ON INTEGER MATRICES AND INCIDENCE MATRICES OF CERTAIN COMBINATORIAL CONFIGURATIONS, III: RECTANGULAR MATRICES

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Introduction. In this paper, we give a connection between incidence matrices of affine resolvable balanced incomplete block designs and rectangular integer matrices subject to certain arithmetical conditions. The definition of these terms can be found in paper II of this series or in **(2)**. For some necessary conditions on the parameters of affine resolvable balanced incomplete block designs and their properties see **(2)**.

We begin with a lemma.

LEMMA. *Let A be a $v \times b$ matrix and*

$$A'A = \begin{bmatrix} B_{11} & B_{12} & B_{13} & \dots & B_{1r} \\ B_{21} & B_{22} & B_{23} & \dots & B_{2r} \\ \dots & \dots & \dots & \dots & \dots \\ B_{r1} & B_{r2} & B_{r3} & \dots & B_{rr} \end{bmatrix}, \tag{3.1}$$

where B_{ii} is an $n_i \times n_i$ matrix with k_i down its main diagonal and $\alpha_i \neq k_i$ elsewhere $(i = 1, 2, \dots, r)$ and B_{ij} is an $n_i \times n_j$ matrix with $c \neq 0$ everywhere $(i \neq j; i, j = 1, 2, \dots, r; \sum_{i=1}^{r} n_i = b)$. If $b \geqslant v + r - 1$ then $b = v + r - 1$ and $cn_i = k_i + (n_i - 1)\,\alpha_i, \quad (i = 1, \dots, r)$.

Proof. We use the notations introduced in the proof of Theorem 2 of paper II.

Write $d_i = k_i - \alpha_i$ and $m_i = k_i + (n_i - 1)d_i$.

In view of the special pattern of the matrix $A'A$, it is not difficult to determine its rank. For this purpose, subtract its N_ith row from its $(N_{i-1} + 1)$-th, $(N_{i-1} + 2)$-th, $\dots$, $(N_{i-1} + n_{i-1})$-th rows, $(i = 1, 2, \dots, r)$. After these operations, add the $(N_{i-1} + 1)$-th, $(N_{i-1} + 2)$-th, $\dots$, $(N_{i-1} + n_{i-1})$-th columns to the N_ith column $(i = 1, 2, \dots, r)$. Then on permuting some of its rows and columns it appears as

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$$
(3.2)\qquad \begin{bmatrix}
d_1 \\
& d_1 \\
&& \ddots \\
&&& d_1 \\
&&&& d_2 \\
&&&&& d_2 \\
&&&&&& \ddots \\
&&&&&&& d_2 \\
&&&&&&&& \ddots \\
&&&&&&&&& d_r \\
&&&&&&&&&& d_r \\
&&&&&&&&&&& \ddots \\
&&&&&&&&&&&& d_r \\
&&&&&&&&&&&&& m_1 & cn_2 & cn_3 & \dots & cn_r \\
&&&&&&&&&&&&& cn_1 & m_2 & cn_3 & \dots & cn_r \\
&&&&&&&&&&&&& \cdot & \cdot & \cdot & \dots & \dots \\
&&&&&&&&&&&&& cn_1 & cn_2 & cn_3 & \dots & m_r
\end{bmatrix},
$$

where the unwritten elements in the

$$\sum_{i=1}^{r} (n_i - 1) = b - r$$

first rows and columns are all 0. As $d_i \neq 0$, the rank of this matrix is $b - r$ plus the rank of the submatrix at the lower corner. As the operations performed have not changed the rank of $A'A$, the rank of this new matrix equals the rank of $A'A$ = rank of $A \leqslant v$. Now remembering that $c \neq 0$, the rank of the submatrix is seen to be at least one. By (i) of the hypothesis we immediately infer that the rank of the submatrix is exactly one and then $b = v + r - 1$. Considering the submatrix again, we obtain the relation

$$cn_i = m_i = d_i + n_i \alpha_i \qquad (i = 1, 2, \dots, r).$$

This completes the proof.

THEOREM 3. *Let A be a $v \times b$ integer matrix such that*

$$(3.3)\qquad A'A = \begin{bmatrix} B_{11} & B_{12} & \dots & B_{1r} \\ B_{21} & B_{22} & \dots & B_{2r} \\ \dots & \dots & \dots & \dots \\ B_{r1} & B_{r2} & \dots & B_{rr} \end{bmatrix},$$

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where B_{ii} *is an* $n_i \times n_i$ *matrix with* k_i *down the main diagonal, and* α_i *elsewhere* $(i = 1, 2, \ldots, r)$ *and* B_{ij} *is an* $n_i \times n_j$ *matrix with* $c \neq 0$ *everywhere* $(i \neq j; i, j = 1, 2, \ldots, r; \sum_{i=1}^{r} n_i = b$. *Moreover, suppose that*

(i) $b \geqslant v + r - 1$,
(ii) $v \geqslant l(\delta - 1)(\lambda q)^{-1} \neq 0$,
(iii) $1 \leqslant q(\tilde{r} - \lambda)l^{-1}$,
(iv) $q\tilde{r}v \leqslant b$,
(v) *all the* $(k_i - \alpha_i)$'s *are odd*,

where

$$\sum_{i=1}^{r} c(c - \alpha_i)^{-1} = \delta,$$

l *is the least common multiple of the* $(k_i - \alpha_i)$'s, $l > 0$, *and*

$$l\delta^2 c^{-1}(\delta - 1)^{-1} = \tilde{r}^2 q\lambda^{-1},$$

$\tilde{r}$, q, λ *denoting integers*, $\tilde{r} > 0$, $\lambda > 0$, *and* q *being square-free and* $(\tilde{r}^2, q\lambda) = 1$.

Then either A *is the incidence matrix of an affine resolvable b.i.b. design or becomes one when some of its rows are multiplied by* -1.

Proof. Use the notations given in the proofs of Theorem 2 and the above lemma. Put

$$\sum_{\nu=N_{t-1}+1}^{N_t} a_{i\nu}{}^2 = r_{it}, \qquad \sum_{\nu=N_{t-1}+1}^{N_t} a_{i\nu} a_{j\nu} = r_{ijt} \qquad (i \neq j),$$
$$h_t = (c - \alpha_t)^{\frac{1}{2}} \qquad (i, j = 1, 2, \ldots, v;\ t = 1, 2, \ldots, r).$$

As by (v) of the hypothesis $d_i \neq 0$, we infer, with the aid of the lemma, that

$$b = v + r - 1, \qquad c = m_i n_i^{-1},$$
$$h_i^{-2} = (c - \alpha_i)^{-1} = n_i d_i^{-1} > 0 \qquad (i = 1, 2, \ldots, r). \tag{3.4}$$

It follows that $\delta \geqslant r$.

To prove our theorem, we augment A by introducing r new row vectors at its bottom and one column vector on its right-hand side. The ith row of the new row vectors is

$$(0, 0, \ldots, 0, h_i, h_i, \ldots, h_i, 0, 0, \ldots, 0), \tag{3.5}$$

where the elements at the $(N_{i-1} + 1)$-th, $(N_{i-1} + 2)$-th, $\ldots$, N_ith positions are all h_i $(i = 1, 2, \ldots, r)$. The new column vector has 0 in its first v positions and then $c/h_1, c/h_2, \ldots, c/h_r$.

Let the square matrix thus obtained be B. Then, using the hypothesis, we see that the off-diagonal elements of $B'B$ are equal to c while the jth diagonal element is

$$k_i - \alpha_i + c \geqslant c + 1 \quad \text{when} \quad N_{i-1} < j \leqslant N_i \quad (i = 1, 2, \ldots, r).$$

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It is $c\delta \neq 0$ when $j = b + 1$. Hence $|B'B| = c\delta^2 \neq 0$. So for some column vector ξ, $\xi' B = (c, c, \ldots, c)$. Write the elements of ξ as $x_1, x_2, \ldots, x_v, x_{v+1} h_1, x_{v+2} h_2, \ldots, x_{v+r} h_r$. Then we see that these x's are rational numbers.

Let A_0 be the $(b + 2) \times (b + 2)$ matrix obtained by adjoining ξ to the right of B and a $(b + 2)$-vector $((-c)^{\frac{1}{2}}, (-c)^{\frac{1}{2}}, \ldots, (-c)^{\frac{1}{2}})$ at its bottom. Then A_0 has the form

$$(3.6) \qquad A_0 = \begin{bmatrix} A_1 & A_2 & \ldots & A_r & 0 & x_1 \\ & & & & 0 & x_2 \\ & & & & \vdots & \vdots \\ & & & & 0 & x_v \\ h_1 h_1 \ldots h_1 & & & & c/h_1 & x_{v+1} h_1 \\ & h_2 h_2 \ldots h_2 & \ldots & & c/h_2 & x_{v+2} h_2 \\ & & & & \vdots & \vdots \\ & & \ldots & h_r h_r \ldots h_r & c/h_r & x_{v+r} h_r \\ (-c)^{\frac{1}{2}} (-c)^{\frac{1}{2}} \ldots & \ldots & \ldots & \ldots & (-c)^{\frac{1}{2}} & (-c)^{\frac{1}{2}} \end{bmatrix},$$

where the empty spaces are to be filled by 0's.

It is easy to see that all the x's above can be assumed to be non-negative, if necessary by multiplying some of the rows of A by -1. Thus if x_i is negative for some i $(1 \leqslant i \leqslant v)$, we multiply the ith row of A by -1. If for some i $(1 \leqslant v \leqslant r)$, x_{v+i} is negative, we interpret the corresponding h_i as the negative square root of $c - \alpha_i$. The matrix thus modified satisfies the hypothesis of the theorem. Without changing the notation, we shall suppose that A is the modified matrix, and all the x_i's are non-negative.

By (3.3) the scalar product of any two of the first $b + 1$ columns of A_0 is 0. It is now easily seen that the column vectors of A_0 are linearly independent in the field of complex numbers. So $|A_0|^2 = A_0' A_0 \neq 0$. But $A_0' A_0$ is a diagonal matrix D with $d_1, d_1, \ldots, d_1; d_2, d_2, \ldots, d_2; d_r, d_r, \ldots, d_r; u, w$ as its elements, where

$$(3.7) \qquad u = \sum_{i=1}^{r} c^2 h_i^{-2} - c = c(\delta - 1) \neq 0,$$

$$(3.8) \qquad w = \sum_{i=1}^{v} x_i^2 + \sum_{i=1}^{r} h_i^2 x_{v+i}^2 - c \neq 0.$$

Now from $A_0 D^{-1} A_0 = I$, where I is the identity matrix, comparing elements on both sides, we derive

$$(3.9) \qquad r_{i1} d_1^{-1} + r_{i2} d_2^{-1} + \ldots + r_{ir} d_r^{-1} + x_i^2 w = 1 \qquad (i = 1, 2, \ldots, v),$$

$$(3.10) \qquad r_{ij1} d_1^{-1} + r_{ij2} d_2^{-1} + \ldots + r_{ijr} d_r^{-1} + x_i x_j w^{-1} = 0 \qquad (i \neq j; i, j = 1, 2, \ldots, v),$$

$$(3.11) \qquad s_{i1} d_1^{-1} + s_{i2} d_2^{-1} + \ldots + s_{ir} d_r^{-1} + x_i w^{-1} - 0 \qquad (i = 1, 2, \ldots, v),$$

$$(3.12) \qquad h_i n_i d_i^{-1} + c u^{-1} h_i^{-1} + x_{v+i} h_i w^{-1} = 0 \qquad (i = 1, 2, \ldots, r),$$

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(3.13) $$-cn_1 d_1^{-1} - cn_2 d_2^{-1} - \ldots - cn_r d_r^{-1} - cu^{-1} - cw^{-1} = 1.$$

From (3.13) and (3.7) and the hypothesis, one obtains

(3.14) $$w^{-1} = \delta^2 c^{-1}(1-\delta)^{-1} = -\tilde{r}^2 q(l\lambda)^{-1}.$$

From (3.12), (3.4), (3.7), and (3.14), we obtain

(3.15) $$x_{v+i} = c\,\delta^{-1} h_i^2 \qquad (i = 1, 2, \ldots, r).$$

Multiplying (3.9) by the least common multiple l of the d_i's, we see that for each i, $-x_i^2 lw^{-1}$ is an integer, i.e. $x_i^2 \tilde{r}^2 q\lambda^{-1}$ is an integer by (3.14). As q is square-free, we must have $rx_i = y_i$, where y_i is an integer $(i = 1, 2, \ldots, v)$. Also multiplying (3.11) by l, we see that

$$-x_i lw^{-1} = x_i \tilde{r}^2 q\lambda^{-1} = y_i \tilde{r} q\lambda^{-1}$$

is also an integer. By hypothesis $(\tilde{r}q, \lambda) = 1$, so $\lambda | y_i$ and thus $x_i = \lambda z_i(\tilde{r})^{-1}$, where z_i is an integer $(i = 1, 2, \ldots, v)$.

Moreover, no x_i can be 0, for then we can derive a contradiction from (3.9) and (3.11) as $r_{ij} \equiv s_{ij} \pmod 2$ and $d_i \equiv 1 \pmod 2$, exactly as in the proof of Theorem 1.

Now from (3.8), (3.15), (3.7), and (3.14), we have

(3.16) $$\sum_{i=1}^{v} x_i^2 = w + c(1-\delta^{-1}) = c(1-\delta)\delta^{-2} + c(1-\delta^{-1})$$
$$= c(\delta-1)^2\delta^{-2} = (\delta-1)l\lambda(\tilde{r}^2 q)^{-1}$$

and so

(3.17) $$\sum_{i=1}^{v} z_i^2 = (\delta-1)l\lambda^{-1}q^{-1} \qquad \text{as } x_i = \lambda z_i \tilde{r}^{-1}.$$

As the z_i's are non-zero integers, (3.17) in conjunction with (ii) of the hypothesis gives $z_i^2 = 1$ $(i = 1, 2, \ldots, v)$ and $l(\delta - 1)(\lambda q)^{-1} = v$. Thus $\tilde{r}x_i = \lambda$, as the x_i's are non-negative by virtue of our modification.

As the scalar product of the last column vector of A_0 with the others is 0, it now follows from (3.6) and (3.15) that

(3.18) each column sum of $A = (-c\delta^{-1} + c)\tilde{r}\lambda^{-1} = c(1-\delta^{-1})\tilde{r}\lambda^{-1} = k$, say.

Subtracting (3.11) from (3.9) we obtain, using (3.14),

(3.19) $$\sum_{j=1}^{r} (r_{ij} - s_{ij})d_j^{-1} = 1 - x_i^2 w^{-1} + x_i w^{-1} = 1 + \lambda q l^{-1} - \tilde{r} q l^{-1}.$$

As $r_{ij} \geqslant s_{ij}$, (3.19) together with (iii) of the hypothesis gives $1 = (\tilde{r} - \lambda)ql^{-1}$ and $r_{ij} = s_{ij}$ $(i = 1, 2, \ldots, v; j = 1, 2, \ldots, r)$. From this, we immediately derive the useful fact that the elements of A are equal to 1 or 0.

Summing up (3.11) over i and using (3.18), we obtain

(3.20) $$\sum_{i=1}^{r} kn_i d_i^{-1} = -\sum_{i=1}^{r} x_i w^{-1} = (\lambda v\tilde{r}^{-1})(\tilde{r}^2 q)(l\lambda)^{-1} = \tilde{r}vql^{-1},$$

i.e.,

$$\sum_{i=1}^{r} n_i(kd_i^{-1} - 1) = \tilde{r}vql^{-1} - b. \tag{3.21}$$

As the α_i's are non-negative integers and $d_i = k_i - \alpha_i = k - \alpha_i \leqslant k$, the terms on the left are non-negative; (3.21) and (iv) of the hypothesis enable us to deduce that $\tilde{r}vql^{-1} = b$ and $d_i = k$ for $i = 1, 2, \ldots, r$ and then $\alpha_i = 0$ $(i = 1, 2, \ldots, r)$. Consequently, $m_i = k$ and from (3.4) all the n_i's are equal, to n say, and then $nc = k$. Also $l = k$, $nr = b$, and

$$\delta = \sum_{i=1}^{r} c(c - \alpha_i)^{-1} = r.$$

From (3.10) it follows now that the scalar product of any two rows of A is

$$-x_i x_j kw^{-1} = \lambda^2\tilde{r}^{-2}(\tilde{r}^2qk)(l\lambda)^{-1} = \lambda q. \tag{3.22}$$

Next consider the quantity

$$\sum_{\nu=1}^{v} (s_{\nu i} - s_{\nu j})^2.$$

Clearly it is equal to $(\xi_i - \xi_j)^2$, where ξ_i and ξ_j are respectively the sums of the column vectors of A_i and A_j. Now $\xi_i^2 = \xi_j^2 = nk$ and $n\xi_i \xi_j = n^2k$. Hence $s_{\nu i} = s_\nu$, say $(\nu = 1, 2, \ldots, v; i = 1, 2, \ldots, r)$.

Further, no s_ν can be zero. If possible, let $s_1 = 0$, without loss of generality. Then all the elements in the first row of A are zero, requiring its scalar product with the other rows to be zero and contradicting (3.22). It therefore follows that $s_\nu = 1$ $(\nu = 1, 2, \ldots, v)$. This means that every row of A_i contains just one 1 and the rest of the elements are 0. Consequently any row sum of A is r.

Remembering that $l = k$, $\delta = r$, $z_i^2 = 1$, we obtain, from (3.17),

$$nk = v = (r - 1)k(\lambda q)^{-1},$$

when $n|(r - 1)$. One also has

$$l\delta^2c^{-1}(\delta - 1)^{-1} = kr^2(k(r - 1))^{-1} n = r^2(r - 1)^{-1}n = \tilde{r}^2q\lambda^{-1}. \tag{3.23}$$

But as $1 = (r\,r - 1) = (r^2, r - 1) = (r^2\ (r - 1)/n)$ we infer by virtue of our assumption that $r = \tilde{r}$, $q = 1$, $n\lambda = (r - 1)$. So the scalar product of any two rows of A is $\lambda q = \lambda$. Hence A is the incidence matrix of an affine resolvable b.i.b. design with the parameters $v = nk$, $b = nr$, r, k, λ with $b = v + r - 1$. This completes the proof.

We now show by examples that the conditions of the theorem cannot be relaxed. Let

$$A = \begin{bmatrix} 2 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -2 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

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Here $v = 4$, $b = 6$, $r = 3$, $n_1 = n_2 = n_3 = 2$, $k_1 = k_2 = k_3 = 5$, $\alpha_1 = \alpha_2 = \alpha_3 = -3$, $d_1 = d_2 = d_3 = 8$, $l = 8$, $c = 1$, $\delta = \frac{3}{4}$, $r = 3$, $q = -2$, $\lambda = 1$. All the conditions of the theorem are satisfied except (iii) and (v). Clearly A cannot be the incidence matrix of a b.i.b. design.

Consider another example. From each element of the existent incidence matrix of an affine resolvable b.i.b. design with $v = 27$, $b = 39$, $r = 13$, $k = 9$, $\lambda = 4$, subtract 1. Let the matrix thus obtained be A. For this matrix, we have $v = 27$, $b = 39$, $r = 13$, $n_i = 13$, $k_i = 18$, $\alpha_i = 9$, $d_i = 9$, $l = 9$, $c = 12$, $\delta = 52$, $\tilde{r} = 26$, $q = 1$, $\lambda = 17$. So all the conditions of the theorem hold except (iv). It is easy to see that A or its modification cannot be the incidence matrix of any affine resolvable b.i.b. design.

THEOREM 4. *Let A be an integer matrix such that*

$$A'A = \begin{bmatrix} B_{11} & B_{12} & \dots & B_{1r} \\ B_{21} & B_{22} & \dots & B_{2r} \\ B_{r1} & B_{r2} & \dots & B_{rr} \end{bmatrix},$$

where B_{ii} are $n_i \times n_i$ matrices with k_i down the main diagonal and $\alpha_i \neq k_i$ elsewhere $(i = 1, 2, \dots, r)$ and B_{ij} is an $n_i \times n_j$ matrix with $c \neq 0$ everywhere $(i \neq j;\ i, j = 1, 2, \dots, r;\ \sum_{i=1}^{r} (n_i - b)$.

Moreover, assume that

(i) $b \geqslant v + r - 1$;

(ii) $k_j + (n_j - 1)\alpha_j > k_i + (n_i - 1)\alpha_i$ *if* $n_i > n_j$;

(iii) $\sum_{i=1}^{r} \alpha_i \geqslant 0$;

(iv) *at least for some* k_j, *say* $k_1, k_1, n_1 \leqslant v$;

(v) *the square of the length of any row vector of A is odd.*

Then either A is the incidence matrix of an affine resolvable b.i.b. design or becomes one when some of its rows are multiplied by -1.

Proof. We use the notations of the proofs of Theorems 2 and 3 and the lemma. Nore also that the lemma can be utilized in this proof.

As before we have

$$\sum_{\nu=1}^{v} (s_{\nu i} - s_{\nu j})^2 = (\xi_i - \xi_j)^2,$$

where ξ_i and ξ_j are the sums of the columns of A_i and A_j respectively. Now by the hypothesis of this theorem

$$\xi_i^2 = (k_i + (n_i - 1)\alpha_i)n_i = m_i n_i, \qquad \xi_j^2 = m_j n_j,$$

and

$$\xi_i \xi_j = cn_i n_j = m_i n_j = m_j n_i$$

by virtue of the lemma. Consequently, we have by our hypothesis,

$$\sum_{\nu=1}^{v} (s_{\nu i} - s_{\nu j})^2 = m_i n_i + m_j n_j - m_i n_j - m_j n_i = (m_i - m_j)(n_i - n_j) \leqslant 0. \tag{3.24}$$

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This relation implies that $n_i = n$, say, and $s_{\nu i} = s_\nu$, say $(i = 1, 2, \ldots, r)$. Furthermore, no s_ν can be zero; otherwise we can derive a contradiction from (v) of our hypothesis exactly as in the proof of Theorem 1 of paper I.

Next, by (iv) of the hypothesis,

$$\sum_{\nu=1}^{n} a_{1\nu}^2 + \sum_{\nu=1}^{n} a_{2\nu}^2 + \ldots + \sum_{\nu=1}^{n} a_{v\nu}^2 = k_1 n_1 = k_1 n \leqslant v. \tag{3.25}$$

Now none of the v sums on the left can be zero. For if, say, the first one vanishes, then

$$0 = \sum_{\nu=1}^{n} a_{1\nu}^2 = \sum_{\nu=1}^{n} a_{1\nu} = s_1, \tag{3.26}$$

and this cannot be true. Consequently, (3.25) holds with the equality sign and each sum on the left is one. This implies that each row of A_1 contains precisely one non-zero element. This element can be 1 or -1. Suppose in the ith row of A_1 it is -1. Then all the row sums of the ith rows of A_2, A_3, $\ldots, A_r$ are also -1 by what we have already proved. Consequently, the ith row sum of A is negative. Multiplying all the rows of A, whose sums are negative, by -1, we can make all the row sums of A non-negative and simultaneously make all the non-zero elements of A_1 equal to 1. If initially some of the row sums of A were negative, this modification will be necessary; otherwise not. We show that A thus modified is an incidence matrix of an affine resolvable b.i.b. design. Clearly the modified matrix satisfies all the conditions of the theorem. Without changing notation we shall henceforth suppose that A itself is the modified matrix.

We now readily obtain $s_\nu = 1$ $(\nu = 1, 2, \ldots, v)$. Also $\alpha_1 = 0$, $m_i = k_i = k$, say, and then by the lemma, $c = k/n$. Considering the scalar product of any column vector of A_i $(i = 2, 3, 4, \ldots, r)$ with the sum of column vectors of A_1, we conclude that the sum of the elements of each column vector of A is equal to $cn = k$. Further, by the lemma, $k_i + (n-1)\alpha_i = m_i = m_1 = k$ $(i = 1, 2, \ldots, r)$. But by our hypothesis and what we have just shown, we have

$$k_i = \sum_{\nu=1}^{v} a_{\nu j}^2 \geqslant \sum_{\nu=1}^{v} a_{\nu j} = k, \qquad N_{i-1} < j \leqslant N_i \qquad (i = 1, 2, \ldots, r). \tag{3.27}$$

As $k_i - k \geqslant 0$, we must have $\alpha_i \leqslant 0$ $(i = 1, 2, 3, 4, \ldots, r)$. But this, in conjunction with (iii) of the hypothesis, implies that $\alpha_i = 0$ for all i, and thus $k_i = d_i = k$ for $i = 1, 2, \ldots, r$. This in its turn implies that the elements of each column vector of A are not different from 0 and 1. Consequently, A is a 0–1 matrix. It now follows that each row sum of A is r. That the scalar product of any two of its rows is a non-zero constant follows from the relations (3.11) and (3.10) in the proof of the preceding theorem, when we use the facts proved above that $d_i = k$ and

$$\sum_{j=1}^{r} s_{ij} = r \qquad (i = 1, 2, \ldots, v).$$

These make the x_i's equal in (3.10), and then (3.11) gives the constancy of the scalar product. Hence the (modified) $v \times (v + r - 1)$ matrix A is the incidence matrix of an affine resolvable b.i.b. design.

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