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NORM DECREASING HOMOMORPHISMS BETWEEN IDEALS OF $L^p(G)$

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1. Introduction. Let G_1 and G_2 be compact groups and $T : L_p(G_1) \rightarrow L_p(G_2)$ $(1 \leqq p \leqq \infty)$ be an algebra homomorphism. If $||T|| \leqq 1$ and T is either a monomorphism of an epimorphism then T can in many cases be explicitly characterized (see [**4**; **8**; **9**; **11**; **13**; **14**]). Excluding $p = 2$, the outstanding cases are $1 < p < \infty$ for monomorphisms and $2 < p < \infty$ for epimorphisms (cf. [**14**]). One aim of the present note is to complete this work. We also consider the problem of extending these results in some form to homomorphisms on ideals of group algebras; the only known result in this area is for abelian groups [**3**].

The characterization of isometries on subspaces of L^p-spaces that preserve the constant functions has recently been completed by Rudin [**10**]. However, in this paper we use the techniques in the earlier work of Forelli [**2**]. We extend his ideas in Section 2, mainly with applications to group algebras in mind, although we believe there is some independent value for these results. The main theorems are in Section 3.

2. L_p-norm decreasing operators. Throughout this section X and Y will be compact Hausdorff spaces and μ and ν will be probability measures on X and Y respectively such that whenever $U \subset X$ and $V \subset Y$ are open and non-empty then $\mu(U) > 0$ and $\nu(V) > 0$. We denote by $L_p(X)$ and $L_p(Y)$, $(1 \leqq p \leqq \infty)$ the spaces of complex L_p-functions on (X, μ) and (Y, ν). As usual

$$||f||_p = \left\{ \int_X |f|^p d\mu \right\}^{1/p} \quad f \in L_p(X), p < \infty$$

$$||f||_\infty = \mu - \text{ess. sup } |f| \quad f \in L_\infty(X)$$

and similarly for $L_p(Y)$. We shall identify functions which are equal μ-a.e. or ν-a.e.

Our first lemma is a trivial generalization of a result of Forelli [**2**, Proposition 1]. We omit the proof, which is identical to Forelli's. We denote the constantly one function on X or Y by 1.

LEMMA 2.1. *Suppose* $1 \leqq p < \infty$, $f \in L_p(X)$ *and* $g \in L_p(Y)$ *and*

$$||1 + zf||_p \geqq ||1 + zg||_p \quad \textit{for all } z \in \mathbf{C}.$$

Then $||f||_2 \geqq ||g||_2$ *(in particular, if* $f \in L_2(X)$ *then* $g \in L_2(Y)$*).*

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As an immediate application, we have

THEOREM 2.2. *Suppose* $1 \leqq p < \infty$ *and* $T : L_p(X) \to L_p(X)$ *is a linear operator satisfying* $T1 = 1$ *and* $||T|| = 1$. *Then there is a sub-*σ*-algebra* Σ *of the Borel sets of* X *such that* $Tf = f$ *if and only if* f *is* Σ*-measurable.*

Proof. Let $A_n = (1/n)(T + \ldots + T^n)$. For $p > 1$, A_n converges in the strong operator topology to a contractive projection onto $E = \{f : Tf = f\}$, by the Ergodic Theorem [**1**, p. 662]. Since E is the range of the contractive projection, the result follows from the known form of such projections [**6**, p. 162]. For $p = 1$, we observe that, by Lemma 2.1, $||Tf||_2 \leqq ||f||_2$, and hence applying the Ergodic Theorem to $L_2(X)$, $A_n f$ converges for $f \in L_2$. Hence (see [**1**, p. 662, Cor. 2 & 3]) A_n converges to a contractive projection on L_1 and the result again follows.

Remark. It is not difficult to deduce from Theorem 2.2 the more general

THEOREM 2.3. *Let* (S, Σ, λ) *be a probability space and* $T : L_p(S, \Sigma, \lambda) \to L_p(S, \Sigma, \lambda)$ *be a linear operator with* $||T|| = 1$. *Then there is a sub-*σ*-algebra* Σ_0 *of* Σ *and* $B \in \Sigma_0$ *such that* $Tf = f$ *if and only if* f *is* Σ_0*-measurable and*

$$\lambda\{s : |f(s)| > 0, s \notin B\} = 0.$$

For our next theorem, we need the following technical lemma.

LEMMA 2.4. *Suppose* $\{a_{jk} : j, k = 0, 1, 2, \ldots\}$ *and* $\{b_{jk} ; j, k = 0, 1, 2, \ldots\}$ *are complex numbers satisfying*

(i) $a_{jk} = \bar{a}_{kj}$, $b_{jk} = \bar{b}_{kj}$;

(ii) $a_{jj}b_{jj} \geqq 0$ *for all* j, *and* $a_{jj}b_{jj} = 0$ *implies* $a_{jj} = b_{jj} = 0$;

(iii) *there exists* $r > 0$ *such that for* $|z| < r$, *the doubly infinite series* $\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{jk}z^j\bar{z}^k$ *and* $\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} b_{jk}z^j\bar{z}^k$ *converge absolutely, and*

$$\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{jk}z^j\bar{z}^k \geqq 0 \geqq \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} b_{jk}z^j\bar{z}^k.$$

Then $a_{jk} = 0$ *for all* j, k.

Proof. For $0 < \rho < r$

$$(1) \quad \int_0^{2\pi} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{jk}\rho^{j+k} \exp i(j-k)\theta d\theta = \sum_{j=0}^{\infty} a_{jj}\rho^{2j} \geqq 0$$

and similarly

$$\sum_{j=0}^{\infty} b_{jj}\rho^{2j} \leqq 0.$$

It follows easily by induction that $a_{jj} = b_{jj} = 0$ for all j. Hence, as the integrand in (1) is everywhere non-negative, we conclude

$$\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{jk}z^j\bar{z}^k \equiv 0 \quad |z| < r.$$

Thus $a_{jk} = 0$ for all j, k.

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THEOREM 2.5. *Let L be a subspace of $C(X)$ containing 1 and separating the points of X. Suppose $T : L \to C(Y)$ is a linear operator satisfying $T1 = 1$ and such that for some p, r $1 \leqq p < \infty$, $1 \leqq r < \infty$ where neither p nor r is an even integer,*

$$||Tf||_p \leqq ||f||_p \quad f \in L$$
$$||Tf||_r \geqq ||f||_r \quad f \in L.$$

Then there is a continuous map $\alpha : Y \to X$, satisfying $\nu(\alpha^{-1}(B)) = \mu(B)$ whenever B is a Borel subset of X, such that

$$Tf(y) = f(\alpha(y)) \quad y \in Y, f \in L.$$

In particular T extends to a multiplicative homomorphism of $C(X)$ into $C(Y)$.

Remark. This theorem may be regarded as a generalization of a theorem of Forelli [**2**]. The most obvious application is the case $p = r$ when T is an isometry on L.

Proof. For $f \in L$ and $z \in \mathbf{C}$,

$$\int_Y |1 + zTf|^p d\nu \leqq \int_X |1 + zf|^p d\mu.$$

For small enough $|z|$, since f and Tf are bounded,

$$|1 + zf|^p = (1 + zf)^{p/2}(1 + \bar{z}\bar{f})^{p/2}$$
$$= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \binom{p/2}{j} \binom{p/2}{k} z^j \bar{z}^k f^j \bar{f}^k$$

and

$$|1 + zTf|^p = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \binom{p/2}{j} \binom{p/2}{k} z^j \bar{z}^k (Tf)^j (\overline{Tf})^k.$$

Hence for small $|z|$,

$$\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{jk} z^j \bar{z}^k \geqq 0$$

where

$$a_{jk} = \binom{p/2}{j} \binom{p/2}{k} \left(\int_X f^j \bar{f}^k d\mu - \int_Y (Tf)^j (\overline{Tf})^k d\nu \right).$$

Similarly, for small $|z|$,

$$\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} b_{jk} z^j \bar{z}^k \leqq 0$$

where

$$b_{jk} = \binom{r/2}{j} \binom{r/2}{k} \left(\int_X f^j \bar{f}^k d\mu - \int_Y (Tf)^j (\overline{Tf})^k d\nu \right).$$

Clearly

$$a_{jj}b_{jj} = \binom{p/2}{j}^2\binom{r/2}{j}^2\left[\int_X f^j\bar{f}^j d\mu - \int_Y (Tf)^j(\overline{Tf})^j d\nu\right]^2 \geqq 0$$

and so we may apply Lemma 2.4 to conclude that for $f \in L$

$$\int_X f^j\bar{f}^k d\mu = \int_Y (Tf)^j(\overline{Tf})^k d\nu \quad j, k = 0, 1, 2, \ldots$$

This implies (by expanding)

$$\int_X |1 + f|^{2m} = \int_Y |1 + Tf|^{2m} \quad \text{for all } m \geqq 1, f \in L.$$

In particular for $f_1 \ldots f_n \in L$, $z_1 \ldots z_n \in \mathbf{C}$ and $m \in \mathbf{N}$,

$$\int_X |1 + z_1f_1 + \ldots + z_nf_n|^{2m}d\mu = \int_Y |1 + z_1Tf_1 + \ldots + z_nTf_n|^{2m}d\nu$$

and so expanding and equating coefficients

$$\int_X f_1^{\beta_1}f_2^{\beta_2}\ldots f_n^{\beta_n}\bar{f}_1^{\gamma_1}\ldots\bar{f}_n^{\gamma_n}d\mu = \int_Y (Tf_1)^{\beta_1}(Tf_2)^{\beta_2}\ldots(\overline{Tf_n})^{\gamma_n}d\nu$$

where $\beta_j, \gamma_j = 0, 1, 2, \ldots$.

Hence if P is any polynomial in $2n$-variables

$$\int_X P(f_1\ldots f_n, \bar{f}_1\ldots\bar{f}_n)d\mu = \int_Y P(Tf_1\ldots Tf_n, \overline{Tf_1}\ldots\overline{Tf_n})d\nu.$$

Let A be the subalgebra of $C(X)$ generated by L and its complex conjugates, i.e. the space of all polynomials $P(f_1\ldots f_n, \bar{f}_1\ldots\bar{f}_n)$ for $f_1\ldots f_n \in L$. We define $S : A \to C(Y)$ by

$$S(P(f_1\ldots f_n, \bar{f}_1\ldots\bar{f}_n)) = P(Tf_1\ldots Tf_n, \overline{Tf_1}\ldots\overline{Tf_n})$$

S is well-defined since if $P(f_1\ldots f_n, \bar{f}_1\ldots\bar{f}_n) = 0$,

$$\int |P(f_1\ldots f_n, \bar{f}_1\ldots\bar{f}_n)|^2 d\mu$$
$$= \int P(f_1\ldots f_n, \bar{f}_1\ldots\bar{f}_n)\bar{P}(\bar{f}_1\ldots\bar{f}_n, f_1\ldots f_n)d\mu$$
$$= \int |P(Tf_1\ldots Tf_n, \overline{Tf_1}\ldots\overline{Tf_n})|^2 d\nu.$$

Hence $P(Tf_1\ldots Tf_n, \overline{Tf_1}\ldots\overline{Tf_n}) = 0$ ν-almost everywhere, and by our assumptions on ν, $P(Tf_1\ldots Tf_n, \overline{Tf_1}\ldots\overline{Tf_n}) = 0$.

Similarly to this calculation we may show

$$\int_Y |Sa|^{2m}d\nu = \int_X |a|^{2m}d\mu \quad a \in A, m \in \mathbf{N}.$$

Hence $||Sa||_\infty = \lim_{m\to\infty} ||Sa||_{2m} = \lim_{m\to\infty} ||a||_{2m} = ||a||_\infty$. Thus S is a $||\cdot||_\infty$ isometry on A. A is dense in $C(X)$ by the Stone-Weierstrass Theorem and so S may be extended to an algebra homomorphism $\tilde{S} : C(X) \to C(Y)$. Hence

$$\tilde{S}f(y) = f(\alpha y)$$

for some continuous map $\alpha : Y \to X$. It is trivial to see that

$$\int_Y \tilde{S}f d\nu = \int_X f d\mu \quad f \in C(X)$$

and so α is measure-preserving (The last part of this theorem would follow from [**10**, Theorem II]).

THEOREM 2.6. *Suppose* $1 \leqq p < \infty$, $p \neq 2$ *and* $T : C(X) \to L_p(Y)$ *is a linear operator satisfying* $T1 = 1$ *and* $||Tf||_p \leqq ||f||_p$ $(f \in C(X))$. *Let* $L = \{f : Tf \in C(Y); ||Tf||_2 = ||f||_2\}$, *and suppose* L *separates the points of* X. *Then there is a continuous map* $\alpha : Y \to X$ *such that if* B *is a Borel subset of* X, $\nu(\alpha^{-1}(B)) = \mu(B)$, *and*

$$Tf(y) = f(\alpha(y)) \quad \nu - \text{a.e.}, f \in C(X), y \in Y.$$

Proof. By Lemma 2.1, $||T||_2 \leqq 1$. By the Riesz Convexity Theorem, for any r between 2 and p, $||T||_r \leqq 1$. Thus, without loss of generality, we can assume that neither p nor $q = p/p - 1$ is an even integer or ∞.

Suppose $f \in L$ and $g \in C(X)$. Then

$$\int |T(f + re^{i\theta}g)|^2 d\nu \leqq \int |f + re^{i\theta}g|^2 d\mu \quad r > 0, 0 \leqq \theta < 2\pi$$

and hence, letting $r \to 0$,

$$\text{Re}\, e^{i\theta} \int \overline{Tf}Tg d\nu \leqq \text{Re}\, e^{i\theta} \int \bar{f}g d\mu.$$

Hence

$$\int \overline{Tf}Tg d\nu = \int \bar{f}g d\mu.$$

In particular it follows that if $g \in L$ then $z_1 f + z_2 g \in L$, i.e. L is a linear subspace of $C(X)$. Also if $f \in L$, then

$$||f||_p^{q-1} = ||\,|f|^{q-1} \text{sgn } f||_p \geqq ||T(|f|^{q-1} \text{sgn } f)||_p$$

where $\text{sgn } z = e^{i\theta}$ if $z = re^{i\theta}$ $r > 0$, $0 \leqq \theta < 2\pi$, and $\text{sgn } 0 = 0$. Therefore

$$\begin{aligned} ||Tf||_q ||f||_q^{q-1} &\geqq \left| \int \overline{Tf}T(|f|^{q-1} \text{sgn } f) d\nu \right| \quad \text{(by Holder's inequality)} \\ &= \left| \int \bar{f}|f|^{q-1} \text{sgn } f d\mu \right| \\ &= ||f||_q^q. \end{aligned}$$

Hence $||Tf||_q \geqq ||f||_q$ for $f \in L$.

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Now by Theorem 2.5, there is a continuous map $\alpha : Y \to X$ such that α is measure preserving and

$$Tf(y) = f(\alpha(y)) \quad y \in Y, f \in L.$$

Define $S : L_1(X) \to L_1(Y)$ by

$$Sf(y) = f(\alpha(y)).$$

Then S is an isometry on $L_p(X)$ for all p. $S(L_1(X))$ is a closed subspace of $L_1(Y)$ on which there is a conditional expectation projection P with $||P|| = 1$. (See [**6**, p. 158]) For each p, the restriction of P to $L_p(Y)$ will also have norm one as a map between L_p-Spaces. Let $\tilde{T}$ be the natural extension of T to $L_p(X)$, and consider $U = S^{-1}P\tilde{T} : L_p(X) \to L_p(X)$. Then $||U||_p \leqq 1$, and so by Theorem 2.2 the set $\{f \in C(X) : Uf = f\}$ is a closed subalgebra containing 1 and complex conjugates, and separating points. Hence $Uf = f$ for all $f \in C(X)$. That is $PT = S$ on $C(X)$. Since P is an orthogonal projection on $L_2(X)$ and $||PTf||_2 \geqq ||f||_2 \geqq ||Tf||_2 \geqq ||PTf||_2$, for all $f \in C(X)$, we have $PT = T$ on $C(X)$. Thus $S = T$ on $C(X)$ and the result is proved.

Example. Let $\{\varphi_n : n \geqq 0\}$ and $\{\psi_n : n \geqq 0\}$ be two orthonormal sequences in $L_2(0, 1)$ consisting of continuous functions, such that $\varphi_0 = \psi_0 = 1$, both $\{\varphi_n\}$ and $\{\psi_n\}$ separate the points of $[0, 1]$. Suppose for some $p \neq 2$,

$$\int_0^1 \left| \sum_{n=0}^{\infty} (f, \varphi_n)\psi_n \right|^p dx \leqq \int_0^1 |f|^p dx \quad f \in C[0, 1].$$

Then both sequences are complete and either

$$\varphi_n(x) = \psi_n(x), n = 1, 2, \ldots \quad \text{or} \quad \varphi_n(x) = \psi_n(1 - x), n = 1, 2, \ldots.$$

This is an immediate deduction from the preceeding theorem applied to the map $T : C[0, 1] \to L_p[0, 1]$ defined by

$$Tf = \sum_{n=0}^{\infty} (f, \varphi_n)\psi_n.$$

3. Application to group algebras. Let G_1 and G_2 be compact groups with identities e_1 and e_2, and with normalized Haar measure. We denote by Soc (G_i) the set of continuous functions on $G_i (i = 1, 2)$ whose translates generate finite-dimensional vector spaces. Then Soc (G_i) is the socle of the convolution algebra $L_p(G_i)$ for $1 \leqq p < \infty$ and of $C(G_i)$. Let $\hat{G}_i$ denote the set of continuous homomorphisms from G_i into the circle group.

LEMMA 3.1. *Let N be a minimal two-sided ideal in $L_2(G_1)$ and $T : N \to L_2(G_2)$ be a convolution algebra homomorphism with $||T|| \leqq 1$. Then if $T \neq 0$, T is an isometry.*

Proof. In this proof, we use the fact (implicit in [**7**]) that if e is an idempotent in a minimal ideal of $L^2(G)$ of dimension n^2, then $||e||_2 \geqq \sqrt{n}$ with equality if and only if e is minimal and self-adjoint. We see this as follows. Certainly minimal self-adjoint idempotents have norm $\sqrt{n}$ [**7**, p. 158]. Since any self-adjoint idempotent is the sum of minimal ones which are mutually orthogonal [**7**, p. 102], the result is clear for self-adjoint idempotents. Finally for a general idempotent e, let f be a non-zero self-adjoint idempotent in the left ideal generated by e (see [**7**, p. 101]) and $g = e - f$. Then $f * g = 0$. Since f is self-adjoint, this means $f \perp g$ and so $||e||^2 = ||f||^2 + ||g||^2 \geqq ||f||^2$ with equality only if e is self-adjoint.

Now N is algebraically isomorphic to a full matrix algebra of dimension m^2, say. $L_2(G_2)$ is the l_2-sum of its minimal two-sided ideals $\{J_\alpha : \alpha \in A\}$ where each J_α is a full matrix algebra of dimension $m_\alpha{}^2$. Let P_α be the orthogonal projection of $L_2(G_2)$ onto J_α; P_α is an algebra homomorphism. If $P_\alpha T \neq 0$ then, since N is simple, $P_\alpha T$ is injective and hence $m_\alpha \geqq m$.

Let ϵ be a minimal self-adjoint idempotent of N. Then $||\epsilon||_2 = \sqrt{m}$. Hence $||T\epsilon||_2 \leqq \sqrt{m}$. However $T\epsilon = \sum_B \epsilon_\alpha$ where B is the set of α such that $P_\alpha T \neq 0$ and ϵ_α is a non-zero idempotent in J_α. Hence

$$||T\epsilon||_2{}^2 = \sum_B ||\epsilon_\alpha||^2 \geqq \sum_B m_\alpha$$

and as each $m_\alpha \geqq m$ for $\alpha \in B$, B consists of one member $\bar{\alpha}$ and $m_{\bar{\alpha}} = m$. Thus $||T\epsilon||_2 = \sqrt{m_{\bar{\alpha}}}$ and so $T\epsilon$ is a self-adjoint minimal idempotent in $J_{\bar{\alpha}}$. Hence T is a $*$-map since N is the span of its minimal self-adjoint idempotents, and $T(N) = J_{\bar{\alpha}}$ since the dimensions of N and $J_{\bar{\alpha}}$ are equal. Let τ denote the trace on N or $J_{\bar{\alpha}}$. Clearly $\tau(Tf) = \tau(f)$. Hence $Tf(e_2) = f(e_1)$ for $f \in N$. (c.f. [**12**, Lemma 1 and Corollary]).

Thus if $f \in N$,

$$\begin{aligned} ||Tf||_2{}^2 &= \int |Tf(x)|^2 dx \\ &= (Tf)^* * (Tf)(e_2) \\ &= f^* * f(e_2) \\ &= ||f||_2{}^2. \end{aligned}$$

If G_1 and G_2 are compact groups and $\theta : G_1 \to G_2$ is an epimorphism then θ induces two natural algebra homomorphisms:

$$\Lambda_\theta : L_p(G_2) \to L_p(G_1) \quad (1 \leqq p < \infty)$$
$$\Lambda_\theta f(x) = f(\theta x)$$

and

$$\Pi_\theta : L_p(G_1) \to L_p(G_2) \quad (1 \leqq p < \infty)$$
$$\Pi_\theta f(\theta x) = \int_{\ker\theta} f(xy)dy$$

where the integration is with respect to the invariant measure on $\ker \theta$.

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THEOREM 3.2. *Let G_1 and G_2 be compact groups and $1 \leqq p < \infty$ $(p \neq 2)$. Let $T : L_p(G_1) \to L_p(G_2)$ be a norm-decreasing algebra homomorphism such that $T1 = 1$. Then there is a compact group H and epimorphisms $\theta_1 : G_1 \to H$, $\theta_2 : G_2 \to H$ such that $T = \Lambda_{\theta_2} \circ \Pi_{\theta_1}$.*

Proof. Let J be the linear span of the minimal two-sided ideals not included in the kernel of T.

(a) Suppose J separates the points of G_1; then by Lemma 3.1 and Theorem 2.6, we obtain a map $\theta : G_2 \to G_1$ which is continuous and surjective and such that $Tf(x) = f(\theta x)$. It is easy to show that θ is an epimorphism and so $T = \Lambda_\theta$; in this case $\theta = \theta_2$ and θ_1 is the identity map.

(b) In general, let

$$K = \{x : f(x) = f(e_1); f \in J\}.$$

Then K is a closed normal subgroup of G_1; let $H = G_1/K$ and $\theta_1 : G_1 \to G_1/K$ be the natural quotient map. Then $T = S \circ \Pi_{\theta_1}$ where $S : L_p(H) \to L_p(G_2)$ is a norm-decreasing algebra homomorphism ($S = T \circ \Lambda_{\theta_1}$). If J' is the linear span of the minimal two-sided ideals in $L_p(H)$ not included in the kernel of S, then $J' = \Pi_{\theta_1}(J)$ separates the points of H. Now apply (a) to S.

Theorem 3.2 can be applied when $T(\hat{G}_1) \neq 0$, and this is the case when T is either a monomorphism or an epimorphism. Let λ be a character on G_1; then by A_λ we denote the automorphism of $L_p(G_1)$ defined by $A_\lambda f(x) = \lambda(x)f(x)$.

THEOREM 3.3. *Let G_1 and G_2 be compact groups and $1 \leqq p < \infty$ $(p \neq 2)$. Suppose $T : L_p(G_1) \to L_p(G_2)$ is a norm-decreasing algebra homomorphism.*

(i) *If T is an epimorphism, then $T = \Pi_\theta \circ A_\lambda$ where $\lambda \in \hat{G}_1$ and $\theta : G_1 \to G_2$ is an epimorphism.*

(ii) *If T is a monomorphism and $1 < p < \infty$, then $T = A_\lambda \circ \Lambda_\theta$ where $\lambda \in \hat{G}_2$ and $\theta : G_2 \to G_1$ is an epimorphism.*

(iii) *If T is a monomorphism and $p = 1$, there is an open subgroup H of G_2 of index n, $\lambda \in \hat{H}$ and $\theta : H \to G_1$ an epimorphism such that*

$$Tf(x) = n\lambda(x)f(\theta x) \quad x \in H$$
$$= 0 \quad x \notin H.$$

Proof. (i) $T^{-1}(\mathbf{C}.1)$ is an ideal of $L_p(G_1)$ strictly larger than $T^{-1}(0)$. Hence there is a minimal ideal J such that $J \cap T^{-1}(0) = (0)$ and $T(J) \subset \mathbf{C}.1$. Thus J has dimension one and there exists $\lambda^{-1} \in \hat{G}_1$ such that $T\lambda^{-1} = 1$. Thus $T \circ A_{\lambda^{-1}}(1) = 1$, and so by the preceding theorem $T \circ A_{\lambda^{-1}} = \Pi_\theta$ as required (θ_2 is an isomorphism since T is surjective) and hence $T = \Pi_\theta \circ A_\lambda$.

We omit (ii) in view of its similarity to (iii) (which is more difficult).

(iii) By [**14**, Lemma 2], since $T1$ is a norm one idempotent in $L_1(G_2)$,

$$T1 = n\lambda(x) \quad x \in H$$
$$= 0 \quad x \notin H$$

where H is an open subgroup of index n and $\lambda \in \hat{H}$.

Now if $f \in \text{Soc } G_1$, then $Tf \in \text{Soc } G_2$ since Soc G_i is the linear span of idempotents. (Since Soc G_i is dense in $L^1(G_i)$, if $f \in L^1(G_i)$, multiplication by f is a compact operator on $L^1(G)$. Thus if f is an idempotent it must act as a finite rank operator i.e. $f \in \text{Soc } (G_i)$.) Now we have

$$\frac{1}{2\pi}\int_0^{2\pi} ||1 + re^{i\theta}f||_1 d\theta = 1 + o(r)$$

by expanding as in Section 2. Hence

$$\frac{1}{2\pi}\int_0^{2\pi} ||T1 + re^{i\theta}Tf||_1 d\theta \leqq 1 + o(r).$$

However

$$\frac{1}{2\pi}\int_0^{2\pi} ||T1 + re^{i\theta}Tf||_1 d\theta$$
$$= \frac{1}{2\pi}\int_0^{2\pi}\int_H |n + re^{i\theta}\overline{\lambda(x)}Tf(x)|dxd\theta + \frac{|r|}{2\pi}\int_0^{2\pi}\int_{G_2 - H} |Tf(x)|dxd\theta$$
$$= 1 + |r|\int_{G_2-H} |Tf(x)|dx + o(r).$$

Hence $|Tf(x)| = 0$ for $x \notin H$. If we define

$$Sf(x) = \frac{1}{n}\bar{\lambda}(x)Tf(x) \quad x \in H$$

then $S : L_p(G_1) \to L_p(H)$ is a norm-decreasing monomorphism such that $S1 = 1$. Now apply Theorem 3.2.

THEOREM 3.4. *For $1 \leqq p < \infty$, p not an even integer, let J be a closed ideal in $L_p(G_1)$ with $1 \in J$. Let T be an isometry of J into $L_p(G_2)$ such that $T1 = 1$. Then there is a compact group H and epimorphism $\theta_1 : G_1 \to H$ and $\theta_2 : G_2 \to H$ such that $Tf = \Lambda_{\theta_2} \circ \Pi_{\theta_1} f$ ($f \in J$).*

Proof. Let $K = \{x : f(x) = f(e_1)\ f \in J\}$; K is a closed normal subgroup of G_1. Let $H = G_1/K$ and $\theta_1 : G_1 \to H$ be the natural epimorphism. Let $S = T \circ \Lambda_{\theta_1}$ on $\Lambda_{\theta_1}^{-1}(J)$. Then $\Lambda_{\theta_1}^{-1}(J)$ is a closed ideal of $L_p(H)$ separating the points of H. Now apply Theorem 2.5 to deduce that $S = \Lambda_{\theta_2}$ where $\theta_2 : G_2 \to H$ is an epimorphism. Hence $T = \Lambda_{\theta_2} \circ \Pi_{\theta_1}$.

Remark. The condition $T1 = 1$ can be relaxed to the condition that $T1$ is a norm one idempotent. However, it cannot be removed altogether. In [**5**], there are examples of norm-decreasing homomorphisms between ideals of $L^p(G)$ without the condition $T1 = 1$ and which have quite a different form to those in Theorem 3.2. That paper is primarily concerned with the corresponding problem for $C(G)$, where different techniques are required.

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