{-7}$)	A_{24}	$24!/2$

Partition (K3 surface)	Cartographic group	Cartographic group order
$9, 7, 3, 2, 2, 1$ (cubic)	A_{24}	$24!/2$
$9, 7, 3, 2, 2, 1$ (cubic)	A_{24}	$24!/2$
$9, 7, 3, 2, 2, 1$ (cubic)	A_{24}	$24!/2$
$9, 6, 5, 2, 1, 1$ (cubic)	A_{24}	$24!/2$
$9, 6, 5, 2, 1, 1$ (cubic)	A_{24}	$24!/2$
$9, 6, 5, 2, 1, 1$ (cubic)	A_{24}	$24!/2$
$9, 6, 4, 3, 1, 1 \, (\mathbb{Q})$	$(A_6 \wr A_4) : C_2$	$2^{15} \cdot 3^9 \cdot 5^4$
$9, 6, 3, 3, 2, 1$ (cubic)	$(A_6 \wr A_4) : C_2$	$2^{15} \cdot 3^9 \cdot 5^4$
$9, 6, 3, 3, 2, 1$ (cubic)	$(A_6 \wr A_4) : C_2$	$2^{15} \cdot 3^9 \cdot 5^4$
$9, 6, 3, 3, 2, 1$ (cubic)	$(A_6 \wr A_4) : C_2$	$2^{15} \cdot 3^9 \cdot 5^4$
$9, 5, 5, 2, 2, 1 \, (\mathbb{Q})$	A_{24}	$24!/2$
$9, 5, 4, 3, 2, 1$ (cubic)	A_{24}	$24!/2$
$9, 5, 4, 3, 2, 1$ (cubic)	A_{24}	$24!/2$
$9, 5, 4, 3, 2, 1$ (cubic)	A_{24}	$24!/2$
$9, 5, 3, 3, 3, 1 \, (\mathbb{Q})$	$A_6 \wr A_4$	$2^{14} \cdot 3^9 \cdot 5^4$
$9, 4, 3, 3, 3, 2 \, (\mathbb{Q})$	$A_6 \wr A_4$	$2^{14} \cdot 3^9 \cdot 5^4$
$8, 8, 4, 2, 1, 1 \, (\mathbb{Q})$	$(((C_4 \times C_2) : C_4) : C_3) : C_2$	$2^6 \cdot 3$
$8, 8, 3, 3, 1, 1 \, (\mathbb{Q})$	t24n14598 (Index 2 in t8n14 $\wr S_3$)	$2^9 \cdot 3^4$
$8, 8, 3, 3, 1, 1 \, (\sqrt{-2})$	t24n24970	$2^{21} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11$
$8, 8, 3, 3, 1, 1 \, (\sqrt{-2})$	t24n24970	$2^{21} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11$
$8, 8, 2, 2, 2, 2 \, (\mathbb{Q})$	$((C_4 \times C_4) : C_3) : C_2$	$2^5 \cdot 3$
$8, 7, 6, 1, 1, 1 \, (\sqrt{-3})$	A_{24}	$24!/2$
$8, 7, 6, 1, 1, 1 \, (\sqrt{-3})$	A_{24}	$24!/2$
$8, 7, 5, 2, 1, 1$ (cubic)	A_{24}	$24!/2$
$8, 7, 5, 2, 1, 1$ (cubic)	A_{24}	$24!/2$
$8, 7, 5, 2, 1, 1$ (cubic)	A_{24}	$24!/2$
$8, 7, 4, 3, 1, 1 \, (\sqrt{-6})$	A_{24}	$24!/2$
$8, 7, 4, 3, 1, 1 \, (\sqrt{-6})$	A_{24}	$24!/2$
$8, 7, 4, 2, 2, 1 \, (\sqrt{-7})$	$(E_8 : L(7)) \wr S_3 = \mathrm{ASL}(3, 2) \wr S_3$	$2^{19} \cdot 3^4 \cdot 7^3$
$8, 7, 4, 2, 2, 1 \, (\sqrt{-7})$	$(E_8 : L(7)) \wr S_3 = \mathrm{ASL}(3, 2) \wr S_3$	$2^{19} \cdot 3^4 \cdot 7^3$
$8, 7, 3, 3, 2, 1 \, (\mathbb{Q})$	A_{24}	$24!/2$
$8, 6, 6, 2, 1, 1 \, (\sqrt{-3})$	$(S_8 \wr S_3) \cap A_{24}$	$2^{21} \cdot 3^7 \cdot 5^3 \cdot 7^3$
$8, 6, 6, 2, 1, 1 \, (\sqrt{-3})$	$(S_8 \wr S_3) \cap A_{24}$	$2^{21} \cdot 3^7 \cdot 5^3 \cdot 7^3$
$8, 6, 5, 3, 1, 1 \, (\sqrt{5})$	A_{24}	$24!/2$
$8, 6, 5, 3, 1, 1 \, (\sqrt{5})$	A_{24}	$24!/2$
$8, 6, 5, 2, 2, 1 \, (\mathbb{Q})$	$(S_8 \wr S_3) \cap A_{24}$	$2^{21} \cdot 3^7 \cdot 5^3 \cdot 7^3$
$8, 6, 4, 3, 2, 1 \, (\sqrt{2})$	$(S_8 \wr S_3) \cap A_{24}$	$2^{21} \cdot 3^7 \cdot 5^3 \cdot 7^3$
$8, 6, 4, 3, 2, 1 \, (\sqrt{2})$	$(S_8 \wr S_3) \cap A_{24}$	$2^{21} \cdot 3^7 \cdot 5^3 \cdot 7^3$
$8, 6, 4, 2, 2, 2 \, (\mathbb{Q})$	t24n22747 $= 2^{12} \wr$ t18n217	$2^{15} \cdot 3^4$
$8, 5, 4, 3, 3, 1 \, (\sqrt{10})$	A_{24}	$24!/2$
$8, 5, 4, 3, 3, 1 \, (\sqrt{10})$	A_{24}	$24!/2$
$8, 5, 4, 3, 2, 2 \, (\mathbb{Q})$	$A_8 \wr S_3$	$2^{19} \cdot 3^7 \cdot 5^3 \cdot 7^3$
$8, 4, 4, 4, 3, 1 \, (\mathbb{Q})$	t24n24710 $= 2^{12} : $ t18n762	$2^{17} \cdot 3^7$
$8, 4, 4, 4, 2, 2 \, (\mathbb{Q})$	$(((C_4 \times C_2) : C_4) : C_3) : C_2$	$2^6 \cdot 3$
$7, 7, 7, 1, 1, 1 \, (\mathbb{Q})$	$\mathrm{PSL}(3, 2)$	$2^3 \cdot 3 \cdot 7$
$7, 7, 5, 3, 1, 1 \, (\sqrt{21})$	A_{24}	$24!/2$
$7, 7, 5, 3, 1, 1 \, (\sqrt{21})$	A_{24}	$24!/2$
$7, 7, 4, 4, 1, 1 \, (\sqrt{-7})$	t24n24970	$2^{21} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11$
$7, 7, 4, 4, 1, 1 \, (\sqrt{-7})$	t24n24970	$2^{21} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11$
$7, 7, 3, 3, 2, 2 \, (\sqrt{7})$	t24n24970	$2^{21} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11$
$7, 7, 3, 3, 2, 2 \, (\sqrt{7})$	t24n24970	$2^{21} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11$

Partition (K3 surface)	Cartographic group	Cartographic group order
$7, 6, 6, 2, 2, 1$ ($\mathbb{Q}$)	$L(8) \wr S_3 = \mathrm{PSL}(2, 7) \wr S_3$	$2^{10} \cdot 3^4 \cdot 7^3$
$7, 6, 5, 4, 1, 1$ ($\mathbb{Q}$)	A_{24}	$24!/2$
$7, 6, 5, 3, 2, 1$ (cubic)	A_{24}	$24!/2$
$7, 6, 5, 3, 2, 1$ (cubic)	A_{24}	$24!/2$
$7, 6, 5, 3, 2, 1$ (cubic)	A_{24}	$24!/2$
$7, 6, 4, 4, 2, 1$ ($\mathbb{Q}$)	$A_8 \wr S_3$	$2^{19} \cdot 3^7 \cdot 5^3 \cdot 7^3$
$7, 5, 5, 4, 2, 1$ ($\sqrt{2}$)	A_{24}	$24!/2$
$7, 5, 5, 4, 2, 1$ ($\sqrt{2}$)	A_{24}	$24!/2$
$7, 5, 5, 3, 2, 2$ ($\mathbb{Q}$)	A_{24}	$24!/2$
$7, 5, 4, 4, 3, 1$ ($\mathbb{Q}$)	A_{24}	$24!/2$
$7, 5, 4, 3, 3, 2$ ($\mathbb{Q}$)	A_{24}	$24!/2$
$6, 6, 6, 4, 1, 1$ ($\mathbb{Q}$)	t24n12261	$2^{11} \cdot 3^2$
$6, 6, 6, 2, 2, 2$ ($\mathbb{Q}$)	$A_4 \times S_3$	72
$6, 6, 5, 5, 1, 1$ ($\sqrt{3}$)	t24n24970	$2^{21} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11$
$6, 6, 5, 5, 1, 1$ ($\sqrt{3}$)	t24n24970	$2^{21} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11$
$6, 6, 5, 4, 2, 1$ ($\mathbb{Q}$)	$(S_8 \wr S_3) \cap A_{24}$	$2^{21} \cdot 3^7 \cdot 5^3 \cdot 7^3$
$6, 6, 5, 3, 3, 1$ ($\mathbb{Q}$)	$L(6) \wr A_4 = \mathrm{PSL}(2, 5) \wr A_4$	$2^{10} \cdot 3^5 \cdot 5^4$
$6, 6, 4, 4, 2, 2$ ($\mathbb{Q}$)	$(((C_2 \times C_2((C_2 \times C_2 \times C_2 \times C_2) : C_3)) : C_3) : C_3) : C_2$	$2^7 \cdot 3^3$
$6, 6, 4, 3, 3, 2$ ($\mathbb{Q}$)	t24n12260	$2^{11} \cdot 3^2$
$6, 6, 3, 3, 3, 3$ ($\mathbb{Q}$)	$((C_4 \times C_4) : C_3) : C_2$	$2^5 \cdot 3$
$6, 5, 4, 4, 3, 2$ ($\mathbb{Q}$)	$A_8 \wr S_3$	$2^{19} \cdot 3^7 \cdot 5^3 \cdot 7^3$
$5, 5, 5, 5, 2, 2$ ($\mathbb{Q}$)	$((C_2 \times C_2 \times C_2 \times C_2 \times C_2) : A_5) : C_2$	$2^8 \cdot 3 \cdot 5$
$5, 5, 4, 4, 3, 3$ ($\mathbb{Q}$)	t24n24970	$2^{21} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11$
$4, 4, 4, 4, 4, 4$ ($\mathbb{Q}$)	S_4	$4!$

3.2. *Generators for the modular subgroups*

There is a so-called *Reidemeister–Schreier process* (discussed in [**9**]) which enables one to compute presentations of subgroups H of finite index in a group G defined by finite presentations. We can use this process to find the presentations of all of the modular subgroups of interest. In our case, G is $\Gamma = \mathrm{PSL}(2, \mathbb{Z})$, the presentation of which is $\langle x, y \mid x^2, y^3 \rangle$. For all our conjugacy classes of modular subgroups, the Reidemeister–Schreier process yields the following group presentation, when the permutation data (here σ_0, σ_1) are input in the manner detailed in [**9**]:

$$\langle h, i, j, k, l | \!\!-\!\!- \rangle . \tag{3.2}$$

Thus, in every case we have a free group on five generators. Considering the specific example of $\Gamma(4)$ for illustration, this should not be surprising, since the modular curve $X(4)$ is $\mathbb{P}^1$ with six cusps and no elliptic points, so its fundamental group is the free group on five generators.

We can find the generators for a representative of all of the conjugacy classes of subgroups of interest, which we now know to be free groups on five generators, using GAP[†]. First, we use the permutation data σ_0, σ_1 obtained from each of the Schreier coset graphs (in turn obtained from each of the *dessins*) to find the group homomorphism by images between G and H. We then use this to define the H in question. Finally, we use the GAP command IsomorphismFpGroup(G), which returns an isomorphism from the given H to a finitely presented group isomorphic to H. This function first chooses a set of generators of H and then computes a presentation in terms of these generators.

To give an example, consider the partition $4, 4, 4, 4, 4, 4$, which corresponds to the conjugacy class of subgroups $\Gamma(4)$. We can find a set of generators as 2×2 matrices for a representative of this class of subgroups by implementing the following code in GAP:

[†]We thank Alexander Hulpke for guidance here.

```
gap> f:=FreeGroup(''x'',''y'');
<free group on the generators [ x, y ]>
gap>gamma:=f/[f.1^2,f.2^3];
<fp group on the generators [ x, y ]>
gap> hom:=GroupHomomorphismByImages(gamma,Group(sigma_0,sigma_1),
>GeneratorsOfGroup(gamma),[sigma_0,sigma_1]);
[x, y] ->
[(1,10)(2,4)(3,24)(5,7)(6,21)(8,12)(9,18)(11,14)(13,16)(15,23)
(17,19)(20,22), (1,2,3)(4,5,6)(7,8,9)(10,11,12)(13,14,15)
(16,17,18)(19,20,21)(22,23,24)]
gap> gamma4:=PreImage(hom,Stabilizer(Image(hom),1));
Group(<fp, no generators known>)
gap> iso:=IsomorphismFpGroup(gamma4);
[ <[ [ 1, 1 ] ]|y*x*y*x*y*x^-1*y*x^-1>,
<[ [ 2, 1 ] ]|y^-1*x*y^-1*x*y^-1*x^ -1*y^-1*x^-1>,
<[ [ 3, 1 ] ]|y*x*y^-1*x*y^-1*x^-1*y^-1*x^-1*y>,
<[ [ 4, 1 ] ]|x*y*x*y^-1*x*y^-1*x^-1*y^-1*x^-1*y*x^-1>,
<[ [ 5, 1 ] ]|y*x*y^-1*x*y*x*y^-1*x^-1*y*x^-1*y^-1*x^-1> ]
-> [ F1, F2, F3, F4, F5 ]
```

Here, the only inputs in each case are the permutation data σ_0, σ_1. Once the output has been obtained, we will see the five generators $[F1, F2, F3, F4, F5]$, as indicated in the last line of the output, as functions of x and y, i.e. the generators of Γ. Now, returning to the matrices (2.4), the only thing left to do is to multiply together the matrices and their inverses as indicated. This will produce the generators of each H, as desired. Note that one can achieve the same results here in Magma though use of the Rewrite command[†] [**4**].

3.2.1. *Catalogue of generators as explicit 2×2 matrices.* In this section we give the generators for a representative of each of the conjugacy classes of modular subgroups corresponding to the entries in the table at the end of §3.1.1. For each of the 112 K3 surfaces, denoted as a 6-tuple partition of 24 (the notation is as before, we also include the number field of definition for the associated j-function as a Belyi map), we write the 5 matrix generators explicitly.

Partition	Generators
$19, 1, 1, 1, 1, 1$ ($\mathbb{Q}$)	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 5 & 4 \\ -9 & -7 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ -16 & -5 \end{pmatrix} \begin{pmatrix} 18 & -5 \\ 25 & 0 \end{pmatrix} \begin{pmatrix} 20 & 9 \\ -49 & -22 \end{pmatrix}$
$18, 2, 1, 1, 1, 1$ ($\mathbb{Q}$)	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 5 & 4 \\ -9 & -7 \end{pmatrix} \begin{pmatrix} -5 & -2 \\ 18 & 7 \end{pmatrix} \begin{pmatrix} 5 & 1 \\ -36 & -7 \end{pmatrix} \begin{pmatrix} 17 & 4 \\ -81 & -19 \end{pmatrix}$
$18, 2, 1, 1, 1, 1$ ($\sqrt{-3}$)	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 5 & 4 \\ -9 & -7 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ -16 & -5 \end{pmatrix} \begin{pmatrix} -19 & -8 \\ 50 & 21 \end{pmatrix} \begin{pmatrix} 43 & 16 \\ -121 & -45 \end{pmatrix}$
$18, 2, 1, 1, 1, 1$ ($\sqrt{-3}$)	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 5 & 4 \\ -9 & -7 \end{pmatrix} \begin{pmatrix} 9 & 4 \\ -25 & -11 \end{pmatrix} \begin{pmatrix} -7 & -2 \\ 32 & 9 \end{pmatrix} \begin{pmatrix} 5 & 1 \\ -36 & -7 \end{pmatrix}$
$17, 3, 1, 1, 1, 1$ ($\mathbb{Q}$)	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 5 & 4 \\ -9 & -7 \end{pmatrix} \begin{pmatrix} -8 & -3 \\ 27 & 10 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ -25 & -6 \end{pmatrix} \begin{pmatrix} 20 & 9 \\ -49 & -22 \end{pmatrix}$

[†]We thank an anonymous referee for this observation.

Partition	Generators
$17, 2, 2, 1, 1, 1\ (\sqrt{17})$	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -3 & -2 \\ 8 & 5 \end{pmatrix} \begin{pmatrix} -7 & -2 \\ 32 & 9 \end{pmatrix} \begin{pmatrix} 6 & 1 \\ -49 & -8 \end{pmatrix} \begin{pmatrix} 21 & 4 \\ -121 & -23 \end{pmatrix}$
$17, 2, 2, 1, 1, 1\ (\sqrt{17})$	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -3 & -2 \\ 8 & 5 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ -25 & -6 \end{pmatrix} \begin{pmatrix} -27 & -8 \\ 98 & 29 \end{pmatrix} \begin{pmatrix} 59 & 16 \\ -225 & -61 \end{pmatrix}$
$16, 4, 1, 1, 1, 1\ (\mathbb{Q})$	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 7 & 4 \\ -16 & -9 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ -16 & -5 \end{pmatrix} \begin{pmatrix} 11 & 9 \\ -16 & -13 \end{pmatrix} \begin{pmatrix} 23 & 9 \\ -64 & -25 \end{pmatrix}$
$16, 3, 2, 1, 1, 1\ (\mathbb{Q})$	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -3 & -2 \\ 8 & 5 \end{pmatrix} \begin{pmatrix} -11 & -3 \\ 48 & 13 \end{pmatrix} \begin{pmatrix} 5 & 1 \\ -36 & -7 \end{pmatrix} \begin{pmatrix} 29 & 9 \\ -100 & -31 \end{pmatrix}$
$16, 3, 2, 1, 1, 1\ (\sqrt{-2})$	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 5 & 4 \\ -9 & -7 \end{pmatrix} \begin{pmatrix} 9 & 4 \\ -25 & -11 \end{pmatrix} \begin{pmatrix} -3 & -1 \\ 22 & 7 \end{pmatrix} \begin{pmatrix} -11 & -3 \\ 48 & 13 \end{pmatrix}$
$16, 3, 2, 1, 1, 1\ (\sqrt{-2})$	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 5 & 4 \\ -9 & -7 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ -16 & -5 \end{pmatrix} \begin{pmatrix} -13 & -6 \\ 37 & 17 \end{pmatrix} \begin{pmatrix} -29 & -12 \\ 75 & 31 \end{pmatrix}$
$16, 2, 2, 2, 1, 1\ (\mathbb{Q})$	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -3 & -2 \\ 8 & 5 \end{pmatrix} \begin{pmatrix} -7 & -2 \\ 32 & 9 \end{pmatrix} \begin{pmatrix} -11 & -2 \\ 72 & 13 \end{pmatrix} \begin{pmatrix} 7 & 1 \\ -64 & -9 \end{pmatrix}$
$15, 5, 1, 1, 1, 1\ (\sqrt{-15})$	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 5 & 4 \\ -9 & -7 \end{pmatrix} \begin{pmatrix} 9 & 4 \\ -25 & -11 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ -19 & -6 \end{pmatrix} \begin{pmatrix} 17 & 4 \\ -81 & -19 \end{pmatrix}$
$15, 5, 1, 1, 1, 1\ (\sqrt{-15})$	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 5 & 4 \\ -9 & -7 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ -16 & -5 \end{pmatrix} \begin{pmatrix} 11 & 5 \\ -31 & -14 \end{pmatrix} \begin{pmatrix} 64 & 25 \\ -169 & -66 \end{pmatrix}$
$15, 4, 2, 1, 1, 1\ (\sqrt{-15})$	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ -14 & -9 \end{pmatrix} \begin{pmatrix} -5 & -2 \\ 18 & 7 \end{pmatrix} \begin{pmatrix} 34 & 25 \\ -49 & -36 \end{pmatrix} \begin{pmatrix} 19 & 16 \\ -25 & -21 \end{pmatrix}$
$15, 4, 2, 1, 1, 1\ (\sqrt{-15})$	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 5 & 4 \\ -14 & -11 \end{pmatrix} \begin{pmatrix} 7 & 4 \\ -16 & -9 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ -25 & -6 \end{pmatrix} \begin{pmatrix} 13 & 4 \\ -49 & -15 \end{pmatrix}$
$15, 3, 3, 1, 1, 1\ (\mathbb{Q})$	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -5 & -3 \\ 12 & 7 \end{pmatrix} \begin{pmatrix} 4 & 3 \\ -15 & -11 \end{pmatrix} \begin{pmatrix} 5 & 1 \\ -36 & -7 \end{pmatrix} \begin{pmatrix} 17 & 4 \\ -81 & -19 \end{pmatrix}$
$15, 3, 2, 2, 1, 1\ (\mathbb{Q})$	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 5 & 4 \\ -9 & -7 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -11 & -5 \end{pmatrix} \begin{pmatrix} -19 & -6 \\ 73 & 23 \end{pmatrix} \begin{pmatrix} -41 & -12 \\ 147 & 43 \end{pmatrix}$
$15, 3, 2, 2, 1, 1\ (\sqrt{-15})$	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -3 & -2 \\ 8 & 5 \end{pmatrix} \begin{pmatrix} 13 & 4 \\ -49 & -15 \end{pmatrix} \begin{pmatrix} -4 & -1 \\ 33 & 8 \end{pmatrix} \begin{pmatrix} -14 & -3 \\ 75 & 16 \end{pmatrix}$
$15, 3, 2, 2, 1, 1\ (\sqrt{-15})$	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -3 & -2 \\ 8 & 5 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ -25 & -6 \end{pmatrix} \begin{pmatrix} -19 & -6 \\ 73 & 23 \end{pmatrix} \begin{pmatrix} -41 & -12 \\ 147 & 43 \end{pmatrix}$
$14, 6, 1, 1, 1, 1\ (\sqrt{-3})$	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ -11 & -7 \end{pmatrix} \begin{pmatrix} 9 & 4 \\ -25 & -11 \end{pmatrix} \begin{pmatrix} 34 & 25 \\ -49 & -36 \end{pmatrix} \begin{pmatrix} 19 & 16 \\ -25 & -21 \end{pmatrix}$
$14, 6, 1, 1, 1, 1\ (\sqrt{-3})$	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 4 & 3 \\ -11 & -8 \end{pmatrix} \begin{pmatrix} 14 & 9 \\ -25 & -16 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ -25 & -6 \end{pmatrix} \begin{pmatrix} 13 & 4 \\ -49 & -15 \end{pmatrix}$

Partition	Generators
$14, 5, 2, 1, 1, 1$ ($\mathbb{Q}$)	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -5 & -2 \\ 18 & 7 \end{pmatrix} \begin{pmatrix} 5 & 3 \\ -22 & -13 \end{pmatrix} \begin{pmatrix} 11 & 9 \\ -16 & -13 \end{pmatrix} \begin{pmatrix} 39 & 25 \\ -64 & -41 \end{pmatrix}$
$14, 5, 2, 1, 1, 1$ (cubic)	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 5 & 4 \\ -9 & -7 \end{pmatrix} \begin{pmatrix} -5 & -2 \\ 18 & 7 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ -29 & -7 \end{pmatrix} \begin{pmatrix} 21 & 4 \\ -121 & -23 \end{pmatrix}$
$14, 5, 2, 1, 1, 1$ (cubic)	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -3 & -2 \\ 8 & 5 \end{pmatrix} \begin{pmatrix} 13 & 4 \\ -49 & -15 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ -29 & -7 \end{pmatrix} \begin{pmatrix} 21 & 4 \\ -121 & -23 \end{pmatrix}$
$14, 5, 2, 1, 1, 1$ (cubic)	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -3 & -2 \\ 8 & 5 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ -25 & -6 \end{pmatrix} \begin{pmatrix} 16 & 5 \\ -61 & -19 \end{pmatrix} \begin{pmatrix} 89 & 25 \\ -324 & -91 \end{pmatrix}$
$14, 4, 3, 1, 1, 1$ ($\mathbb{Q}$)	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 7 & 4 \\ -16 & -9 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ -16 & -5 \end{pmatrix} \begin{pmatrix} 11 & 8 \\ -29 & -21 \end{pmatrix} \begin{pmatrix} 19 & 16 \\ -25 & -21 \end{pmatrix}$
$14, 4, 2, 2, 1, 1$ ($\sqrt{-7}$)	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -3 & -2 \\ 8 & 5 \end{pmatrix} \begin{pmatrix} 13 & 4 \\ -62 & -19 \end{pmatrix} \begin{pmatrix} 15 & 4 \\ -64 & -17 \end{pmatrix} \begin{pmatrix} 5 & 1 \\ -36 & -7 \end{pmatrix}$
$14, 4, 2, 2, 1, 1$ ($\sqrt{-7}$)	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -3 & -2 \\ 8 & 5 \end{pmatrix} \begin{pmatrix} 7 & 2 \\ -46 & -13 \end{pmatrix} \begin{pmatrix} -9 & -2 \\ 50 & 11 \end{pmatrix} \begin{pmatrix} 29 & 9 \\ -100 & -31 \end{pmatrix}$
$14, 3, 3, 2, 1, 1$ ($\mathbb{Q}$)	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -5 & -3 \\ 12 & 7 \end{pmatrix} \begin{pmatrix} 11 & 9 \\ -16 & -13 \end{pmatrix} \begin{pmatrix} -3 & -1 \\ 22 & 7 \end{pmatrix} \begin{pmatrix} -11 & -3 \\ 48 & 13 \end{pmatrix}$
$14, 3, 3, 2, 1, 1$ ($\mathbb{Q}$)	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -5 & -3 \\ 12 & 7 \end{pmatrix} \begin{pmatrix} 4 & 3 \\ -15 & -11 \end{pmatrix} \begin{pmatrix} -9 & -2 \\ 50 & 11 \end{pmatrix} \begin{pmatrix} 6 & 1 \\ -49 & -8 \end{pmatrix}$
$14, 3, 2, 2, 2, 1$ ($\mathbb{Q}$)	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -3 & -2 \\ 8 & 5 \end{pmatrix} \begin{pmatrix} -7 & -2 \\ 32 & 9 \end{pmatrix} \begin{pmatrix} -5 & -1 \\ 46 & 9 \end{pmatrix} \begin{pmatrix} -17 & -3 \\ 108 & 19 \end{pmatrix}$
$13, 7, 1, 1, 1, 1$ ($\mathbb{Q}$)	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -8 & -5 \\ 21 & 13 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ -16 & -5 \end{pmatrix} \begin{pmatrix} 11 & 9 \\ -16 & -13 \end{pmatrix} \begin{pmatrix} 20 & 9 \\ -49 & -22 \end{pmatrix}$
$13, 6, 2, 1, 1, 1$ ($\sqrt{-3}$)	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -3 & -2 \\ 8 & 5 \end{pmatrix} \begin{pmatrix} 10 & 3 \\ -47 & -14 \end{pmatrix} \begin{pmatrix} 5 & 1 \\ -36 & -7 \end{pmatrix} \begin{pmatrix} 32 & 9 \\ -121 & -34 \end{pmatrix}$
$13, 6, 2, 1, 1, 1$ ($\sqrt{-3}$)	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -3 & -2 \\ 8 & 5 \end{pmatrix} \begin{pmatrix} 7 & 2 \\ -39 & -11 \end{pmatrix} \begin{pmatrix} 17 & 4 \\ -81 & -19 \end{pmatrix} \begin{pmatrix} 29 & 9 \\ -100 & -31 \end{pmatrix}$
$13, 5, 3, 1, 1, 1$ ($\mathbb{Q}$)	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -5 & -3 \\ 12 & 7 \end{pmatrix} \begin{pmatrix} 11 & 9 \\ -16 & -13 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ -19 & -6 \end{pmatrix} \begin{pmatrix} 17 & 4 \\ -81 & -19 \end{pmatrix}$
$13, 5, 2, 2, 1, 1$ ($\sqrt{65}$)	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 5 & 4 \\ -9 & -7 \end{pmatrix} \begin{pmatrix} -7 & -2 \\ 32 & 9 \end{pmatrix} \begin{pmatrix} -19 & -8 \\ 50 & 21 \end{pmatrix} \begin{pmatrix} 8 & 3 \\ -43 & -16 \end{pmatrix}$
$13, 5, 2, 2, 1, 1$ ($\sqrt{65}$)	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -3 & -2 \\ 8 & 5 \end{pmatrix} \begin{pmatrix} -7 & -2 \\ 32 & 9 \end{pmatrix} \begin{pmatrix} 5 & 1 \\ -41 & -8 \end{pmatrix} \begin{pmatrix} 25 & 4 \\ -169 & -27 \end{pmatrix}$
$13, 4, 3, 2, 1, 1$ ($\mathbb{Q}$)	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 5 & 4 \\ -9 & -7 \end{pmatrix} \begin{pmatrix} -8 & -3 \\ 27 & 10 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -17 & -8 \end{pmatrix} \begin{pmatrix} 12 & 5 \\ -53 & -22 \end{pmatrix}$

Partition	Generators
13, 3, 3, 2, 2, 1 ($\mathbb{Q}$)	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -2 & -1 \\ 13 & 6 \end{pmatrix} \begin{pmatrix} -8 & -3 \\ 27 & 10 \end{pmatrix} \begin{pmatrix} -7 & -6 \\ 13 & 11 \end{pmatrix} \begin{pmatrix} -17 & -12 \\ 27 & 19 \end{pmatrix}$
12, 7, 2, 1, 1, 1 ($\sqrt{-3}$)	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -9 & -4 \end{pmatrix} \begin{pmatrix} -8 & -7 \\ 15 & 13 \end{pmatrix} \begin{pmatrix} -24 & -17 \\ 41 & 29 \end{pmatrix} \begin{pmatrix} 39 & 25 \\ -64 & -41 \end{pmatrix}$
12, 7, 2, 1, 1, 1 ($\sqrt{-3}$)	$\begin{pmatrix} -1 & -2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 3 & -4 \end{pmatrix} \begin{pmatrix} -3 & 1 \\ -4 & 1 \end{pmatrix} \begin{pmatrix} -5 & 16 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} -11 & 25 \\ -4 & 9 \end{pmatrix}$
12, 6, 3, 1, 1, 1 ($\mathbb{Q}$)	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 5 & 4 \\ -9 & -7 \end{pmatrix} \begin{pmatrix} -8 & -3 \\ 27 & 10 \end{pmatrix} \begin{pmatrix} -2 & -1 \\ 15 & 7 \end{pmatrix} \begin{pmatrix} 17 & 4 \\ -81 & -19 \end{pmatrix}$
12, 6, 2, 2, 1, 1 ($\mathbb{Q}$)	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -5 & -2 \\ 18 & 7 \end{pmatrix} \begin{pmatrix} -11 & -8 \\ 18 & 13 \end{pmatrix} \begin{pmatrix} -7 & -4 \\ 30 & 17 \end{pmatrix} \begin{pmatrix} 5 & 1 \\ -36 & -7 \end{pmatrix}$
12, 6, 2, 2, 1, 1 ($\mathbb{Q}$)	$\begin{pmatrix} -1 & -2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 4 & -9 \end{pmatrix} \begin{pmatrix} 9 & 4 \\ -25 & -11 \end{pmatrix} \begin{pmatrix} -5 & 16 \\ -1 & 3 \end{pmatrix}$
12, 5, 3, 2, 1, 1 (quartic)	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -5 & -3 \\ 12 & 7 \end{pmatrix} \begin{pmatrix} -6 & -5 \\ 23 & 19 \end{pmatrix} \begin{pmatrix} -10 & -7 \\ 33 & 23 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ -25 & -6 \end{pmatrix}$
12, 5, 3, 2, 1, 1 (quartic)	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -5 & -3 \\ 12 & 7 \end{pmatrix} \begin{pmatrix} -4 & -3 \\ 23 & 17 \end{pmatrix} \begin{pmatrix} -7 & -2 \\ 32 & 9 \end{pmatrix} \begin{pmatrix} 19 & 16 \\ -25 & -21 \end{pmatrix}$
12, 5, 3, 2, 1, 1 (quartic)	$\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -4 & -3 \end{pmatrix} \begin{pmatrix} -7 & 9 \\ -4 & 5 \end{pmatrix} \begin{pmatrix} 6 & -13 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 10 & -27 \\ 3 & -8 \end{pmatrix}$
12, 5, 3, 2, 1, 1 (quartic)	$\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -4 & -3 \end{pmatrix} \begin{pmatrix} -4 & 9 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 11 & -13 \\ 6 & -7 \end{pmatrix} \begin{pmatrix} 19 & -27 \\ 12 & -17 \end{pmatrix}$
12, 5, 2, 2, 2, 1 ($\mathbb{Q}$)	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -3 & -2 \\ 8 & 5 \end{pmatrix} \begin{pmatrix} -9 & -2 \\ 50 & 11 \end{pmatrix} \begin{pmatrix} -27 & -8 \\ 98 & 29 \end{pmatrix} \begin{pmatrix} 11 & 3 \\ -70 & -19 \end{pmatrix}$
12, 4, 4, 2, 1, 1 ($\mathbb{Q}$)	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 7 & 4 \\ -16 & -9 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ -16 & -5 \end{pmatrix} \begin{pmatrix} -7 & -6 \\ 20 & 17 \end{pmatrix} \begin{pmatrix} -17 & -12 \\ 44 & 31 \end{pmatrix}$
12, 4, 3, 3, 1, 1 ($\mathbb{Q}$)	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -5 & -3 \\ 12 & 7 \end{pmatrix} \begin{pmatrix} 7 & 5 \\ -24 & -17 \end{pmatrix} \begin{pmatrix} -5 & -4 \\ 24 & 19 \end{pmatrix} \begin{pmatrix} 5 & 1 \\ -36 & -7 \end{pmatrix}$
12, 4, 3, 3, 1, 1 ($\mathbb{Q}$)	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -9 & -4 \end{pmatrix} \begin{pmatrix} -8 & -7 \\ 15 & 13 \end{pmatrix} \begin{pmatrix} 23 & 16 \\ -36 & -25 \end{pmatrix} \begin{pmatrix} -44 & -27 \\ 75 & 46 \end{pmatrix}$
12, 4, 3, 2, 2, 1 ($\mathbb{Q}$)	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -3 & -2 \\ 8 & 5 \end{pmatrix} \begin{pmatrix} -11 & -3 \\ 48 & 13 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ -28 & -9 \end{pmatrix} \begin{pmatrix} 17 & 5 \\ -92 & -27 \end{pmatrix}$
12, 4, 3, 2, 2, 1 ($\sqrt{-3}$)	$\begin{pmatrix} -1 & 0 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} -1 & -2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} -9 & 14 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 7 & -18 \\ 2 & -5 \end{pmatrix} \begin{pmatrix} -13 & 16 \\ -9 & 11 \end{pmatrix}$
12, 4, 3, 2, 2, 1 ($\sqrt{-3}$)	$\begin{pmatrix} -1 & 0 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} -1 & -2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} -11 & 14 \\ -4 & 5 \end{pmatrix} \begin{pmatrix} -9 & 16 \\ -4 & 7 \end{pmatrix} \begin{pmatrix} -5 & 16 \\ -1 & 3 \end{pmatrix}$

Partition	Generators
$12, 3, 3, 3, 2, 1$ ($\mathbb{Q}$)	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -5 & -3 \\ 12 & 7 \end{pmatrix} \begin{pmatrix} 4 & 3 \\ -15 & -11 \end{pmatrix} \begin{pmatrix} -4 & -1 \\ 33 & 8 \end{pmatrix} \begin{pmatrix} -14 & -3 \\ 75 & 16 \end{pmatrix}$
$12, 3, 3, 2, 2, 2$ ($\mathbb{Q}$)	$\begin{pmatrix} -1 & 0 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} -1 & -2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} 7 & -22 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} 13 & -48 \\ 3 & -11 \end{pmatrix}$
$11, 9, 1, 1, 1, 1$ ($\mathbb{Q}$)	$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 9 \\ -1 & -4 \end{pmatrix} \begin{pmatrix} -7 & -13 \\ 6 & 11 \end{pmatrix} \begin{pmatrix} 34 & 49 \\ -25 & -36 \end{pmatrix} \begin{pmatrix} 39 & 64 \\ -25 & -41 \end{pmatrix}$
$11, 8, 2, 1, 1, 1$ (cubic)	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -5 & -2 \\ 18 & 7 \end{pmatrix} \begin{pmatrix} -4 & -3 \\ 19 & 14 \end{pmatrix} \begin{pmatrix} 14 & 9 \\ -25 & -16 \end{pmatrix} \begin{pmatrix} 19 & 16 \\ -25 & -21 \end{pmatrix}$
$11, 8, 2, 1, 1, 1$ (cubic)	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -3 & -2 \\ 17 & 11 \end{pmatrix} \begin{pmatrix} -7 & -4 \\ 23 & 13 \end{pmatrix} \begin{pmatrix} 9 & 4 \\ -25 & -11 \end{pmatrix} \begin{pmatrix} 11 & 9 \\ -16 & -13 \end{pmatrix}$
$11, 8, 2, 1, 1, 1$ (cubic)	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ -16 & -5 \end{pmatrix} \begin{pmatrix} -6 & -5 \\ 17 & 14 \end{pmatrix} \begin{pmatrix} -10 & -7 \\ 23 & 16 \end{pmatrix} \begin{pmatrix} 14 & 9 \\ -25 & -16 \end{pmatrix}$
$11, 7, 3, 1, 1, 1$ (cubic)	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 5 & 4 \\ -9 & -7 \end{pmatrix} \begin{pmatrix} -2 & -1 \\ 13 & 6 \end{pmatrix} \begin{pmatrix} -13 & -5 \\ 47 & 18 \end{pmatrix} \begin{pmatrix} 29 & 9 \\ -100 & -31 \end{pmatrix}$
$11, 7, 3, 1, 1, 1$ (cubic)	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -5 & -3 \\ 17 & 10 \end{pmatrix} \begin{pmatrix} -4 & -3 \\ 19 & 14 \end{pmatrix} \begin{pmatrix} 9 & 4 \\ -25 & -11 \end{pmatrix} \begin{pmatrix} 19 & 16 \\ -25 & -21 \end{pmatrix}$
$11, 7, 3, 1, 1, 1$ (cubic)	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -7 & -5 \\ 17 & 12 \end{pmatrix} \begin{pmatrix} -8 & -3 \\ 27 & 10 \end{pmatrix} \begin{pmatrix} 14 & 9 \\ -25 & -16 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ -25 & -6 \end{pmatrix}$
$11, 7, 2, 2, 1, 1$ ($\sqrt{-7}$)	$\begin{pmatrix} -1 & -2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 3 & -4 \end{pmatrix} \begin{pmatrix} -3 & 1 \\ -4 & 1 \end{pmatrix} \begin{pmatrix} 7 & -18 \\ 2 & -5 \end{pmatrix} \begin{pmatrix} -6 & 25 \\ -1 & 4 \end{pmatrix}$
$11, 7, 2, 2, 1, 1$ ($\sqrt{-7}$)	$\begin{pmatrix} -1 & -2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ 5 & -3 \end{pmatrix} \begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} -4 & 1 \\ -9 & 2 \end{pmatrix} \begin{pmatrix} -5 & 16 \\ -1 & 3 \end{pmatrix}$
$11, 6, 4, 1, 1, 1$ ($\sqrt{33}$)	$\begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} -5 & 4 \\ -4 & 3 \end{pmatrix} \begin{pmatrix} -4 & 9 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 9 & -2 \\ 5 & -1 \end{pmatrix} \begin{pmatrix} -11 & 4 \\ -25 & 9 \end{pmatrix}$
$11, 6, 4, 1, 1, 1$ ($\sqrt{33}$)	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -9 & -4 \end{pmatrix} \begin{pmatrix} -7 & -6 \\ 13 & 11 \end{pmatrix} \begin{pmatrix} 23 & 16 \\ -36 & -25 \end{pmatrix} \begin{pmatrix} 103 & 64 \\ -169 & -105 \end{pmatrix}$
$11, 6, 3, 2, 1, 1$ (cubic)	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -3 & -2 \\ 8 & 5 \end{pmatrix} \begin{pmatrix} -11 & -3 \\ 48 & 13 \end{pmatrix} \begin{pmatrix} -3 & -1 \\ 25 & 8 \end{pmatrix} \begin{pmatrix} 21 & 4 \\ -121 & -23 \end{pmatrix}$
$11, 6, 3, 2, 1, 1$ (cubic)	$\begin{pmatrix} -1 & 0 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} -1 & -2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} -7 & 11 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} -11 & 25 \\ -4 & 9 \end{pmatrix} \begin{pmatrix} -13 & 16 \\ -9 & 11 \end{pmatrix}$
$11, 6, 3, 2, 1, 1$ (cubic)	$\begin{pmatrix} -1 & 0 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} -1 & -2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} -8 & 11 \\ -3 & 4 \end{pmatrix} \begin{pmatrix} -5 & 16 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} -16 & 25 \\ -9 & 14 \end{pmatrix}$
$11, 5, 5, 1, 1, 1$ ($\mathbb{Q}$)	$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 6 & 11 \\ -5 & -9 \end{pmatrix} \begin{pmatrix} -5 & -18 \\ 2 & 7 \end{pmatrix} \begin{pmatrix} 34 & 49 \\ -25 & -36 \end{pmatrix} \begin{pmatrix} 20 & 49 \\ -9 & -22 \end{pmatrix}$

Partition	Generators
11, 5, 4, 2, 1, 1 (cubic)	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 5 & 4 \\ -9 & -7 \end{pmatrix} \begin{pmatrix} -2 & -1 \\ 13 & 6 \end{pmatrix} \begin{pmatrix} 18 & 7 \\ -67 & -26 \end{pmatrix} \begin{pmatrix} 14 & 5 \\ -45 & -16 \end{pmatrix}$
11, 5, 4, 2, 1, 1 (cubic)	$\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -4 & -3 \end{pmatrix} \begin{pmatrix} -9 & 14 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 7 & -18 \\ 2 & -5 \end{pmatrix} \begin{pmatrix} -13 & 16 \\ -9 & 11 \end{pmatrix}$
11, 5, 4, 2, 1, 1 (cubic)	$\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -4 & -3 \end{pmatrix} \begin{pmatrix} -11 & 14 \\ -4 & 5 \end{pmatrix} \begin{pmatrix} -9 & 16 \\ -4 & 7 \end{pmatrix} \begin{pmatrix} -5 & 16 \\ -1 & 3 \end{pmatrix}$
11, 5, 3, 3, 1, 1 ($\sqrt{5}$)	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -5 & -3 \\ 12 & 7 \end{pmatrix} \begin{pmatrix} 4 & 3 \\ -15 & -11 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ -29 & -7 \end{pmatrix} \begin{pmatrix} 21 & 4 \\ -121 & -23 \end{pmatrix}$
11, 5, 3, 3, 1, 1 ($\sqrt{5}$)	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ -16 & -5 \end{pmatrix} \begin{pmatrix} -6 & -5 \\ 17 & 14 \end{pmatrix} \begin{pmatrix} 9 & 5 \\ -20 & -11 \end{pmatrix} \begin{pmatrix} -17 & -12 \\ 27 & 19 \end{pmatrix}$
11, 5, 3, 2, 2, 1 ($\mathbb{Q}$)	$\begin{pmatrix} -1 & 0 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} -1 & -2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} -6 & 19 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} -19 & 81 \\ -4 & 17 \end{pmatrix}$
11, 4, 4, 3, 1, 1 ($\mathbb{Q}$)	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 7 & 4 \\ -16 & -9 \end{pmatrix} \begin{pmatrix} -5 & -4 \\ 19 & 15 \end{pmatrix} \begin{pmatrix} 11 & 8 \\ -29 & -21 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ -25 & -6 \end{pmatrix}$
11, 4, 3, 3, 2, 1 ($\mathbb{Q}$)	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -3 & -2 \\ 8 & 5 \end{pmatrix} \begin{pmatrix} -3 & -1 \\ 25 & 8 \end{pmatrix} \begin{pmatrix} 15 & 4 \\ -64 & -17 \end{pmatrix} \begin{pmatrix} -14 & -3 \\ 75 & 16 \end{pmatrix}$
10, 10, 1, 1, 1, 1 ($\mathbb{Q}$)	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 4 & 3 \\ -15 & -11 \end{pmatrix} \begin{pmatrix} 9 & 4 \\ -25 & -11 \end{pmatrix} \begin{pmatrix} 14 & 9 \\ -25 & -16 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ -25 & -6 \end{pmatrix}$
10, 10, 1, 1, 1, 1 ($\sqrt{5}$)	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 5 & 4 \\ -9 & -7 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -11 & -5 \end{pmatrix} \begin{pmatrix} 32 & 9 \\ -121 & -34 \end{pmatrix} \begin{pmatrix} 29 & 9 \\ -100 & -31 \end{pmatrix}$
10, 10, 1, 1, 1, 1 ($\sqrt{5}$)	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 5 & 4 \\ -14 & -11 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ -16 & -5 \end{pmatrix} \begin{pmatrix} 14 & 9 \\ -25 & -16 \end{pmatrix} \begin{pmatrix} 20 & 9 \\ -49 & -22 \end{pmatrix}$
10, 9, 2, 1, 1, 1 ($\sqrt{5}$)	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 5 & 4 \\ -9 & -7 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -11 & -5 \end{pmatrix} \begin{pmatrix} -27 & -8 \\ 98 & 29 \end{pmatrix} \begin{pmatrix} 59 & 16 \\ -225 & -61 \end{pmatrix}$
10, 9, 2, 1, 1, 1 ($\sqrt{5}$)	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -3 & -2 \\ 8 & 5 \end{pmatrix} \begin{pmatrix} -3 & -1 \\ 22 & 7 \end{pmatrix} \begin{pmatrix} 17 & 4 \\ -81 & -19 \end{pmatrix} \begin{pmatrix} 32 & 9 \\ -121 & -34 \end{pmatrix}$
10, 8, 3, 1, 1, 1 ($\mathbb{Q}$)	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 5 & 4 \\ -14 & -11 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ -16 & -5 \end{pmatrix} \begin{pmatrix} -13 & -8 \\ 31 & 19 \end{pmatrix} \begin{pmatrix} 27 & 16 \\ -49 & -29 \end{pmatrix}$
10, 7, 4, 1, 1, 1 ($\mathbb{Q}$)	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 4 & 3 \\ -15 & -11 \end{pmatrix} \begin{pmatrix} -8 & -5 \\ 21 & 13 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ -25 & -6 \end{pmatrix} \begin{pmatrix} 20 & 9 \\ -49 & -22 \end{pmatrix}$
10, 7, 3, 2, 1, 1 ($\sqrt{21}$)	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 5 & 4 \\ -9 & -7 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -11 & -5 \end{pmatrix} \begin{pmatrix} -19 & -6 \\ 73 & 23 \end{pmatrix} \begin{pmatrix} -41 & -12 \\ 147 & 43 \end{pmatrix}$
10, 7, 3, 2, 1, 1 ($\sqrt{21}$)	$\begin{pmatrix} -3 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} -2 & -3 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -9 & -4 \end{pmatrix} \begin{pmatrix} -5 & 16 \\ -1 & 3 \end{pmatrix}$

Partition	Generators
10, 7, 2, 2, 2, 1 ($\mathbb{Q}$)	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -5 & -2 \\ 18 & 7 \end{pmatrix} \begin{pmatrix} -11 & -8 \\ 18 & 13 \end{pmatrix} \begin{pmatrix} -5 & -3 \\ 32 & 19 \end{pmatrix} \begin{pmatrix} -9 & -5 \\ 38 & 21 \end{pmatrix}$
10, 6, 5, 1, 1, 1 ($\mathbb{Q}$)	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 5 & 4 \\ -14 & -11 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ -16 & -5 \end{pmatrix} \begin{pmatrix} 9 & 5 \\ -20 & -11 \end{pmatrix} \begin{pmatrix} 39 & 25 \\ -64 & -41 \end{pmatrix}$
10, 6, 5, 1, 1, 1 (cubic)	$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 9 \\ -1 & -4 \end{pmatrix} \begin{pmatrix} 6 & 11 \\ -5 & -9 \end{pmatrix} \begin{pmatrix} 42 & 61 \\ -31 & -45 \end{pmatrix} \begin{pmatrix} 233 & 324 \\ -169 & -235 \end{pmatrix}$
10, 6, 5, 1, 1, 1 (cubic)	$\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -4 & -3 \end{pmatrix} \begin{pmatrix} -8 & 11 \\ -3 & 4 \end{pmatrix} \begin{pmatrix} -5 & 16 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} -16 & 25 \\ -9 & 14 \end{pmatrix}$
10, 6, 5, 1, 1, 1 (cubic)	$\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -4 & -3 \end{pmatrix} \begin{pmatrix} -7 & 11 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} -11 & 25 \\ -4 & 9 \end{pmatrix} \begin{pmatrix} -13 & 16 \\ -9 & 11 \end{pmatrix}$
10, 6, 4, 2, 1, 1 ($\mathbb{Q}$)	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 5 & 4 \\ -14 & -11 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ -16 & -5 \end{pmatrix} \begin{pmatrix} -19 & -12 \\ 46 & 29 \end{pmatrix} \begin{pmatrix} -11 & -6 \\ 24 & 13 \end{pmatrix}$
10, 6, 4, 2, 1, 1 ($\mathbb{Q}$)	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -3 & -2 \\ 8 & 5 \end{pmatrix} \begin{pmatrix} -3 & -1 \\ 22 & 7 \end{pmatrix} \begin{pmatrix} 15 & 4 \\ -64 & -17 \end{pmatrix} \begin{pmatrix} 41 & 9 \\ -196 & -43 \end{pmatrix}$
10, 6, 3, 2, 2, 1 ($\mathbb{Q}$)	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -5 & -3 \\ 12 & 7 \end{pmatrix} \begin{pmatrix} -7 & -2 \\ 32 & 9 \end{pmatrix} \begin{pmatrix} -23 & -18 \\ 32 & 25 \end{pmatrix} \begin{pmatrix} -7 & -5 \\ 38 & 27 \end{pmatrix}$
10, 5, 5, 2, 1, 1 ($\sqrt{5}$)	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 4 & 3 \\ -15 & -11 \end{pmatrix} \begin{pmatrix} 11 & 7 \\ -30 & -19 \end{pmatrix} \begin{pmatrix} 9 & 5 \\ -20 & -11 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ -25 & -6 \end{pmatrix}$
10, 5, 5, 2, 1, 1 ($\sqrt{5}$)	$\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -4 & -3 \end{pmatrix} \begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} -6 & 19 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} -19 & 81 \\ -4 & 17 \end{pmatrix}$
10, 5, 4, 2, 2, 1 (cubic)	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -3 & -2 \\ 8 & 5 \end{pmatrix} \begin{pmatrix} -3 & -1 \\ 22 & 7 \end{pmatrix} \begin{pmatrix} 25 & 7 \\ -118 & -33 \end{pmatrix} \begin{pmatrix} 19 & 5 \\ -80 & -21 \end{pmatrix}$
10, 5, 4, 2, 2, 1 (cubic)	$\begin{pmatrix} -1 & -2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -4 & 1 \end{pmatrix} \begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} -10 & 7 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} -16 & 9 \\ -25 & 14 \end{pmatrix}$
10, 5, 4, 2, 2, 1 (cubic)	$\begin{pmatrix} -1 & -2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -4 & 1 \end{pmatrix} \begin{pmatrix} -4 & 9 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} -12 & 7 \\ -7 & 4 \end{pmatrix} \begin{pmatrix} -6 & 5 \\ -5 & 4 \end{pmatrix}$
10, 5, 3, 2, 2, 2 ($\mathbb{Q}$)	$\begin{pmatrix} -1 & 0 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} -1 & -2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 7 & -18 \\ 2 & -5 \end{pmatrix} \begin{pmatrix} 13 & -18 \\ 8 & -11 \end{pmatrix} \begin{pmatrix} -13 & 22 \\ -3 & 5 \end{pmatrix}$
10, 4, 4, 3, 2, 1 ($\mathbb{Q}$)	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -5 & -3 \\ 12 & 7 \end{pmatrix} \begin{pmatrix} 7 & 5 \\ -24 & -17 \end{pmatrix} \begin{pmatrix} -1 & -1 \\ 10 & 9 \end{pmatrix} \begin{pmatrix} -9 & -7 \\ 40 & 31 \end{pmatrix}$
10, 4, 3, 3, 2, 2 ($\mathbb{Q}$)	$\begin{pmatrix} -1 & 0 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} -1 & -2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 7 & -12 \\ 3 & -5 \end{pmatrix} \begin{pmatrix} -7 & 8 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} -17 & 24 \\ -5 & 7 \end{pmatrix}$
9, 9, 2, 2, 1, 1 ($\mathbb{Q}$)	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -5 & -2 \\ 18 & 7 \end{pmatrix} \begin{pmatrix} -11 & -8 \\ 18 & 13 \end{pmatrix} \begin{pmatrix} 5 & 3 \\ -27 & -16 \end{pmatrix} \begin{pmatrix} 17 & 4 \\ -81 & -19 \end{pmatrix}$

Partition	Generators
9, 9, 2, 2, 1, 1 (cubic)	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -3 & -2 \\ 8 & 5 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ -19 & -6 \end{pmatrix} \begin{pmatrix} -35 & -8 \\ 162 & 37 \end{pmatrix} \begin{pmatrix} 75 & 16 \\ -361 & -77 \end{pmatrix}$
9, 9, 2, 2, 1, 1 (cubic)	$\begin{pmatrix} -1 & -2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 2 & -5 \\ -5 & 13 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ -16 & -5 \end{pmatrix} \begin{pmatrix} -5 & 16 \\ -1 & 3 \end{pmatrix}$
9, 9, 2, 2, 1, 1 (cubic)	$\begin{pmatrix} -1 & -2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} -3 & 1 \\ -4 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -5 & 6 \end{pmatrix} \begin{pmatrix} 3 & -5 \\ -7 & 12 \end{pmatrix} \begin{pmatrix} -4 & 9 \\ -1 & 2 \end{pmatrix}$
9, 8, 3, 2, 1, 1 (cubic)	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ -11 & -7 \end{pmatrix} \begin{pmatrix} 11 & 9 \\ -16 & -13 \end{pmatrix} \begin{pmatrix} -13 & -6 \\ 37 & 17 \end{pmatrix} \begin{pmatrix} -29 & -12 \\ 75 & 31 \end{pmatrix}$
9, 8, 3, 2, 1, 1 (cubic)	$\begin{pmatrix} -1 & -2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} -3 & 1 \\ -4 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 4 & -5 \end{pmatrix} \begin{pmatrix} 2 & -5 \\ -5 & 13 \end{pmatrix} \begin{pmatrix} -5 & 16 \\ -1 & 3 \end{pmatrix}$
9, 8, 3, 2, 1, 1 (cubic)	$\begin{pmatrix} -1 & -2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 4 & -3 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} -3 & 1 \\ 8 & -3 \end{pmatrix} \begin{pmatrix} -4 & 9 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ -16 & -5 \end{pmatrix}$
9, 7, 5, 1, 1, 1 ($\mathbb{Q}$)	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ -11 & -7 \end{pmatrix} \begin{pmatrix} 11 & 9 \\ -16 & -13 \end{pmatrix} \begin{pmatrix} 11 & 5 \\ -31 & -14 \end{pmatrix} \begin{pmatrix} 64 & 25 \\ -169 & -66 \end{pmatrix}$
9, 7, 4, 2, 1, 1 ($\sqrt{-7}$)	$\begin{pmatrix} -1 & -2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -4 & 1 \end{pmatrix} \begin{pmatrix} 7 & -5 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} -7 & 9 \\ -4 & 5 \end{pmatrix} \begin{pmatrix} -5 & 16 \\ -1 & 3 \end{pmatrix}$
9, 7, 4, 2, 1, 1 ($\sqrt{-7}$)	$\begin{pmatrix} -1 & -2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -4 & 1 \end{pmatrix} \begin{pmatrix} 8 & -5 \\ 5 & -3 \end{pmatrix} \begin{pmatrix} -4 & 9 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} -13 & 16 \\ -9 & 11 \end{pmatrix}$
9, 7, 3, 2, 2, 1 (cubic)	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -3 & -2 \\ 8 & 5 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ -19 & -6 \end{pmatrix} \begin{pmatrix} -25 & -6 \\ 121 & 29 \end{pmatrix} \begin{pmatrix} -53 & -12 \\ 243 & 55 \end{pmatrix}$
9, 7, 3, 2, 2, 1 (cubic)	$\begin{pmatrix} -1 & 0 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} -1 & -2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 6 & -7 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 7 & -18 \\ 2 & -5 \end{pmatrix} \begin{pmatrix} -16 & 25 \\ -9 & 14 \end{pmatrix}$
9, 7, 3, 2, 2, 1 (cubic)	$\begin{pmatrix} -1 & 0 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} -1 & -2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 6 & -7 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 7 & -18 \\ 2 & -5 \end{pmatrix} \begin{pmatrix} -16 & 25 \\ -9 & 14 \end{pmatrix}$
9, 6, 5, 2, 1, 1 (cubic)	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -3 & -2 \\ 8 & 5 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ -19 & -6 \end{pmatrix} \begin{pmatrix} 21 & 5 \\ -101 & -24 \end{pmatrix} \begin{pmatrix} 114 & 25 \\ -529 & -116 \end{pmatrix}$
9, 6, 5, 2, 1, 1 (cubic)	$\begin{pmatrix} -1 & 2 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} 1 & 4 \\ -1 & -3 \end{pmatrix} \begin{pmatrix} 5 & -2 \\ 8 & -3 \end{pmatrix} \begin{pmatrix} 4 & -3 \\ -9 & 7 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ -16 & -5 \end{pmatrix}$
9, 6, 5, 2, 1, 1 (cubic)	$\begin{pmatrix} -1 & -2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} -3 & 4 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -5 & 1 \end{pmatrix} \begin{pmatrix} 4 & -3 \\ -9 & 7 \end{pmatrix} \begin{pmatrix} -16 & 9 \\ -25 & 14 \end{pmatrix}$
9, 6, 4, 3, 1, 1 ($\mathbb{Q}$)	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -5 & -3 \\ 12 & 7 \end{pmatrix} \begin{pmatrix} 7 & 5 \\ -24 & -17 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -9 & -8 \end{pmatrix} \begin{pmatrix} 17 & 4 \\ -81 & -19 \end{pmatrix}$
9, 6, 3, 3, 2, 1 (cubic)	$\begin{pmatrix} -1 & 0 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} -1 & -2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 7 & -12 \\ 3 & -5 \end{pmatrix} \begin{pmatrix} 6 & -7 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} -15 & 49 \\ -4 & 13 \end{pmatrix}$

Partition	Generators
9, 6, 3, 3, 2, 1 (cubic)	$\begin{pmatrix} -1 & -2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 4 & -3 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} -8 & 3 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} 1 & -3 \\ -3 & 10 \end{pmatrix} \begin{pmatrix} 9 & 4 \\ -25 & -11 \end{pmatrix}$
9, 6, 3, 3, 2, 1 (cubic)	$\begin{pmatrix} -1 & -2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} -3 & 4 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 6 & -1 \end{pmatrix} \begin{pmatrix} 4 & -3 \\ -9 & 7 \end{pmatrix} \begin{pmatrix} 7 & -3 \\ 12 & -5 \end{pmatrix}$
9, 5, 5, 2, 2, 1 ($\mathbb{Q}$)	$\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -4 & -3 \end{pmatrix} \begin{pmatrix} 7 & -18 \\ 2 & -5 \end{pmatrix} \begin{pmatrix} 13 & -18 \\ 8 & -11 \end{pmatrix} \begin{pmatrix} -13 & 22 \\ -3 & 5 \end{pmatrix}$
9, 5, 4, 3, 2, 1 (cubic)	$\begin{pmatrix} -1 & -2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -4 & 1 \end{pmatrix} \begin{pmatrix} 4 & -3 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} -5 & 11 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} -15 & 49 \\ -4 & 13 \end{pmatrix}$
9, 5, 4, 3, 2, 1 (cubic)	$\begin{pmatrix} -1 & -2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 4 & -3 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -5 & 1 \end{pmatrix} \begin{pmatrix} 2 & -5 \\ -5 & 13 \end{pmatrix} \begin{pmatrix} -5 & 16 \\ -1 & 3 \end{pmatrix}$
9, 5, 4, 3, 2, 1 (cubic)	$\begin{pmatrix} -1 & -2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -5 & 1 \end{pmatrix} \begin{pmatrix} -5 & 4 \\ -4 & 3 \end{pmatrix} \begin{pmatrix} 3 & -5 \\ -7 & 12 \end{pmatrix} \begin{pmatrix} -4 & 9 \\ -1 & 2 \end{pmatrix}$
9, 5, 3, 3, 3, 1 ($\mathbb{Q}$)	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -5 & -3 \\ 12 & 7 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -9 & -8 \end{pmatrix} \begin{pmatrix} -10 & -7 \\ 33 & 23 \end{pmatrix} \begin{pmatrix} -11 & -3 \\ 48 & 13 \end{pmatrix}$
9, 4, 3, 3, 3, 2 ($\mathbb{Q}$)	$\begin{pmatrix} -1 & 0 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} -1 & -2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 6 & -7 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} -9 & 16 \\ -4 & 7 \end{pmatrix} \begin{pmatrix} 10 & -27 \\ 3 & -8 \end{pmatrix}$
8, 8, 4, 2, 1, 1 ($\mathbb{Q}$)	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 7 & 4 \\ -16 & -9 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -8 & -7 \end{pmatrix} \begin{pmatrix} 7 & 5 \\ -24 & -17 \end{pmatrix} \begin{pmatrix} 23 & 9 \\ -64 & -25 \end{pmatrix}$
8, 8, 3, 3, 1, 1 ($\mathbb{Q}$)	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -5 & -3 \\ 12 & 7 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -8 & -7 \end{pmatrix} \begin{pmatrix} 11 & 8 \\ -40 & -29 \end{pmatrix} \begin{pmatrix} 29 & 9 \\ -100 & -31 \end{pmatrix}$
8, 8, 3, 3, 1, 1 ($\sqrt{-2}$)	$\begin{pmatrix} -1 & 0 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} -2 & -3 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 3 & -8 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} -7 & 9 \\ -4 & 5 \end{pmatrix} \begin{pmatrix} 3 & 16 \\ -1 & -5 \end{pmatrix}$
8, 8, 3, 3, 1, 1 ($\sqrt{-2}$)	$\begin{pmatrix} -1 & 0 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} -2 & -3 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 4 & -7 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} -4 & 9 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 9 & 25 \\ -4 & -11 \end{pmatrix}$
8, 8, 2, 2, 2, 2 ($\mathbb{Q}$)	$\begin{pmatrix} -1 & -2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} -5 & -2 \\ 18 & 7 \end{pmatrix} \begin{pmatrix} 7 & -18 \\ 2 & -5 \end{pmatrix} \begin{pmatrix} 1 & -4 \\ -4 & 17 \end{pmatrix}$
8, 7, 6, 1, 1, 1 ($\sqrt{-3}$)	$\begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -4 & -3 \end{pmatrix} \begin{pmatrix} 3 & -8 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} -7 & 9 \\ -4 & 5 \end{pmatrix} \begin{pmatrix} 3 & 16 \\ -1 & -5 \end{pmatrix}$
8, 7, 6, 1, 1, 1 ($\sqrt{-3}$)	$\begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -4 & -3 \end{pmatrix} \begin{pmatrix} 4 & -7 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} -4 & 9 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 9 & 25 \\ -4 & -11 \end{pmatrix}$
8, 7, 5, 2, 1, 1 (cubic)	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -8 & -5 \\ 21 & 13 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -8 & -7 \end{pmatrix} \begin{pmatrix} 7 & 5 \\ -24 & -17 \end{pmatrix} \begin{pmatrix} 20 & 9 \\ -49 & -22 \end{pmatrix}$
8, 7, 5, 2, 1, 1 (cubic)	$\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -4 & -3 \end{pmatrix} \begin{pmatrix} 6 & -7 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 10 & -17 \\ 3 & -5 \end{pmatrix} \begin{pmatrix} -11 & 25 \\ -4 & 9 \end{pmatrix}$

Partition	Generators
$8, 7, 5, 2, 1, 1$ (cubic)	$\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -4 & -3 \end{pmatrix} \begin{pmatrix} 6 & -7 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 7 & -18 \\ 2 & -5 \end{pmatrix} \begin{pmatrix} -16 & 25 \\ -9 & 14 \end{pmatrix}$
$8, 7, 4, 3, 1, 1$ $(\sqrt{-6})$	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 7 & 4 \\ -16 & -9 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -8 & -7 \end{pmatrix} \begin{pmatrix} 11 & 8 \\ -29 & -21 \end{pmatrix} \begin{pmatrix} 13 & 4 \\ -49 & -15 \end{pmatrix}$
$8, 7, 4, 3, 1, 1$ $(\sqrt{-6})$	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 7 & 4 \\ -16 & -9 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -8 & -7 \end{pmatrix} \begin{pmatrix} -8 & -3 \\ 27 & 10 \end{pmatrix} \begin{pmatrix} 34 & 25 \\ -49 & -36 \end{pmatrix}$
$8, 7, 4, 2, 2, 1$ $(\sqrt{-7})$	$\begin{pmatrix} -1 & -2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} -3 & 4 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 3 & -2 \\ -7 & 5 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 7 & -1 \end{pmatrix} \begin{pmatrix} -5 & 2 \\ 17 & -7 \end{pmatrix}$
$8, 7, 4, 2, 2, 1$ $(\sqrt{-7})$	$\begin{pmatrix} -1 & -2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ -3 & 7 \end{pmatrix} \begin{pmatrix} -7 & 2 \\ -4 & 1 \end{pmatrix} \begin{pmatrix} -5 & 4 \\ -4 & 3 \end{pmatrix} \begin{pmatrix} 9 & 4 \\ -25 & -11 \end{pmatrix}$
$8, 7, 3, 3, 2, 1$ $(\mathbb{Q})$	$\begin{pmatrix} -1 & 0 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} -1 & -2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 5 & -6 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 13 & -21 \\ 5 & -8 \end{pmatrix} \begin{pmatrix} -22 & 49 \\ -9 & 20 \end{pmatrix}$
$8, 6, 6, 2, 1, 1$ $(\sqrt{-3})$	$\begin{pmatrix} -2 & 3 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} -4 & 1 \\ 7 & -2 \end{pmatrix} \begin{pmatrix} -4 & 3 \\ 5 & -4 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -9 & -4 \end{pmatrix} \begin{pmatrix} 2 & 9 \\ -1 & -4 \end{pmatrix}$
$8, 6, 6, 2, 1, 1$ $(\sqrt{-3})$	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -1 & -1 \\ 6 & 5 \end{pmatrix} \begin{pmatrix} -17 & -10 \\ 46 & 27 \end{pmatrix} \begin{pmatrix} -19 & -8 \\ 50 & 21 \end{pmatrix} \begin{pmatrix} 39 & 25 \\ -64 & -41 \end{pmatrix}$
$8, 6, 5, 3, 1, 1$ $(\sqrt{5})$	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -5 & -3 \\ 12 & 7 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -8 & -7 \end{pmatrix} \begin{pmatrix} -10 & -7 \\ 33 & 23 \end{pmatrix} \begin{pmatrix} 32 & 9 \\ -121 & -34 \end{pmatrix}$
$8, 6, 5, 3, 1, 1$ $(\sqrt{5})$	$\begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -4 & -3 \end{pmatrix} \begin{pmatrix} 7 & -3 \\ 12 & -5 \end{pmatrix} \begin{pmatrix} 6 & -1 \\ 7 & -1 \end{pmatrix} \begin{pmatrix} -36 & 25 \\ -49 & 34 \end{pmatrix}$
$8, 6, 5, 2, 2, 1$ $(\mathbb{Q})$	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -8 & -7 \end{pmatrix} \begin{pmatrix} 7 & 5 \\ -24 & -17 \end{pmatrix} \begin{pmatrix} 11 & 7 \\ -30 & -19 \end{pmatrix} \begin{pmatrix} 9 & 5 \\ -20 & -11 \end{pmatrix}$
$8, 6, 4, 3, 2, 1$ $(\sqrt{2})$	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -5 & -3 \\ 12 & 7 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -8 & -7 \end{pmatrix} \begin{pmatrix} -15 & -11 \\ 56 & 41 \end{pmatrix} \begin{pmatrix} -13 & -9 \\ 42 & 29 \end{pmatrix}$
$8, 6, 4, 3, 2, 1$ $(\sqrt{2})$	$\begin{pmatrix} -1 & 0 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} -1 & -2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 5 & -6 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} -9 & 16 \\ -4 & 7 \end{pmatrix} \begin{pmatrix} -25 & 64 \\ -9 & 23 \end{pmatrix}$
$8, 6, 4, 2, 2, 2$ $(\mathbb{Q})$	$\begin{pmatrix} -1 & -2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -4 & 1 \end{pmatrix} \begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} 11 & -6 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 13 & -10 \\ 4 & -3 \end{pmatrix}$
$8, 5, 4, 3, 3, 1$ $(\sqrt{10})$	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -8 & -7 \end{pmatrix} \begin{pmatrix} -8 & -3 \\ 27 & 10 \end{pmatrix} \begin{pmatrix} 9 & 5 \\ -20 & -11 \end{pmatrix} \begin{pmatrix} -17 & -12 \\ 27 & 19 \end{pmatrix}$
$8, 5, 4, 3, 3, 1$ $(\sqrt{10})$	$\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -4 & -3 \end{pmatrix} \begin{pmatrix} 6 & -7 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} -9 & 16 \\ -4 & 7 \end{pmatrix} \begin{pmatrix} 10 & -27 \\ 3 & -8 \end{pmatrix}$
$8, 5, 4, 3, 2, 2$ $(\mathbb{Q})$	$\begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} -1 & -2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 5 & -1 \\ 6 & -1 \end{pmatrix} \begin{pmatrix} -19 & 7 \\ -30 & 11 \end{pmatrix} \begin{pmatrix} -11 & 5 \\ -20 & 9 \end{pmatrix}$

Partition	Generators
$8, 4, 4, 4, 3, 1\ (\mathbb{Q})$	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 7 & 4 \\ -16 & -9 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -8 & -7 \end{pmatrix} \begin{pmatrix} 11 & 8 \\ -40 & -29 \end{pmatrix} \begin{pmatrix} 11 & 4 \\ -36 & -13 \end{pmatrix}$
$8, 4, 4, 4, 2, 2\ (\mathbb{Q})$	$\begin{pmatrix} -1 & -2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -4 & 1 \end{pmatrix} \begin{pmatrix} -5 & 4 \\ -4 & 3 \end{pmatrix} \begin{pmatrix} -9 & 14 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 7 & -18 \\ 2 & -5 \end{pmatrix}$
$7, 7, 7, 1, 1, 1\ (\mathbb{Q})$	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -1 & -1 \\ 7 & 6 \end{pmatrix} \begin{pmatrix} -8 & -5 \\ 21 & 13 \end{pmatrix} \begin{pmatrix} 13 & 4 \\ -49 & -15 \end{pmatrix} \begin{pmatrix} 20 & 9 \\ -49 & -22 \end{pmatrix}$
$7, 7, 5, 3, 1, 1\ (\sqrt{21})$	$\begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -5 & 1 \end{pmatrix} \begin{pmatrix} -2 & -3 \\ 5 & 7 \end{pmatrix} \begin{pmatrix} 5 & 4 \\ -9 & -7 \end{pmatrix} \begin{pmatrix} -7 & 4 \\ -9 & 5 \end{pmatrix}$
$7, 7, 5, 3, 1, 1\ (\sqrt{21})$	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -5 & -3 \\ 12 & 7 \end{pmatrix} \begin{pmatrix} -1 & -1 \\ 7 & 6 \end{pmatrix} \begin{pmatrix} -27 & -8 \\ 98 & 29 \end{pmatrix} \begin{pmatrix} 59 & 16 \\ -225 & -61 \end{pmatrix}$
$7, 7, 4, 4, 1, 1\ (\sqrt{-7})$	$\begin{pmatrix} -2 & 1 \\ 5 & -3 \end{pmatrix} \begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix} \begin{pmatrix} 1 & -4 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 5 & 9 \\ -4 & -7 \end{pmatrix} \begin{pmatrix} -4 & 1 \\ -9 & 2 \end{pmatrix}$
$7, 7, 4, 4, 1, 1\ (\sqrt{-7})$	$\begin{pmatrix} -2 & 1 \\ 5 & -3 \end{pmatrix} \begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix} \begin{pmatrix} 1 & -4 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 5 & 9 \\ -4 & -7 \end{pmatrix} \begin{pmatrix} -4 & 1 \\ -9 & 2 \end{pmatrix}$
$7, 7, 3, 3, 2, 2\ (\sqrt{7})$	$\begin{pmatrix} -1 & -2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 5 & -11 \end{pmatrix} \begin{pmatrix} 1 & -5 \\ -2 & 11 \end{pmatrix} \begin{pmatrix} -8 & -3 \\ 27 & 10 \end{pmatrix}$
$7, 7, 3, 3, 2, 2\ (\sqrt{7})$	$\begin{pmatrix} -1 & 0 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} -1 & -2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} -4 & 5 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 17 & -37 \\ 6 & -13 \end{pmatrix} \begin{pmatrix} 31 & -75 \\ 12 & -29 \end{pmatrix}$
$7, 6, 6, 2, 2, 1\ (\mathbb{Q})$	$\begin{pmatrix} -1 & -2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} -5 & 3 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 6 & -1 \end{pmatrix} \begin{pmatrix} -5 & 2 \\ 12 & -5 \end{pmatrix} \begin{pmatrix} -7 & 9 \\ -4 & 5 \end{pmatrix}$
$7, 6, 5, 4, 1, 1\ (\mathbb{Q})$	$\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -4 & -3 \end{pmatrix} \begin{pmatrix} 5 & -6 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} -9 & 16 \\ -4 & 7 \end{pmatrix} \begin{pmatrix} -25 & 64 \\ -9 & 23 \end{pmatrix}$
$7, 6, 5, 3, 2, 1\ (\text{cubic})$	$\begin{pmatrix} 0 & 1 \\ -1 & -3 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} -4 & 5 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} -14 & 31 \\ -5 & 11 \end{pmatrix} \begin{pmatrix} -66 & 169 \\ -25 & 64 \end{pmatrix}$
$7, 6, 5, 3, 2, 1\ (\text{cubic})$	$\begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -3 & 1 \\ 5 & -2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -9 & -4 \end{pmatrix} \begin{pmatrix} -9 & 5 \\ 7 & -4 \end{pmatrix} \begin{pmatrix} -9 & 7 \\ 5 & -4 \end{pmatrix}$
$7, 6, 5, 3, 2, 1\ (\text{cubic})$	$\begin{pmatrix} -2 & -3 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} -3 & 5 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -5 & 1 \end{pmatrix} \begin{pmatrix} 3 & -2 \\ -7 & 5 \end{pmatrix} \begin{pmatrix} -4 & 9 \\ -1 & 2 \end{pmatrix}$
$7, 6, 4, 4, 2, 1\ (\mathbb{Q})$	$\begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -4 & -3 \end{pmatrix} \begin{pmatrix} -5 & 7 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} 7 & -17 \\ -2 & 5 \end{pmatrix} \begin{pmatrix} 1 & -7 \\ 0 & 1 \end{pmatrix}$
$7, 5, 5, 4, 2, 1\ (\sqrt{2})$	$\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -4 & -3 \end{pmatrix} \begin{pmatrix} 5 & -6 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} -19 & 30 \\ -7 & 11 \end{pmatrix} \begin{pmatrix} -11 & 20 \\ -5 & 9 \end{pmatrix}$
$7, 5, 5, 4, 2, 1\ (\sqrt{2})$	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -1 & -1 \\ 7 & 6 \end{pmatrix} \begin{pmatrix} 7 & 4 \\ -16 & -9 \end{pmatrix} \begin{pmatrix} 18 & 7 \\ -67 & -26 \end{pmatrix} \begin{pmatrix} 14 & 5 \\ -45 & -16 \end{pmatrix}$

Partition	Generators
$7, 5, 5, 3, 2, 2 \ (\mathbb{Q})$	$\begin{pmatrix} -1 & -2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 4 & -3 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -5 & 1 \end{pmatrix} \begin{pmatrix} 3 & -7 \\ -8 & 19 \end{pmatrix} \begin{pmatrix} 1 & -5 \\ -2 & 11 \end{pmatrix}$
$7, 5, 4, 4, 3, 1 \ (\mathbb{Q})$	$\begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} 5 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -6 & 5 \\ -5 & 4 \end{pmatrix} \begin{pmatrix} -19 & 11 \\ -7 & 4 \end{pmatrix} \begin{pmatrix} -9 & 16 \\ -4 & 7 \end{pmatrix}$
$7, 5, 4, 3, 3, 2 \ (\mathbb{Q})$	$\begin{pmatrix} -1 & -2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -4 & 1 \end{pmatrix} \begin{pmatrix} 9 & -5 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 7 & -12 \\ 3 & -5 \end{pmatrix} \begin{pmatrix} -6 & 5 \\ -5 & 4 \end{pmatrix}$
$6, 6, 6, 4, 1, 1 \ (\mathbb{Q})$	$\begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -4 & -3 \end{pmatrix} \begin{pmatrix} -5 & 7 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} -1 & 6 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -11 & 25 \\ -4 & 9 \end{pmatrix}$
$6, 6, 6, 2, 2, 2 \ (\mathbb{Q})$	$\begin{pmatrix} -1 & -2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} 5 & -2 \\ 8 & -3 \end{pmatrix} \begin{pmatrix} -3 & 2 \\ 10 & -7 \end{pmatrix} \begin{pmatrix} -1 & 4 \\ 2 & -9 \end{pmatrix}$
$6, 6, 5, 5, 1, 1 \ (\sqrt{3})$	$\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -4 & -3 \end{pmatrix} \begin{pmatrix} -4 & 5 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} -14 & 31 \\ -5 & 11 \end{pmatrix} \begin{pmatrix} -66 & 169 \\ -25 & 64 \end{pmatrix}$
$6, 6, 5, 5, 1, 1 \ (\sqrt{3})$	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -1 & -1 \\ 6 & 5 \end{pmatrix} \begin{pmatrix} 9 & 5 \\ -20 & -11 \end{pmatrix} \begin{pmatrix} 14 & 9 \\ -39 & -25 \end{pmatrix} \begin{pmatrix} 64 & 25 \\ -169 & -66 \end{pmatrix}$
$6, 6, 5, 4, 2, 1 \ (\mathbb{Q})$	$\begin{pmatrix} -1 & -2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -4 & 1 \end{pmatrix} \begin{pmatrix} -7 & 4 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} -8 & 11 \\ -3 & 4 \end{pmatrix} \begin{pmatrix} -16 & 25 \\ -9 & 14 \end{pmatrix}$
$6, 6, 5, 3, 3, 1 \ (\mathbb{Q})$	$\begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -4 & -3 \end{pmatrix} \begin{pmatrix} 7 & -9 \\ -3 & 4 \end{pmatrix} \begin{pmatrix} -1 & 6 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 7 & -12 \\ 3 & -5 \end{pmatrix}$
$6, 6, 4, 4, 2, 2 \ (\mathbb{Q})$	$\begin{pmatrix} -1 & -2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -4 & 1 \end{pmatrix} \begin{pmatrix} -7 & 4 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} -11 & 14 \\ -4 & 5 \end{pmatrix} \begin{pmatrix} -9 & 16 \\ -4 & 7 \end{pmatrix}$
$6, 6, 4, 3, 3, 2 \ (\mathbb{Q})$	$\begin{pmatrix} -1 & -2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 4 & -3 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 6 & -1 \end{pmatrix} \begin{pmatrix} -1 & 4 \\ 2 & -9 \end{pmatrix} \begin{pmatrix} -12 & 5 \\ -5 & 2 \end{pmatrix}$
$6, 6, 3, 3, 3, 3 \ (\mathbb{Q})$	$\begin{pmatrix} -2 & -3 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 4 & -3 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ -3 & -8 \end{pmatrix} \begin{pmatrix} -8 & 3 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} 1 & -3 \\ -3 & 10 \end{pmatrix}$
$6, 5, 4, 4, 3, 2 \ (\mathbb{Q})$	$\begin{pmatrix} -1 & -2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -5 & 1 \end{pmatrix} \begin{pmatrix} -5 & 4 \\ -4 & 3 \end{pmatrix} \begin{pmatrix} -1 & 4 \\ 2 & -9 \end{pmatrix} \begin{pmatrix} 7 & -12 \\ 3 & -5 \end{pmatrix}$
$5, 5, 5, 5, 2, 2 \ (\mathbb{Q})$	$\begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ -3 & 7 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ -3 & -8 \end{pmatrix} \begin{pmatrix} 4 & 7 \\ -7 & -12 \end{pmatrix} \begin{pmatrix} 4 & 5 \\ -5 & -6 \end{pmatrix}$
$5, 5, 4, 4, 3, 3 \ (\mathbb{Q})$	$\begin{pmatrix} 1 & 0 \\ -4 & 1 \end{pmatrix} \begin{pmatrix} 4 & -3 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} 3 & 4 \\ -4 & -5 \end{pmatrix} \begin{pmatrix} -1 & 5 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 3 & 8 \\ -5 & -13 \end{pmatrix}$
$4, 4, 4, 4, 4, 4 \ (\mathbb{Q})$	$\begin{pmatrix} 1 & 0 \\ -4 & 1 \end{pmatrix} \begin{pmatrix} 1 & -4 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 4 \\ -4 & -5 \end{pmatrix} \begin{pmatrix} -5 & 4 \\ -4 & 3 \end{pmatrix} \begin{pmatrix} 5 & -8 \\ -8 & 13 \end{pmatrix}$

4. *Conclusions and outlook*

In this paper, we have further developed the connections between the torsion-free, genus zero modular subgroups and the theory of *dessins d'enfants*. This was accomplished through construction of the ramification data and the *dessins* for each conjugacy class of subgroups. In addition, we have shown that the index 24 and 36 subgroups are of special interest, since they are naturally associated to K3 surfaces and Calabi–Yau threefolds, respectively.

In a parallel vein, we have found the generating set of a representative of the conjugacy classes of modular subgroups associated to all of the 112 semi-stable elliptic fibrations over $\mathbb{P}^1$ which are extremal K3 surfaces with six singular fibres. Indeed, at the intersection of the 112 and 33 lie the nine K3 surfaces. This result is important not only for developing our understanding of these surfaces and their connection to the theory of modular subgroups, but also because these connections should give us further tools for categorizing these surfaces, and the relations between them.

To our intricate web of correspondences we can add physics. In [**1**], Seiberg–Witten curves for four-dimensional supersymmetric field theories with unitary gauge groups were given an interpretation as clean *dessins*. In [**8**], the nine K3 surfaces were singled out to relate to the recently advanced class of $\mathcal{N} = 2$ Gaiotto theories. Now, we have an extensive catalogue of *dessins* which have underlying Calabi–Yau structure, to construct their associated gauge theory is clearly an interesting story which the authors are currently pursuing.

Appendix. *Dessins from the Miranda–Persson table*

In this appendix we give the *dessins* for all of the entries in [**12**]. These *dessins* match those already given in [**2, 3**], hence their being relegated to an appendix. Nevertheless, there are two advantages to our drawings over those in [**2, 3**]: first, they are drawn electronically; second, we have marked the black/white nodes as an emphasis of the bipartite nature of the graphs.

19, 1, 1, 1, 1, 1 ($\mathbb{Q}$)

18, 2, 1, 1, 1, 1 ($\mathbb{Q}$)


18, 2, 1, 1, 1, 1 ($\sqrt{-3}$)

18, 2, 1, 1, 1, 1 ($\sqrt{-3}$)

17, 3, 1, 1, 1, 1 ($\mathbb{Q}$)
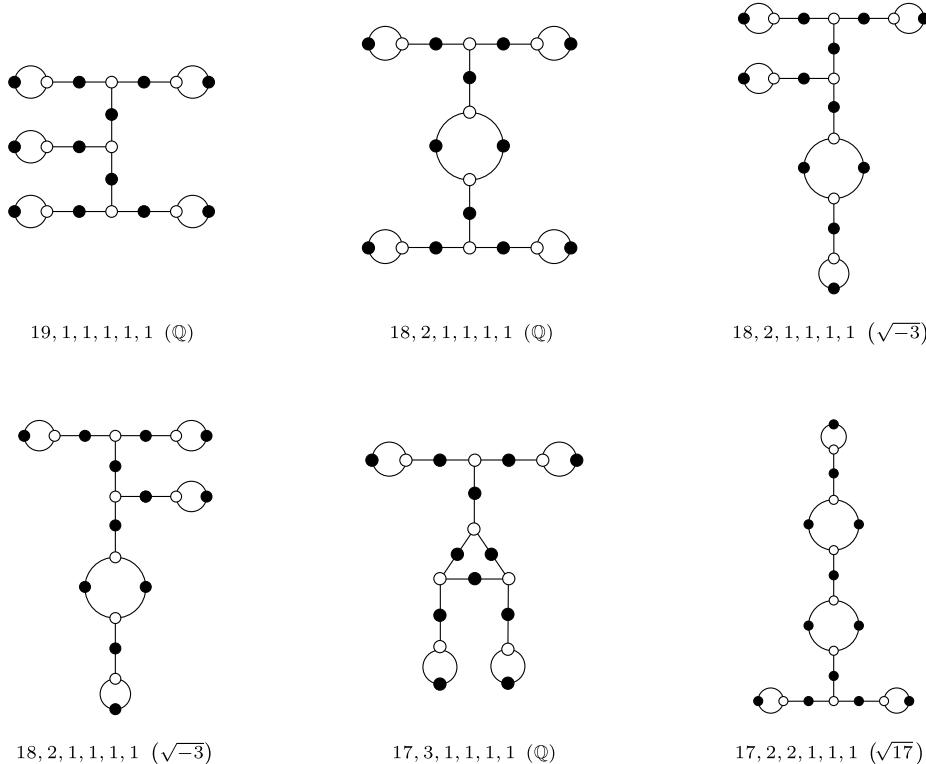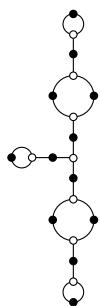
17, 2, 2, 1, 1, 1 ($\sqrt{17}$)

$17, 2, 2, 1, 1, 1 \ (\sqrt{17})$
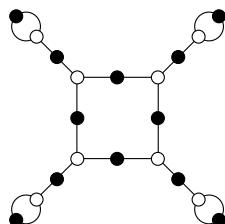
$16, 4, 1, 1, 1, 1 \ (\mathbb{Q})$
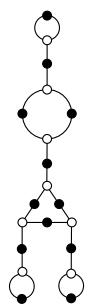
$16, 3, 2, 1, 1, 1 \ (\mathbb{Q})$
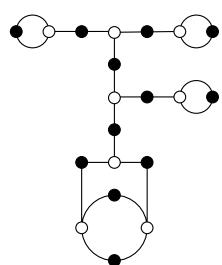
$16, 3, 2, 1, 1, 1 \ (\sqrt{-2})$
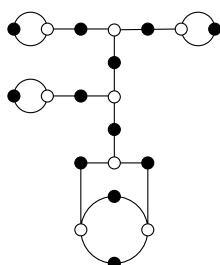
$16, 3, 2, 1, 1, 1 \ (\sqrt{-2})$

$16, 2, 2, 2, 1, 1 \ (\mathbb{Q})$
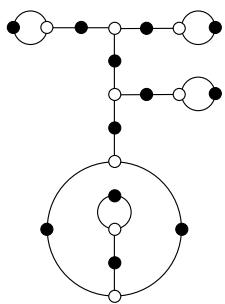
$15, 5, 1, 1, 1, 1 \ (\sqrt{-15})$
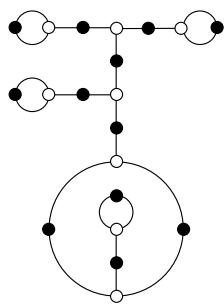
$15, 5, 1, 1, 1, 1 \ (\sqrt{-15})$
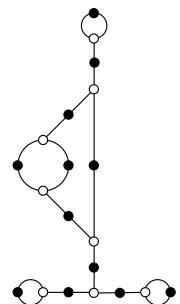
$15, 4, 2, 1, 1, 1 \ (\sqrt{-15})$
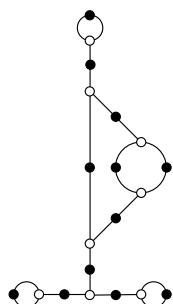
$15, 4, 2, 1, 1, 1 \ (\sqrt{-15})$
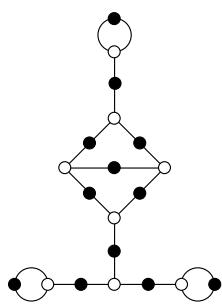
$15, 3, 3, 1, 1, 1 \ (\mathbb{Q})$
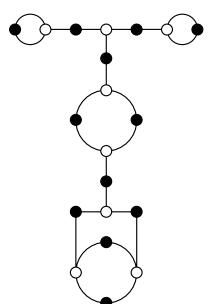
$15, 3, 2, 2, 1, 1 \ (\mathbb{Q})$
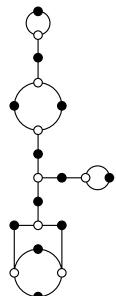
$15, 3, 2, 2, 1, 1 \; \left(\sqrt{-15}\right)$
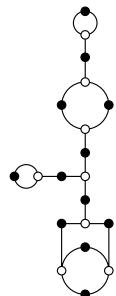
$15, 3, 2, 2, 1, 1 \; \left(\sqrt{-15}\right)$
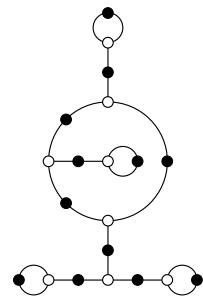
$14, 6, 1, 1, 1, 1 \; \left(\sqrt{-3}\right)$
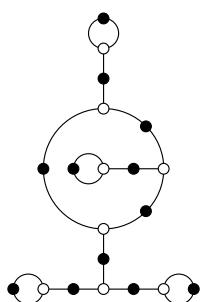
$14, 6, 1, 1, 1, 1 \; \left(\sqrt{-3}\right)$
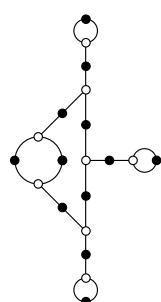
$14, 5, 2, 1, 1, 1 \; (\mathbb{Q})$
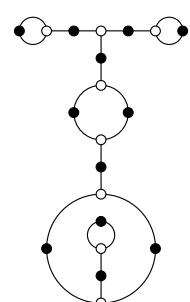
$14, 5, 2, 1, 1, 1 \; (\text{cubic})$
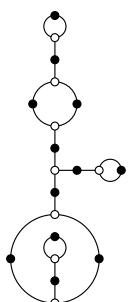
$14, 5, 2, 1, 1, 1 \; (\text{cubic})$
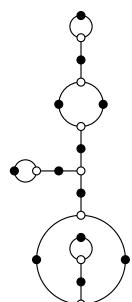
$14, 5, 2, 1, 1, 1 \; (\text{cubic})$
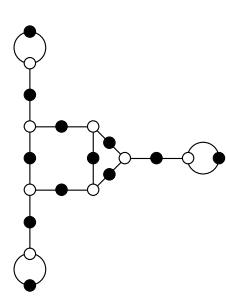
$14, 4, 3, 1, 1, 1 \; (\mathbb{Q})$
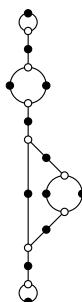
$14, 4, 2, 2, 1, 1 \; \left(\sqrt{-7}\right)$
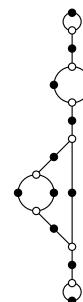
$14, 4, 2, 2, 1, 1 \; \left(\sqrt{-7}\right)$
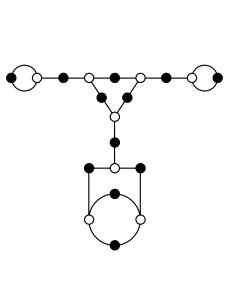
$14, 3, 3, 2, 1, 1 \; (\mathbb{Q})$
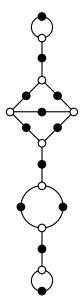
$14, 3, 3, 2, 1, 1 \;(\mathbb{Q})$

$14, 3, 2, 2, 2, 1 \;(\mathbb{Q})$
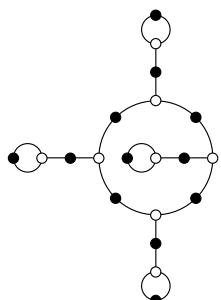
$13, 7, 1, 1, 1, 1 \;(\mathbb{Q})$
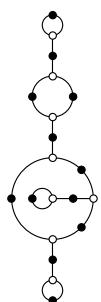
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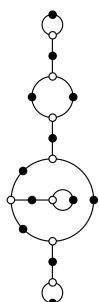
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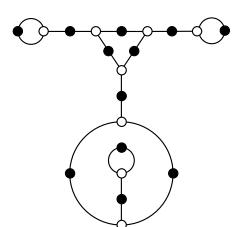
$13, 5, 3, 1, 1, 1 \;(\mathbb{Q})$
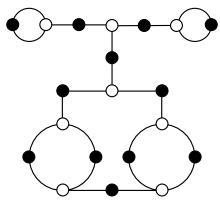
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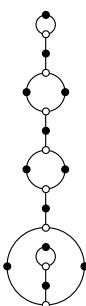
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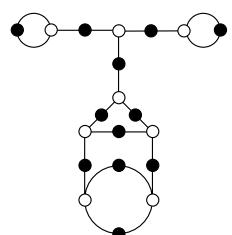
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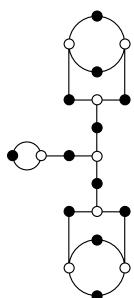
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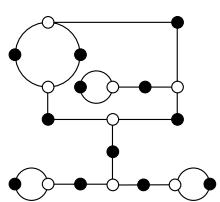
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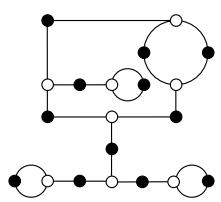
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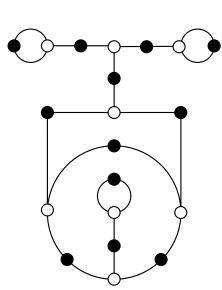
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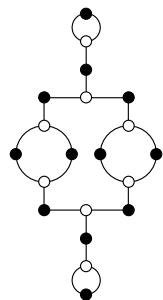
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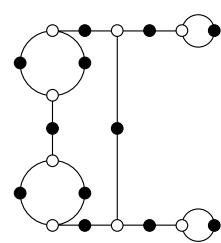
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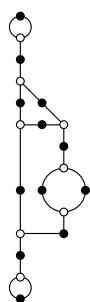
$12, 5, 3, 2, 1, 1 \; (\text{quartic})$
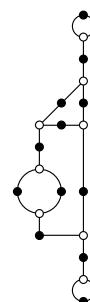
$12, 5, 3, 2, 1, 1 \; (\text{quartic})$
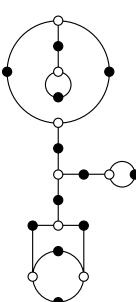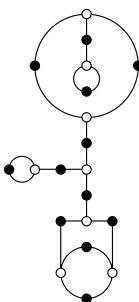
$12, 5, 3, 2, 1, 1 \; (\text{quartic})$

$12, 5, 3, 2, 1, 1 \; (\text{quartic})$
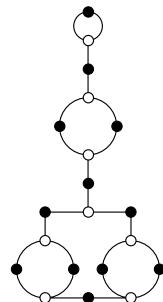
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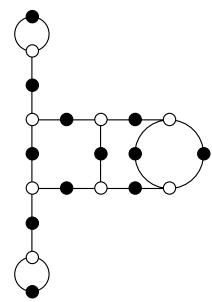
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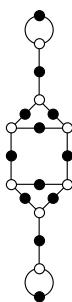
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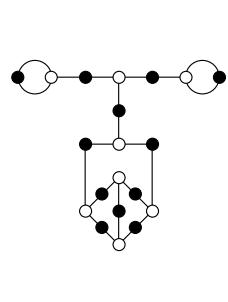
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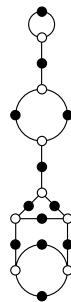
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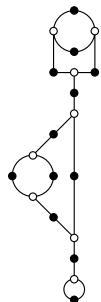
$12, 4, 3, 2, 2, 1 \; \left(\sqrt{-3}\right)$
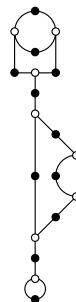
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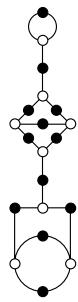
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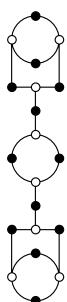
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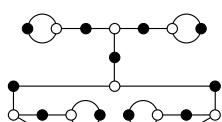
$11, 9, 1, 1, 1, 1 \; (\mathbb{Q})$
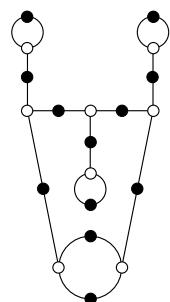
$11, 8, 2, 1, 1, 1 \; (\text{cubic})$
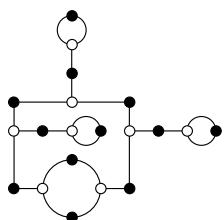
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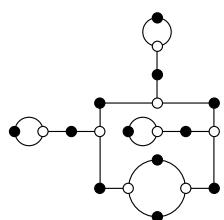
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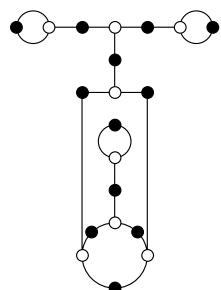
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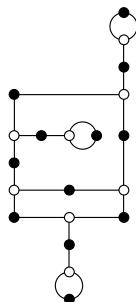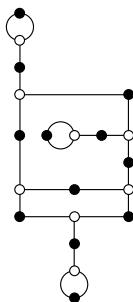
$11, 7, 3, 1, 1, 1 \; (\text{cubic})$

$11, 7, 3, 1, 1, 1 \; (\text{cubic})$
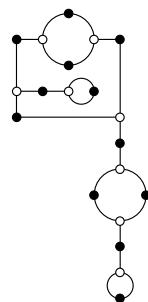
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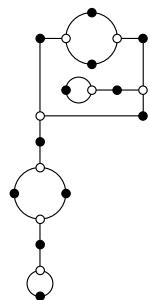
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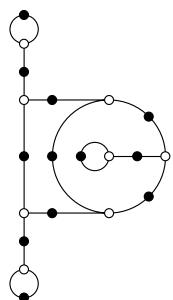
$11, 6, 4, 1, 1, 1 \ (\sqrt{33})$
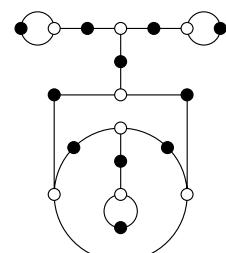
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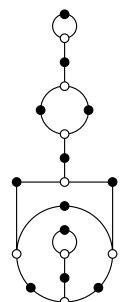
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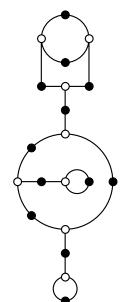
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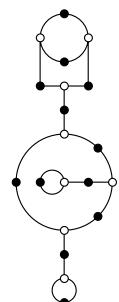
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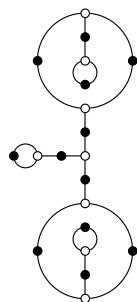
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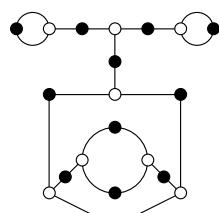
$11, 5, 4, 2, 1, 1 \ (\text{cubic})$
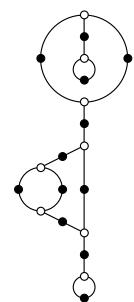
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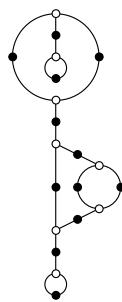
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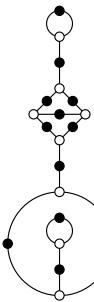
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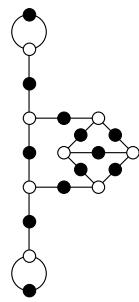
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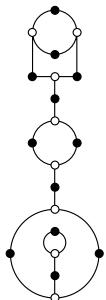
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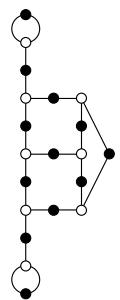
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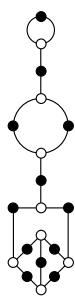
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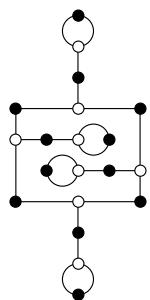
$10, 10, 1, 1, 1, 1 \ (\mathbb{Q})$
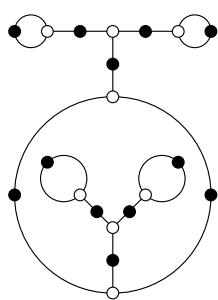
$10, 10, 1, 1, 1, 1 \ (\sqrt{5})$
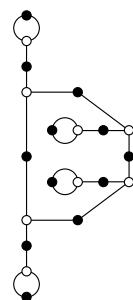
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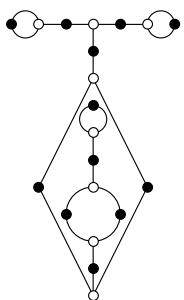
$10, 9, 2, 1, 1, 1 \ (\sqrt{5})$
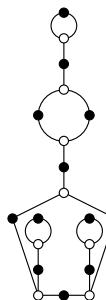
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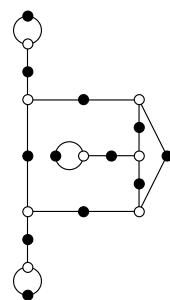
$10, 8, 3, 1, 1, 1 \ (\mathbb{Q})$
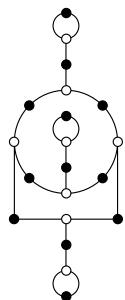
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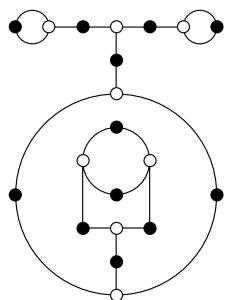
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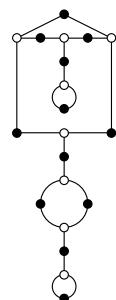
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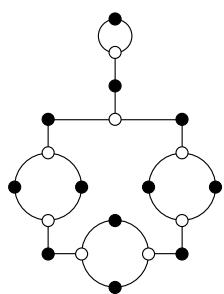
$10, 7, 2, 2, 2, 1 \;(\mathbb{Q})$
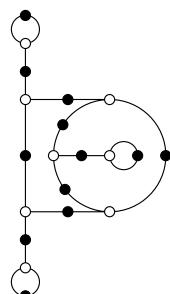
$10, 6, 5, 1, 1, 1 \;(\mathbb{Q})$
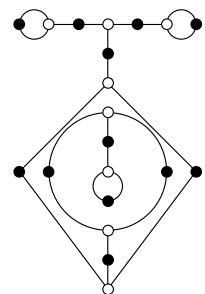
$10, 6, 5, 1, 1, 1 \;(\text{cubic})$
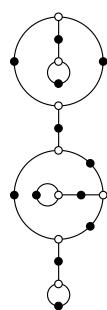
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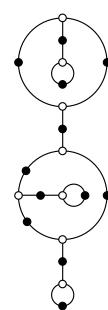
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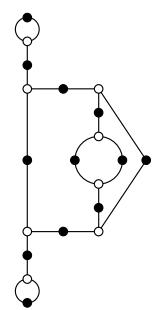
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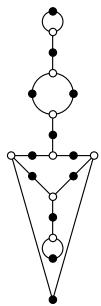
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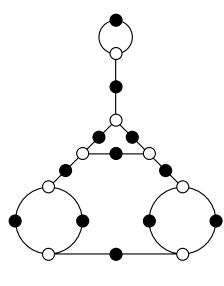
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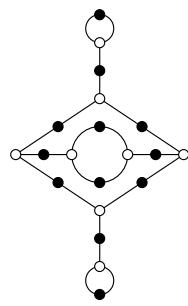
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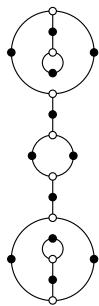
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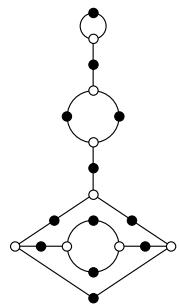
$10, 5, 4, 2, 2, 1 \;(\text{cubic})$
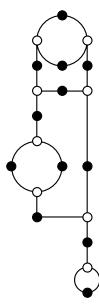
$10, 5, 4, 2, 2, 1 \;(\text{cubic})$
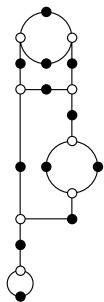
$10, 5, 4, 2, 2, 1$ (cubic)
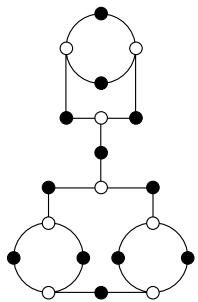
$10, 5, 3, 2, 2, 2$ ($\mathbb{Q}$)
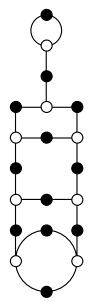
$10, 4, 4, 3, 2, 1$ ($\mathbb{Q}$)
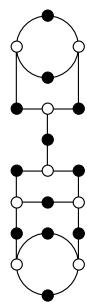
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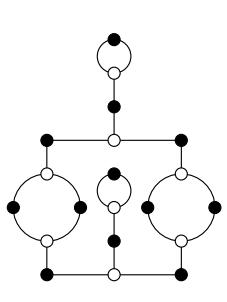
$9, 9, 2, 2, 1, 1$ ($\mathbb{Q}$)
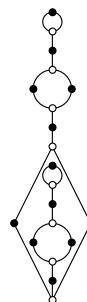
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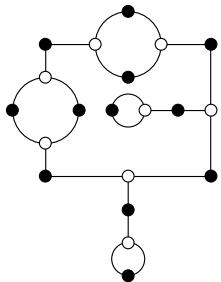
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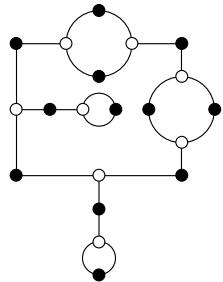
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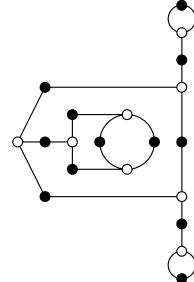
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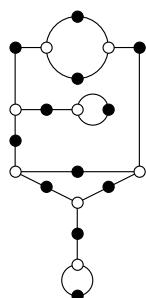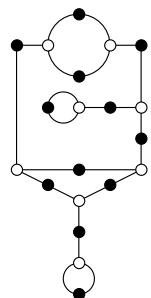
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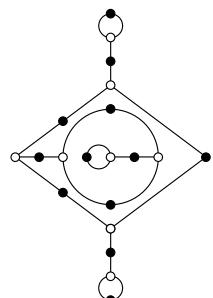
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$9, 7, 5, 1, 1, 1$ ($\mathbb{Q}$)
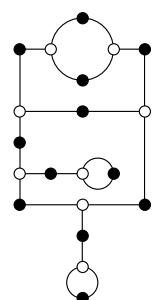
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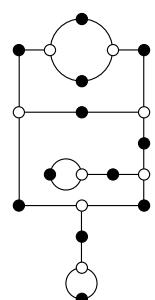
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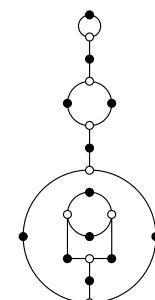
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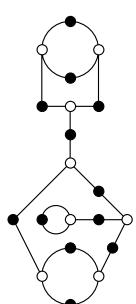
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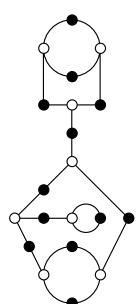
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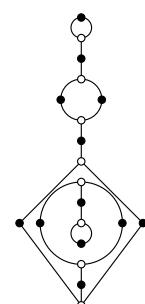
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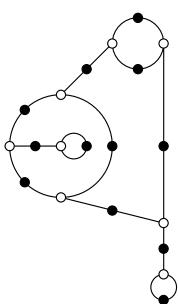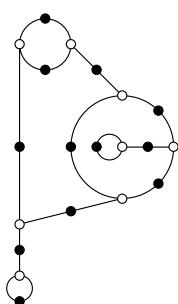
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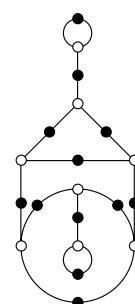
$9, 6, 5, 2, 1, 1$ (cubic)


$9, 6, 4, 3, 1, 1 \ (\mathbb{Q})$
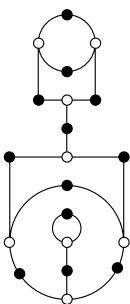
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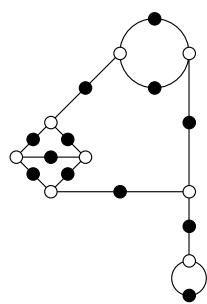
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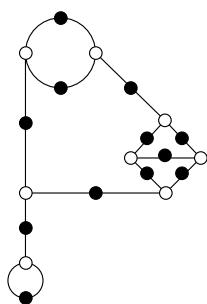
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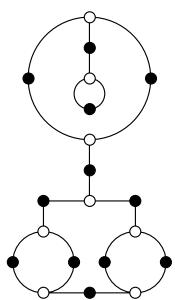
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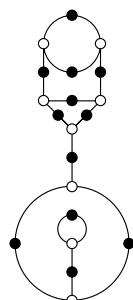
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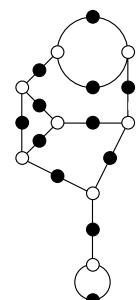
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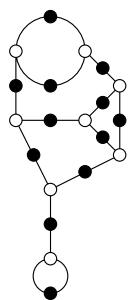
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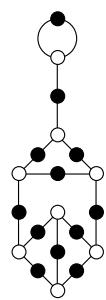
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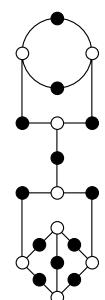
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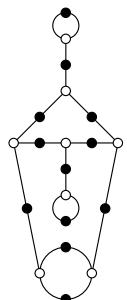
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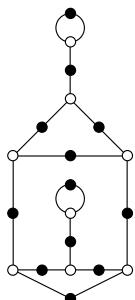
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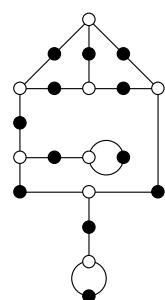
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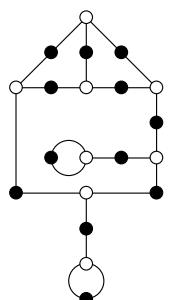
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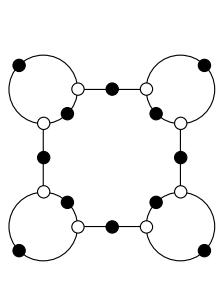
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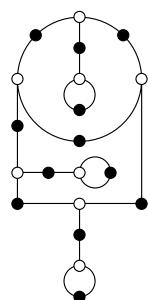
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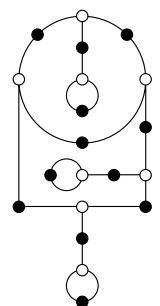
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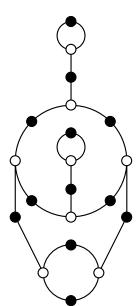
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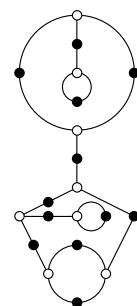
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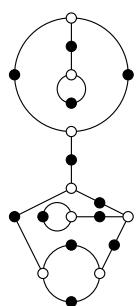
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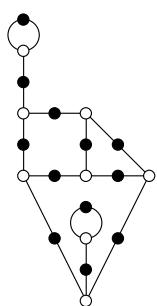
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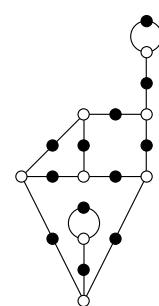
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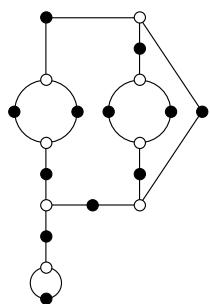
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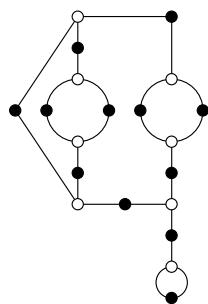
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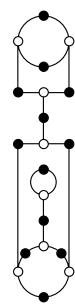
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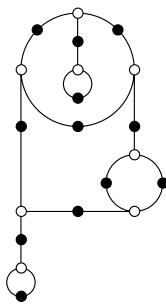
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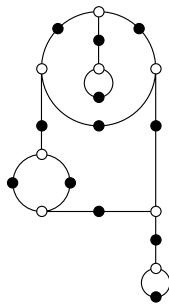
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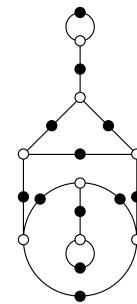
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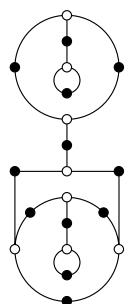
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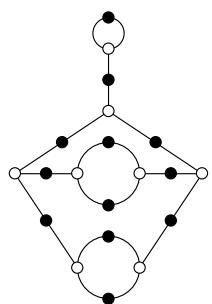
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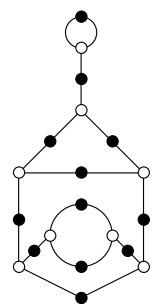
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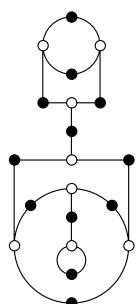
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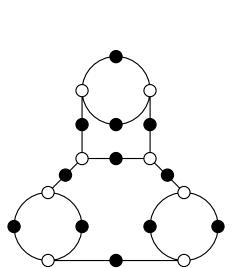
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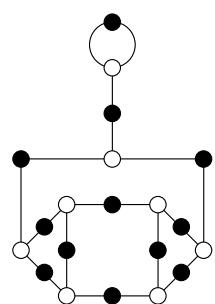
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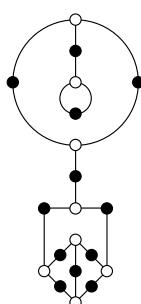
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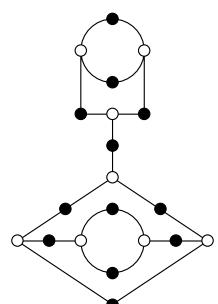
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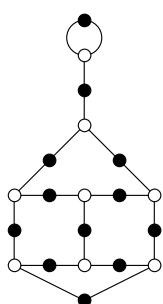
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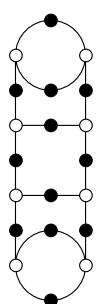
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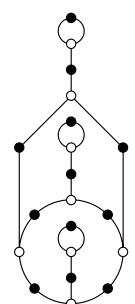
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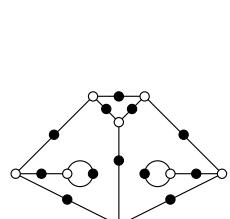
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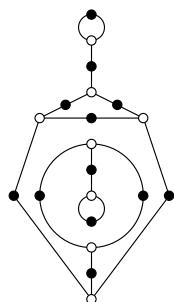
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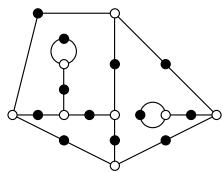
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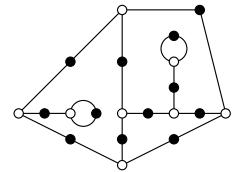
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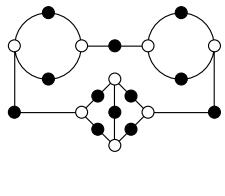
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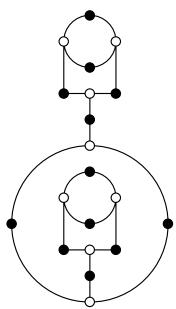
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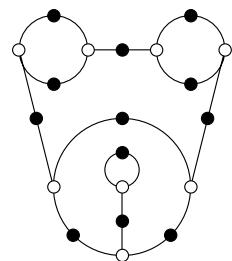
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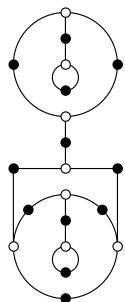
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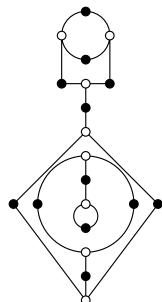
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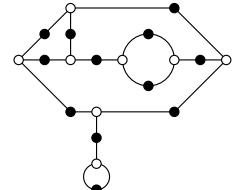
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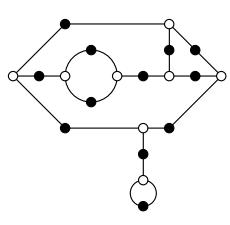
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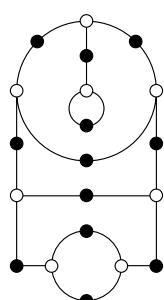
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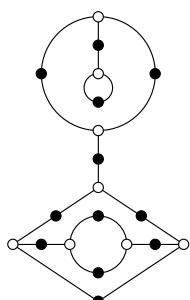
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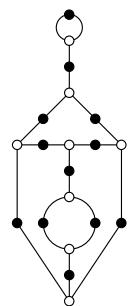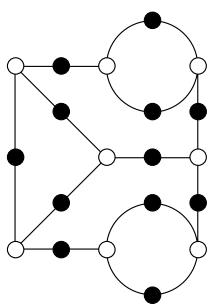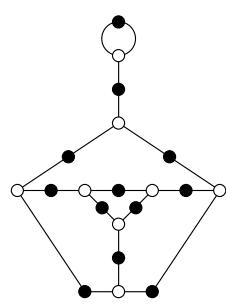
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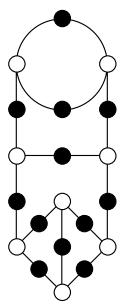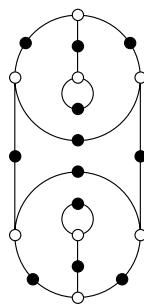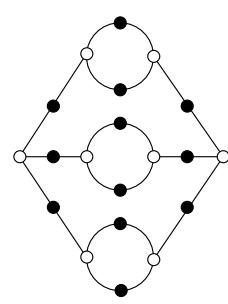
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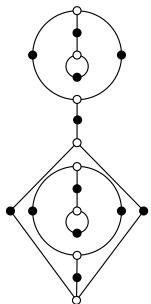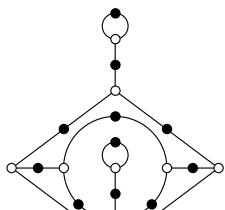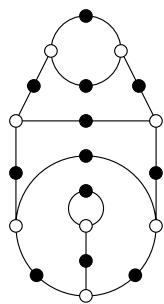
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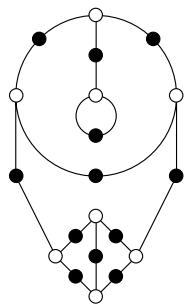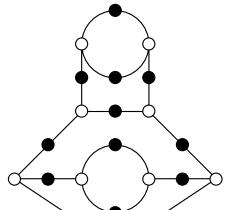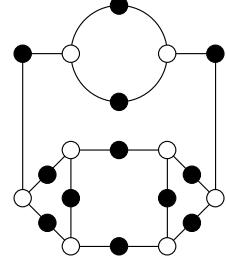
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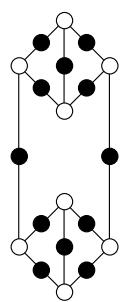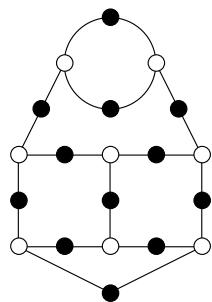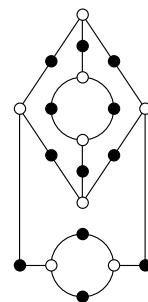
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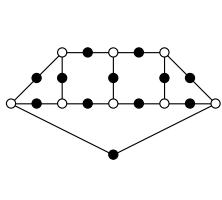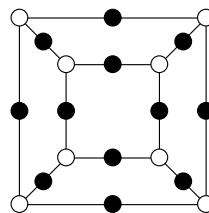
5, 5, 4, 4, 3, 3 ($\mathbb{Q}$)   4, 4, 4, 4, 4, 4 ($\mathbb{Q}$)

References

1. S. Ashok, F. Cachazo and E. Dell'Aquila, 'Children's drawings from Seiberg–Witten curves', Preprint, 2006, arXiv:hep-th/0611082v1.
2. F. Beukers and H. Montanus, 'Explicit calculation of elliptic fibrations of K3-surfaces and their Belyi-maps', *Number theory and polynomials*, LMS Lecture Note Series 352 (Cambridge University Press, Cambridge, 2008) 33–51.
3. F. Beukers, 'Dessins coming from the Miranda–Persson table', 2008, available at http://www.staff.science.uu.nl/~beuke106/mirandapersson/Dessins.html.
4. W. Bosma, J. Cannon and C. Playoust, 'The MAGMA algebra system. I. The user language', *J. Symbolic Comput.* 24 (1997) 235–265.
5. M. Conder, 'Schreier coset graphs and their applications', *RIMS Kokyuroku* 794 (1992) 169–175.
6. J. Conway, A. Hulpke and J. McKay, 'On transitive permutation groups', *LMS J. Comput. Math.* 1 (1996) 1–8.
7. The GAP group, 'GAP—Groups, algorithms, and programming, version 4.5.6', 2012, http://www.gap-system.org.
8. Y.-H. He and J. McKay, '$\mathcal{N} = 2$ gauge theories: congruence subgroups, coset graphs and modular surfaces', *J. Math. Phys.* 54 (2013) 012301.
9. D. Holt, B. Eick and E. O'Brien, *Handbook of computational group theory*, Discrete Mathematics and Its Applications (Chapman and Hall, London, 2005).
10. S. Lando and A. Zvonkin, 'Graphs on surfaces and their applications', *Low-dimensional topology II*, Encyclopaedia of Mathematical Sciences 141 (Springer, New York, 2004).
11. J. McKay and A. Sebbar, 'J-invariants of arithmetic semistable elliptic surfaces and graphs', *Proceedings on Moonshine and related topics,* Montréal, QC, 1999, CRM Proceedings and Lecture Notes 30 (American Mathematical Society, New York, 2001) 119–130.
12. R. Miranda and U. Persson, 'Configurations of in fibers on elliptic K3-surfaces', *Math. Z.* 201 (1989) 339–361.
13. D. Morrison and C. Vafa, 'Compactifications of F theory on Calabi–Yau threefolds - I', *Nucl. Phys.* B 473 (1996).
14. D. Morrison and C. Vafa, 'Compactifications of F theory on Calabi–Yau threefolds - II', *Nucl. Phys.* B 476 (1996).
15. The Sage Group, 'Sage—version 5.3', 2012, http://www.sagemath.org/.
16. L. Schneps (ed.), *The Grothendieck theory of dessins d'enfants*, LMS Lecture Note Series 200 (Cambridge University Press, Cambridge, 1994).
17. A. Sebbar, 'Modular subgroups, forms, curves and surfaces', *Canad. Math. Bull.* 45 (2002) 294–308.

18. T. SHIODA, 'On elliptic modular surfaces', *J. Math. Soc. Japan* 24 (1972) 20–59.
19. D. SINGERMAN and J. WOLFART, 'Cayley graphs, Cori hypermaps, and dessins d'enfants', *Ars Math. Contemp.* 1 (2008) 144–153.
20. J. TOP and N. YUI, 'Explicit equations of some elliptic modular surfaces', *Rocky Mountain J. Math.* 37 (2007) 663–687.
21. A. ZVONKIN, 'Functional composition is a generalized symmetry', *Symmetry: Culture Sci.* 22 (2011) 391–426.

Yang-Hui He
Department of Mathematics
City University
Northampton Square
London EC1V 0HB
United Kingdom

and

School of Physics
NanKai University
Tianjin, 300071
PR China

and

Merton College
University of Oxford
Oxford OX1 4JD
United Kingdom

yang-hui.he@merton.ox.ac.uk

James Read
Oriel College
University of Oxford
Oxford OX1 4EW
United Kingdom

james.read@oriel.ox.ac.uk

John McKay
Department of Mathematics and
  Statistics
Concordia University
1455 de Maisonneuve Boulevard
West Montreal, Quebec, H3G 1M8
Canada

mckay@encs.concordia.ca