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IRREDUCIBLE REPRESENTATIONS OF ALGEBRAS

EDGAR G. GOODAIRE

1. Introduction. The concept of the universal enveloping algebra $\mathscr{U}(X)$ of a (not necessarily associative) algebra X is basic to the study of the representations of X, because there is a one-to-one correspondence between the representations of X and $\mathscr{U}(X)$. If one is only interested in studying a certain class of the representations of X, the thought occurs that there may exist a more suitable universal object. The main result of this paper shows that, provided an associative algebra A possesses a *diagonable subspace* L, there is a one-to-one correspondence between the λ-*weighted representations* of A and the λ-*weighted representations* of the subalgebra $\mathscr{C}$, which is the centralizer of L in A. In particular, any Cartan subalgebra of a finite dimensional simple Lie (Jordan) algebra over an algebraically closed field of characteristic 0 is a diagonable subspace of its universal enveloping algebra (universal multiplication envelope) and so our result applies directly to these cases.

We assume throughout that all fields have characteristic 0. If A is an associative algebra over the field F and $x \in A$, the linear transformation ad x of A is defined by $a \mapsto (a, x) = ax - xa$. This is a derivation of A in the sense that

$$(1) \quad (ab, x) = a(b, x) + (a, x)b$$

for any $a, b \in A$. If W is a vector space over F, the *dual* of W (the space of linear functionals $W \to F$) will be denoted by W^*.

Definitions 1.1 Let L be a subspace of an associative algebra A with identity over a field F. A map $\alpha : L \to F$ such that

$$A_\alpha(L) \equiv \{a \in A : (a, x) = \alpha(x)a, \text{ for every } x \in L\}$$

is non-zero is called a *root* of L in A, and $A_\alpha(L)$ is the corresponding *root space*. We say that L is a *diagonable subspace* of A if it is spanned by commuting elements, and if A has a linear basis relative to which $\{\text{ad } x : x \in L\}$ is a set of simultaneously diagonalizable linear transformations; that is, as a vector space,

$$A = \oplus \sum_{\alpha \in \Delta} A_\alpha(L),$$

where Δ is the complete set of roots of L in A. If V is a right A-module which

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is also a vector space over F, a map $\lambda : L \to F$ is a *weight* of L in V if

$$V_\lambda \equiv \{v \in V : v(x - \lambda(x)1)^n = 0, \text{ for every } x \in L \text{ and some } n = n(x, v) > 0\}$$

is non-zero. V_λ is the *weight space* corresponding to λ and V is said to be *weighted*, or λ-*weighted*, if we wish to emphasize that λ is a weight. A representation of A is weighted if the associated A-module is weighted.

PROPOSITION 1.2. *If λ is any weight of L in an A-module V, and α is any root of L, then both λ and α are in L^*.*

Proof. We show first that for any $u \in L$, $\lambda(u)$ is the only characteristic root of u on V_λ. For this, let $0 \neq v \in V_\lambda$. If for some $t \in F$, $v(u - t1)^m = 0 = v(u - \lambda(u)1)^n$, then $v((u - \lambda(u)1) - (u - t1))^{m+n} = 0$. Thus $v((t - \lambda(u))1)^{m+n} = 0$ and $\lambda(u) = t$. That λ is linear now follows from

(a) $x, y \in L$ implies $\lambda(x) + \lambda(y)$ is a characteristic root of $x + y$ on V_λ, and

(b) $x \in L$ and $t \in F$ implies $t\lambda(x)$ is a characteristic root of tx on V_λ.

To see (a), for any $0 \neq v \in V_\lambda$, there exist positive integers n and m such that $0 = v(x - \lambda(x)1)^n = v(y - \lambda(y)1)^m$. Since $x - \lambda(x)1$ and $y - \lambda(y)1$ commute, $v((x + y) - (\lambda(x) + \lambda(y))1)^{n+m} = v((x - \lambda(x)1) + (y - \lambda(y)1))^{n+m} = 0$. The proof of (b) is similar; the linearity of the map ad x then shows that any root α is in L^* as above.

An element $x \in A$ is called *diagonable* if Fx is a diagonable subspace. In this case, we write $A_{\alpha(x)}(x)$ instead of $A_\alpha(Fx)$, thus identifying the root $\alpha : Fx \to F$ with the scalar $\alpha(x)$ which, by the proposition, completely determines α.

2. Examples.

2.1. *Algebraic elements.* Any algebraic element x whose minimal polynomial $p(t) \in F[t]$ has distinct roots $\alpha_1 \ldots, \alpha_n$ in F is diagonable. First, x can be written as $\sum_{i=1}^n \alpha_i e_i$ where $e_1, \ldots, e_n$ are orthogonal idempotents, for upon defining $h_i(t) = \prod_{j \neq i}(t - \alpha_j), h_1(t), \ldots, h_n(t)$ are relatively prime in $F[t]$, and so $\sum_{i=1}^n a_i(t)h_i(t) = 1$ for some $a_i(t) \in F[t]$. Letting $e_i = a_i(x)h_i(x)$, we obtain $x = \sum_{i=1}^n \alpha_i e_i$. But now it is clear that x is diagonable because if $a \in A$, $a = \sum_{i,j=1}^n e_i a e_j$, and $e_i a e_j \in A_{\alpha_j - \alpha_i}(x)$.

2.2. *Commuting diagonable elements.* By definition, any diagonable subspace is spanned by commuting diagonable elements; conversely, if L is a subspace spanned by finitely many commuting diagonable elements $x_1, \ldots, x_n$, then L is diagonable. Without loss of generality, we can take $x_1, \ldots, x_n$ as a basis for L, for any linear combination of commuting diagonable elements is diagonable. This observation is a consequence of the following more general fact:

If x is diagonable and B is a subspace of A such that B ad $x \subset B$, then B decomposes as $\sum B_\alpha(x)$ relative to x.

For this, let $b = \sum b_\alpha$ be the decomposition of an element $b \in B$ relative to x. Since B ad $x \subset B$,

$$(1) \quad b(\text{ad } x)^k = \sum_{\alpha \neq 0} \alpha^k b_\alpha = b^{(k)}$$

is in B for every integer $k > 0$. Now $b_\alpha = 0$ for all α outside a finite set of cardinality n say. Then letting k run from 1 to n, (1) is a system of n equations in n unknowns whose coefficient matrix is a vandermonde matrix with non-zero determinant. It follows that each $b_\alpha \in B$.

Now if x and y are commuting diagonable elements, the above result with $B = A_\alpha(x)$ implies that

$$A = \oplus \sum_{\alpha \in \Delta} \sum_{\beta \in \Delta'} (A_\alpha(x))_\beta(y)$$

where Δ and Δ' are the sets of roots of Fx and Fy respectively. But $(A_\alpha(x))_\beta(y) \subset A_{\alpha+\beta}(x + y)$ and so $x + y$ is diagonable. In fact if t and s are non-zero scalars, $tx + sy$ is diagonable with roots in the set $\{t\alpha + s\beta : \alpha \in \Delta, \beta \in \Delta'\}$.

Now let L_1 be the subspace of A spanned by $x_1, \ldots, x_{n-1}$. By induction, we may assume that L_1 is diagonable and $A = \oplus \sum_{\alpha \in \Delta_1} A_\alpha(L_1)$ where Δ_1 is the set of roots of L_1 in A. Since $A_\alpha(L_1)$ is invariant under ad x_n, $A_\alpha(L_1)$ decomposes as

$$\oplus \sum_{\beta \in \Delta'} (A_\alpha(L_1))_\beta(x_n)$$

relative to the set Δ' of roots of Fx_n. A direct verification reveals that

$$(A_\alpha(L_1))_\beta(x_n) = A_\alpha(L_1) \cap A_\beta(x_n) = A_\gamma(L)$$

where by Proposition 1.2, $\gamma : L \to F$ is uniquely determined by the conditions $\gamma(x_i) = \alpha(x_i)$ for $1 \leqq i \leqq n - 1$, and $\gamma(x_n) = \beta(x_n)$. It is immediate that L is a diagonable subspace.

2.3. *Lie algebras.* Let A be the universal enveloping algebra of a finite dimensional simple Lie algebra $\mathscr{L}$ over an algebraically closed field F of characteristic 0, and suppose $\mathscr{H}$ is a Cartan subalgebra of $\mathscr{L}$. Then $\mathscr{L}$ possesses a Cartan basis $B = \{e_\alpha, f_\alpha, h_\beta : \alpha \in I, \beta \in J\}$, where I and J are (finite) totally ordered index sets, and $\{h_\beta : \beta \in J\}$ is a basis for $\mathscr{H}$, such that after embedding $\mathscr{L}$ in A, the following multiplicative relations among the elements of B hold:

$$(h_\beta, h_{\beta'}) = 0$$

$$(e_\alpha, h_\beta) = A_{\alpha,\beta} e_\alpha$$

$$(f_\alpha, h_\beta) = B_{\alpha,\beta} f_\alpha, \quad \alpha \in I, \beta, \beta' \in J$$

where $A_{\alpha,\beta}$ and $B_{\alpha,\beta}$ are integers. By the Poincaré-Birkhoff-Witt Theorem

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[**4**, § 5.2], A has a linear basis consisting of all elements of the form

$$(2)\quad \prod_{\alpha\in I} f_\alpha{}^{n(\alpha)} \prod_{\beta\in J} h_\beta{}^{r(\beta)} \prod_{\alpha\in I} e_\alpha{}^{m(\alpha)}$$

where $n(\alpha)$, $r(\beta)$, $m(\alpha)$ are non-negative integers, and the product respects the ordering in I and J. Because of the identity

$$(3)\quad (xy, z) = x(y, z) + (x, z)y$$

which holds in any associative algebra (see equation (1) of § 1), for any fixed $\beta \in J$, we observe that

$$\left(\prod_{\alpha\in I} e_\alpha{}^{m(\alpha)}, h_\beta\right) = \left(\sum_{\alpha\in I} m(\alpha)A_{\alpha,\beta}\right) \prod_{\alpha\in I} e_\alpha{}^{m(\alpha)}$$

and that in fact, for any basis element u of the form (2), $(u, h_\beta) = \alpha(h_\beta)u$ for some integer $\alpha(h_\beta)$. Thus h_β is a diagonable element of A for each $\beta \in J$. Since $\{h_\beta : \beta \in J\}$ is a basis for $\mathscr{H}$, $\mathscr{H}$ is a diagonable subspace of A by the results of 2.2.

2.4. *Jordan algebras*. Suppose now that A is the universal multiplication envelope of a finite dimensional semi-simple Jordan algebra J over the algebraically closed field F of characteristic 0. In [**5**], Jacobson shows that any Cartan subalgebra $\mathscr{H}$ of J is of the form $\mathscr{H} = \sum_{i=1}^{t} J_{ii}$, where $J = \sum_{i,j=1}^{t} J_{ij}$ is the Peirce decomposition of J relative to a set of primitive orthogonal idempotents $e_1, \ldots, e_t$ with sum 1. Here

$$J_{ij} = \{a \in J : ae_i = ae_j = \tfrac{1}{2}a\} \text{ for } i \neq j, \text{ and } J_{ii} = \{a \in J : ae_i = a\}.$$

Moreover, Albert shows in [**1**] that the simplicity of J forces J_{ii} to be just Fe_i. Now denoting by $a \mapsto \bar{a}$ the canonical embedding of J in A, it is well-known [**6**, p. 102] that for any idempotent e of J, $\bar{e}$ satisfies the polynomial $2t^3 - 3t^2 + t \in F[t]$ with the distinct roots $0, \frac{1}{2}, 1$, and hence is diagonable in A by 2.1. Also, if e and f are orthogonal idempotents in J, $\bar{e}$ and $\bar{f}$ commute in A. Thus, as a subspace spanned by the commuting diagonable elements $\bar{e}_1, \ldots, \bar{e}_t$, $\mathscr{H}$ is a diagonable subspace of A by the results of 2.2. We remark in passing, that this similar behaviour of Cartan subalgebras of Lie and Jordan algebras when considered as subalgebras of a universal enveloping algebra is not really surprising in view of the work of Foster [**2**] who proved that the Cartan theory of Lie and Jordan algebras is essentially the same.

3. Weighted modules. In this section, we establish various results of a technical nature which are needed before we can prove our main representation theorem (4.5). In the remainder of this paper, L is a fixed diagonable subspace of the associative algebra A with 1 over F, and $A = \oplus \sum_{\alpha\in\Delta} A_\alpha$, $A_\alpha = A_\alpha(L)$, is the decomposition of A relative to the collection Δ of roots of L.

LEMMA 3.1. *If V is a weighted A-module, then $V_\lambda A_\alpha \subset V_{\lambda+\alpha}$ for any weight λ. Should V in addition be irreducible, V decomposes as $\oplus \sum_{\lambda\Lambda\in} V_\lambda$ relative to the set Λ of all weights of L in V.*

Proof. An easy induction shows that $x^k a = a(x - \alpha(x)1)^k$ for all $a \in A_\alpha$ and $x \in L$. Thus, if $v(x - \lambda(x)1)^n = 0$, we have

$$0 = v(x - \lambda(x)1)^n a = v \sum_{k=0}^{n} \binom{n}{k} x^k(-\lambda(x)1)^{n-k} a$$
$$= v \sum_{k=0}^{n} \binom{n}{k} (-\lambda(x))^{n-k} a(x - \alpha(x)1)^k$$
$$= va(x - (\lambda(x) + \alpha(x))1)^n$$

and hence $va \in V_{\lambda+\alpha}$. It now follows that $\sum_{\lambda\in\Lambda} V_\lambda$ is a non-zero submodule of any weighted module V, so that if V is irreducible, of course we must have $V = \oplus \sum_{\lambda\in\Lambda} V_\lambda$.

Now the identity (3) in section 2 implies $A_\alpha A_\beta \subset A_{\alpha+\beta}$ for any roots α, β of L. In particular, $A_0 = \{a \in A : (a, x) = 0, \text{ for every } x \in L\}$ is a subalgebra of A containing 1 and L; namely, the centralizer of L in A. The lemma shows that any weight space V_λ of a weighted A-module is an A_0-module, and we can further show

LEMMA 3.2. *If V is an irreducible, weighted A-module, V_λ is an irreducible A_0-module, for any weight λ.*

Proof. If W_{λ_0} is a proper A_0-submodule of V_{λ_0}, then

$$W = W_{\lambda_0} A = W_{\lambda_0} \oplus \sum_{0\neq\alpha\in\Delta} W_{\lambda_0} A_\alpha$$

is a proper A-submodule of V because of 3.1.

LEMMA 3.3. *If K is any maximal right ideal of A_0, and $u \in Z(A_0)$, the centre of A_0, then $au \in K$ with $a \in A_0$ implies $a \in K$ or $u \in K$.*

Proof. If $a \notin K$, $K + aA_0 = A_0$, and so $k + ab = 1$ for some $k \in K$ and $b \in A_0$. Hence $ku + abu = u$ with both ku and $abu(= aub)$ in K. So $u \in K$.

Now suppose V is an irreducible weighted A-module and V_λ is a non-zero weight space. Let $0 \neq v \in V_\lambda$. Then $\tau : a \mapsto va$ is an A_0-module homomorphism $A_0 \to V$ which must be surjective by 3.2. Thus $V_\lambda \cong A_0/T_0$ where T_0 is the kernel of τ. Since T_0 is a maximal right ideal of A_0 containing $(x - \lambda(x)1)^n$ for every $x \in L$, and since $x - \lambda(x)1 \in Z(A_0)$, by 3.3, T_0 actually contains $x - \lambda(x)1$, for every $x \in L$. But now, letting T be the kernel of the A-module homomorphism $A \to V$ defined by $a \mapsto va$, we have $V \cong A/T$. Noting that $T \cap A_0 = T_0$, we obtain

THEOREM 3.4. *If V is an irreducible weighted A-module, and λ is any weight of L in V, then $V \cong A/T$, where T is a maximal right ideal of A containing $x - \lambda(x)1$ for every $x \in L$.*

4. The representation theorem. For any linear functional λ on L, denote by $\mathscr{W}_\lambda$ (respectively, $\mathscr{W}_\lambda{}^0$) the collection of all isomorphism classes of irreducible λ-weighted A-modules (respectively, A_0-modules). Notice that Theorem 3.4, and the fact that $x - \lambda(x)1$ is in the centre of A_0 for any $x \in L$ imply

(1) $V \in \mathscr{W}_\lambda{}^0$ if and only if $V(x - \lambda(x)1) = 0$, for every $x \in L$.

Now there is an obvious map $\Phi : \mathscr{W}_\lambda \rightarrow \mathscr{W}_\lambda{}^0$; namely, $\Phi V = V_\lambda$ for $V \in \mathscr{W}_\lambda$. Φ is well-defined because of

PROPOSITION 4.1. *If W is an irreducible weighted A-module, and V is any A-module isomorphic to W, then V is irreducible and weighted, and $W_\lambda \cong V_\lambda$ as A_0-modules for every weight λ.*

Proof. We simply observe that if $\psi : W \rightarrow V$ is the A-module isomorphism, then by restricting ψ to W_λ and the ring of scalars to A_0, we obtain an isomorphism $W_\lambda \rightarrow V_\lambda$ because each module is irreducible and $\psi(W_\lambda) \subset V_\lambda$.

Our objective now is to establish an inverse for Φ, but this requires several preliminary results. We begin with

LEMMA 4.2. *If V is a vector space over a field F, and $T_1, \ldots, T_n$ are distinct elements of V^*, then there is a $v \in V$ such that $\{T_1(v), \ldots, T_n(v)\}$ is a set of n distinct scalars.*

Proof. Since $T_1, \ldots, T_n$ are distinct, $\{\ker (T_i - T_j) : i \neq j\}$ is a finite set of proper subspaces of V, whose union cannot be all of V because F is infinite (it has characteristic 0). Now choose any

$$v \in V \setminus \bigcup_{i \neq j} \ker (T_i - T_j).$$

An ideal (right, left, or two-sided) of $A = \oplus \sum_{\alpha \in \Delta} A_\alpha$ is said to be *homogeneous* if $I = \oplus \sum_{\alpha \in \Delta} I \cap A_\alpha$. In particular, if I is a right ideal which contains $x - \lambda(x)1$ for every $x \in L$ and some $\lambda \in L^*$, then I must be homogeneous. For suppose $a = \sum_{\alpha \in \Delta} a_\alpha \in I$ with $a_\alpha \in A_\alpha$. Then for any $x \in L$, $(a, x) = (a, x - \lambda(x)1) = \sum_{0 \neq \alpha \in \Delta} \alpha(x) a_\alpha \in I$. Similarly, $((a, x), x) = ((a, x), x - \lambda(x)1) = \sum_{0 \neq \alpha \in \Delta} \alpha(x)^2 a_\alpha \in I$, and continuing in this way, we get that for any integer $k > 0$,

(2) $$\sum_{0 \neq \alpha \in \Delta} \alpha(x)^k a_\alpha = i_k \in I.$$

Now $a_\alpha = 0$ for all α not in some finite set $\{\alpha_1, \ldots, \alpha_n\}$ of distinct roots. The $\alpha_i \in L^*$ by 1.2 and so the previous lemma provides $x \in L$ with $\{\alpha_1(x), \ldots, \alpha_n(x)\}$

a set of n distinct scalars. Letting k run from 1 to n, (2) is a system of linear equations over F with coefficient matrix a vandermonde matrix with non-zero determinant. Thus each $a_\alpha \in I$ for $\alpha \neq 0$, but then $a_0 = a - \sum_{0\neq\alpha\in\Delta} a_\alpha$ is in I too.

Incidentally, it is worth noting that only a slight modification of the above argument shows that any two-sided ideal of A is also homogeneous. The key step in defining Φ^{-1} is

PROPOSITION 4.3. *Any maximal right ideal of* A_0 *which contains* $x - \lambda(x)1$ *for every* $x \in L$ *and some* $\lambda \in L^*$ *determines a unique maximal right ideal* I^* *of* A *contained in* $I \oplus \sum_{0\neq\alpha\in\Delta} A_\alpha$.

Proof. The right ideal IA of A which I generates is contained in $I \oplus \sum_{0\neq\alpha\in\Delta} A_\alpha$ because $A_\alpha A_\beta \subset A_{\alpha+\beta}$. The existence of I^* then follows from an easy Zorn's Lemma argument. Now I is contained in $I^* \cap A_0$ which is a proper right ideal of A because $1 \notin I^*$. By maximality of I, $I^* \cap A_0 = I$. We claim that for any (proper) right ideal of A containing I, $J \subset I \oplus \sum_{0\neq\alpha\in\Delta} A_\alpha$. For such J is homogeneous and so contained in $(J \cap A_0) \oplus \sum_{0\neq\alpha\in\Delta} A_\alpha$. Since $I \subset J \cap A_0 \neq A_0$, $J \cap A_0 = I$ by maximality. Thus the sum of all proper right ideals of A containing I is contained in $I \oplus \sum_{0\neq\alpha\in\Delta} A_\alpha$ and so must again be proper. Clearly this is the unique maximal right ideal I^*.

We now define a map $\psi : \mathscr{W}_\lambda{}^0 \to \mathscr{W}_\lambda$ as follows: If $V \in \mathscr{W}_\lambda{}^0$, then by Theorem 3.4, $V \cong A_0/I$ where I is a maximal right ideal of A_0 containing $x - \lambda(x)1$ for all $x \in L$. By the proposition, I extends uniquely to maximal right ideal I^* of A.

Certainly $A/I^* \in \mathscr{W}_\lambda$ because $(I^* + 1)(x - \lambda(x)1) = 0$ for every $x \in L$ and so we can define ψV to be A/I^*. ψ is well-defined because of

LEMMA 4.4. *If* A_0/I_1 *and* A_0/I_2 *are irreducible, isomorphic* A_0*-modules, and if for some* $\lambda \in L^*$, $x - \lambda(x)1 \in I_1 \cap I_2$ *for all* $x \in L$, *then* $A/I_1{}^* \cong A/I_2{}^*$ *as* A*-modules, where* $I_1{}^*$ *and* $I_2{}^*$ *are the right ideals of* A *given by* 4.3.

Proof. Suppose that $\sigma(I_1 + 1) = I_2 + a_0$, where $\sigma : A_0/I_1 \to A_0/I_2$ is the given isomorphism. Lift σ to $\sigma^* : A/I_1{}^* \to A/I_2{}^*$ by $\sigma^* : I_1{}^* + a \mapsto I_2{}^* + a_0 a$. σ^* is well-defined, for if $a \in I_1{}^*$ but $a_0 a \notin I_2{}^*$, then $I_2{}^* + a_0 a A = A$ and so there exists $u \in A$ such that

$$(3) \quad a_0 a u - 1 \in I_2{}^* \subset I_2 \oplus \sum_{0\neq\alpha\in\Delta} A_\alpha.$$

Now because $au \in I_1{}^* \subset I_1 \oplus \sum_{0\neq\alpha\in\Delta} A_\alpha$, we can write $au = b_0 + \sum b_\alpha$ where $b_0 \in I_1$. From (3), $a_0 b_0 - 1 \in I_2$. But this is impossible because $\sigma(I_1 + b_0) = 0 = I_2 + a_0 b_0$. Finally, since $I_2{}^* \subset I_2 \oplus \sum_{0\neq\alpha\in\Delta} A_\alpha$, $a_0 \notin I_2{}^*$. Hence σ^* is non-zero and therefore an isomorphism by Schur's Lemma.

The main theorem on representations is contained in

THEOREM 4.5. *Let L be a diagonable subspace of an associative algebra A over F and let $\mathscr{C}$ be the centralizer of L in A. Then there is a one-to-one correspondence between the isomorphism classes of λ-weighted irreducible A-modules, and the isomorphism classes of λ-weighted irreducible $\mathscr{C}$-modules.*

Proof. We need only prove that Φ and ψ are inverse maps. Given $V \in \mathscr{W}_\lambda$, recall that $\Phi V = V_\lambda \cong A_0/I$ for some maximal right ideal I of A_0 containing $x - \lambda(x)1$ for every $x \in L$. $\psi(\Phi V)$ is then A/I^*, where I^* is that ideal of A given by 4.3. That $\psi(\Phi V) \cong V$ follows from Theorem 3.4 upon observing that the ideal I^* is the ideal T of that theorem by uniqueness. Conversely, given $V \in \mathscr{W}_\lambda{}^0$, $V \cong A_0/J$, where J is a maximal right ideal of A_0 containing $x - \lambda(x)1$ for all $x \in L$. Letting J^* be that ideal of A given by 4.3, $\Phi(\psi V)$ is $(A/J^*)_\lambda$. To see that this is isomorphic to V, define $\sigma : A_0 \to (A/J^*)_\lambda$ by $a_0 \mapsto J^* + a_0$. For any $a_0 \in A_0$, $a_0(x - \lambda(x)1) = (x - \lambda(x)1)a_0$ is in J and so in J^*. Thus $\sigma(A_0) \subset (A/J^*)_\lambda$. σ is surjective because it is non-zero and $(A/J^*)_\lambda$ is irreducible. The kernel of σ is $\{a_0 \in A_0 : a_0 \in J^*\} = J^* \cap A_0 = J$ by the maximality of J. Thus $(A/J^*)_\lambda \cong A_0/J \cong V$ as required.

Because of our observations in sections 2.3 and 2.4, and because weighted irreducible representations of any algebra correspond to weighted irreducible representations of its universal enveloping algebra, we can extend, as a consequence of our theory, a theorem of Lemire [**7**] concerning the representations of a simple Lie algebra to the case of Jordan algebras as well.

THEOREM 4.6. *Let $\mathscr{H}$ be a Cartan subalgebra of a finite dimensional simple Lie or Jordan algebra X over an algebraically closed field of characteristic 0. Let $\mathscr{C}$ denote the centralizer of $\mathscr{H}$ in the universal enveloping algebra $\mathscr{U}(X)$ of X. Then there is a one-to-one correspondence between the isomorphism classes of λ-weighted irreducible representations of X and the isomorphism classes of λ-weighted irreducible representations of $\mathscr{C}$.*

5. Semi-simplicity and primitivity. In this section of our paper, we examine connections between the ring-theoretic properties of an algebra A and the centralizer $\mathscr{C}$ of a diagonable subspace L which it possesses. Our results are used to analyse the algebraic structure of the centralizer of a Cartan subalgebra of a finite dimensional semi-simple Jordan algebra in its universal multiplication envelope. We recall that the Cartan subalgebra is a diagonable subspace of the universal envelope.

Thus, we assume that L is a diagonable subspace of an associative algebra A with 1, and that $A = \oplus \sum_{\alpha \in \Delta} A_\alpha$ is the decomposition of A relative to the collection Δ of roots of L in A. Furthermore, we suppose L to be finitely diagonable in the sense that Δ is a finite set, $\Delta = \{0, \alpha_1, \ldots, \alpha_k\}$. Under these conditions we prove the useful

LEMMA 5.1. *If $V = \oplus \sum_{\lambda \in \Lambda} V_\lambda$ is the decomposition of an irreducible weighted*

A-module V relative to the set Λ of weights of L in V, then Λ is a finite set.

Proof. Fixing a $\lambda_0 \in \Lambda$. we have by 3.4,

$$V \cong A/J = \sum_{\alpha\in\Delta} (A_\alpha + J)/J$$

for some maximal right ideal J of A containing $x - \lambda_0(x)1$ for every $x \in L$. Now for $a_\alpha \in A_\alpha$ and $x \in L$, $a_\alpha(x - \lambda_0(x)1) = (x - \lambda_0(x)1)a_\alpha + \alpha(x)a_\alpha$, so that $a_\alpha(x - (\lambda_0 + \alpha)(x)1) \in J$. By Proposition 4.1, we have $(A_\alpha + J)/J \cong V_{\lambda_0+\alpha}$; i.e., the weights of L in V are in the finite set $\{\lambda_0 + \alpha : \alpha \in \Delta\}$.

Before continuing, we remind the reader that $A_\alpha A_\beta \subset A_{\alpha+\beta}$, where $A_{\alpha+\beta}$ is 0 by definition if $\alpha + \beta \notin \Delta$.

LEMMA 5.2. *Suppose $x_1, \ldots, x_{k+1}$ are $k + 1$ elements of A with each x_i in a root space A_{β_i}. Then the product $x_1 \ldots x_{k+1}$ contains an element of A_0 as a subproduct (i.e. for some integers m and n with $1 \leqq m \leqq n \leqq k + 1$, the product $x_m \ldots x_n$ is in A_0).*

Proof. Define $b_i = \beta_1 + \beta_2 + \ldots + \beta_i$ for each i. Assuming as we may, that each b_i is a non-zero root, we obtain $k + 1$ elements in the set $\{\alpha_1, \ldots, \alpha_k\}$. Thus there are integers r and n with $r < n$ for which $b_r = b_n$. But then, letting $m = r + 1$, it follows that $x_m \ldots x_n \in A_0$.

LEMMA 5.3. *Suppose I is a right ideal of A such that $(I \cap A_0)^2 = 0$. Then I is nilpotent.*

Proof. Let $x_1, \ldots, x_N$ be N elements of I, where $N = (k + 1)(k + 2)$. By 5.2, the product $x_1 \ldots x_N$ contains $k + 2$ subproducts in A_0, and since they are also in I, if any two are adjacent, $x_1 \ldots x_N = 0$. Thus, we can assume that $x_1 \ldots x_N$ contains a subproduct of the form $u_1a_1u_2a_2 \ldots u_{k+1}a_{k+1}u_{k+2}$ where each u_i is in $I \cap A_0$ and each a_i is in a non-zero root space. Again uisng 5.2, some subproduct $a_mu_{m+1} \ldots a_n$ is in A_0. But then $u_ma_mu_{m+1} \ldots a_n$ is in $(I \cap A_0)^2$ and hence 0. Since I^N consists of sums of products of N elements from I, we see that $I^N = 0$.

THEOREM 5.4. *If A is semi-prime, so is A_0. Conversely, if A_0 is semi-prime, then the nilpotent right ideals of A are exactly those contained in $\sum_{0\neq\alpha\in\Delta} A_\alpha$.*

Proof. Suppose A is semi-prime and I is a right ideal of A_0 with $I^2 = 0$. Then IA is a right ideal of A contained in $I + \sum_{0\neq\alpha\in\Delta} A_\alpha$. Also $IA \cap A_0 = I$ has square 0, so by 5.3, IA is nilpotent and hence 0. But $I \subset IA$ implies $I = 0$ too. On the other hand, if A_0 is semi-prime and I is a nilpotent right ideal of A, then $I_1 = I + AI$ is a two-sided ideal of A which is nilpotent because $I_1^m \subset I^m + AI^m$ for any positive integer m. If $I_1^t = 0$, $(I_1 \cap A_0)^t = 0$, so $I_1 \cap A_0 = 0$ and $I_1 \subset \sum_{0\neq\alpha\in\Delta} A_\alpha$ by homogeneity (see § 4.2). Thus

$$I \subset I_1 \subset \sum_{0\neq\alpha\in\Delta} A_\alpha.$$

Finally, we note that any ideal of A contained in $\sum_{0\neq\alpha\in\Delta} A_\alpha$ must be nilpotent by 5.3.

We will denote the Jacobson radical of an associative algebra X by $J(X)$, and call the algebra semi-simple if $J(X) = 0$. Then

COROLLARY 5.5. *If A_0 is semi-simple, $J(A)$ is the sum of all right ideals of A contained in $\sum_{0\neq\alpha\in\Delta} A_\alpha$ and hence it is nilpotent.*

Proof. Let T denote the sum defined above. Certainly $T \subset J(A)$ because $J(A)$ contains all nilpotent ideals of A. The reverse inclusion holds because $J(A) \cap A_0 \subset J(A_0)$. Seeing this requires the fact that the Jacobson radical of a ring with 1 can be characterized as the intersection of all maximal right ideals (see [**3**, Chapter 1]). But in the course of proving Proposition 4.3 we showed that any maximal right ideal of A_0 is contained in a maximal right ideal of A.

Now suppose that V is an irreducible weighted A-module and V decomposes as $\oplus \sum_{\lambda\in\Lambda} V_\lambda$ relative to the set Λ of weights of L in V, (cf. 3.1). If $va_0 = 0$, for some $v = \sum_{\lambda\in\Delta} v_\lambda \in V$ and $a_0 \in A_0$, then $v_\lambda a_0 = 0$ for each λ because $V_\lambda A_\alpha \subset V_{\lambda+\alpha}$. Thus we obtain

$$(0 : V) \cap A_0 = \bigcap_{\lambda\in\Lambda} (0 : V_\lambda)_0$$

where $(0 : V)$ denotes the A-annihilator of V and $(0 : V_\lambda)_0$ the A_0-annihilator of V_λ. If V is faithful, $\{V_\lambda : \lambda \in \Lambda\}$ is a collection of irreducible A_0-modules and $\bigcap_{\lambda\in\Lambda} P_\lambda = 0$ where $P_\lambda = (0 : V_\lambda)_0$ is a primitive ideal of A_0 containing by 3.4, $x - \lambda(x)1$ for every $x \in L$ (see [**3**, Chapter 2]). Moreover, for λ, $\mu \in \Lambda$ and $\lambda \neq \mu$, there is an $x \in L$ for which $\lambda(x) \neq \mu(x)$, and so $(x - \mu(x)1) - (x - \lambda(x)1) = (\lambda - \mu)(x)1 \in P_\lambda + P_\mu$. This implies $P_\lambda + P_\mu = A_0$. Since Λ is a finite set, we can apply the Chinese Remainder Theorem to obtain

$$A_0 \cong \sum_{\lambda\in\Lambda} A_0/P_\lambda.$$

Since the quotient of any ring by a primitive ideal is a primitive ring, we have established

THEOREM 5.6. *Suppose L is a finitely diagonable subspace of a primitive algebra A possessing a faithful irreducible weighted module. Then the centralizer of L is a direct sum of primitive algebras.*

In passing, we remark that the theorem is also true if A is in fact a direct sum of primitive algebras B_i, for it can be shown that the diagonable subspace L of A decomposes into a direct sum of diagonable subspaces L_i of B_i, and that any weighted irreducible A-module is a weighted irreducible B_i-module for each i.

6. Applications to Jordan algebras. Let now J be a finite-dimensional semi-simple Jordan algebra over an algebraically closed field F of characteristic 0. We showed in Section 2.4 that the centralizer $\mathscr{C} = A_0(\mathscr{H})$ of any Cartan subalgebra $\mathscr{H}$ of J is a diagonable subspace of the universal multiplication envelope $A = \mathscr{U}(J)$; hence it plays an important role in the representation theory of J. Now $\mathscr{U}(J)$ is semi-prime (in fact, semi-simple) and L is finitely diagonable because $\mathscr{U}(J)$ is finite-dimensional. Thus, applying Theorem 5.4, we see that A_0 is a semi-prime finite-dimensional algebra over the algebraically closed field F. One part of the following theorem characterizing $\mathscr{C}$ is now proven.

THEOREM 6.1. *The centralizer of a Cartan subalgebra of a finite dimensional semi-simple Jordan algebra over an algebraically closed field F of characteristic* 0 *in the universal multiplication envelope is a direct sum of complete matrix rings over F, and is actually just the centralizer of a single element of the Cartan subalgebra.*

In order to prove that $\mathscr{C} = A_0(x)$ for some $x \in \mathscr{H}$, we will establish a more general result. First, call a diagonable element x *finitely diagonable* if Fx is a finitely diagonable subspace. Note that this implies that the linear transformation ad x is algebraic, for the roots of the minimal polynomial of ad x will be exactly the set $\{\alpha(x) : \alpha \in \Delta\}$, where Δ is the collection of roots of Fx in A. Theorem 6.1 follows immediately from

THEOREM 6.2. *Let L be a finitely diagonable subspace of $A = \sum_{\alpha\in\Delta} A_\alpha$ and assume that any collection of centralizers of finitely diagonable elements from A has a minimal member* (*with respect to inclusion*). *Then there exists $x \in L$ such that $A_0(x) = A_0(L)$.*

Proof. Let $x \in L$ be such that $A_0(x)$ is a minimal member of the set $\{A_0(u) : u \in L\}$ and let y be any element of L. For any $t \in F$, define $y_t = x + t(y - x)$. Now for a fixed $\alpha \in \Delta$, $A_\alpha(x)$ is invariant under ad $(y - x)$, and so we can define $p(\lambda)$ to be the minimal polynomial of the restriction of this transformation to the space $A_\alpha(x)$. Similarly, we let $f_\alpha(\lambda, t)$ denote the minimal polynomial of the restriction of ad y_t to $A_\alpha(x)$. Assuming $t \neq 0$, it is easy to see that β is a root of $p(\lambda)$ if and only if $\alpha + t\beta$ is a root of $f_\alpha(\lambda, t)$ and so

$$f_\alpha(\lambda, t) = \prod_\beta (\lambda - (\alpha + t\beta)) = \lambda^{m_\alpha} + \beta_1{}^\alpha(t)\lambda^{m_\alpha - 1} + \ldots + \beta_{m_\alpha}{}^\alpha(t),$$

where the product is taken over all the roots β of $p(\lambda)$. Here, the $\beta_i{}^\alpha(t)$, $i = 1, \ldots, m_\alpha$, are polynomials in t, and m_α depends only on α, not on t; in fact, m_α is just the number of roots of the polynomial $p(\lambda)$. If $\alpha \neq 0$, $\beta_{m_\alpha}{}^\alpha(0) = (-\alpha)^{m_\alpha} \neq 0$, and so letting α range over the non-zero roots of Fx, we have finitely many polynomials $\beta_{m_\alpha}{}^\alpha(t)$, none of which is identically 0. Since char $F = 0$, F is infinite, and there must be an infinite subset D of F such that $\beta_{m_\alpha}{}^\alpha(t) \neq 0$ for any $t \in D$ and non-zero α. But for $t \in D$, $A_0(y_t) \subset A_0(x)$, because if $a \in A_0(y_t)$ and we write $a = \sum a_\alpha$ relative to Fx, then $(a, y_t) = 0$ implies $(a_\alpha, y_t) = 0$ for each α. But ad y_t is non-singular on $A_\alpha(x)$ for $\alpha \neq 0$. Hence $a = a_0 \in A_0(x)$. By the minimality of $A_0(x)$, $A_0(y_t) = A_0(x)$ and therefore, the minimal polynomial of ad y_t on $A_0(x)$ is λ for $t \in D$; i.e., $\beta_i^0(t) = 0$ for infinitely many t, $i = 1, \ldots, m_0$. It follows that the polynomials $\beta_i^0(t)$ are identically 0 and ad $y_1 =$ ad y has the minimal polynomial λ on $A_0(x)$. Thus $A_0(x) \subset A_0(y)$, and since $A_0(L) = \bigcap_{y \in L} A_0(y)$, we have $A_0(x) \subset A_0(L)$. The reverse inclusion is obvious.

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University of British Columbia,
Vancouver, British Columbia;
Memorial University of Newfoundland,
St. John's, Newfoundland