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The p -adic Gross–Zagier formula on Shimura curves

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ABSTRACT

We prove a general formula for the p -adic heights of Heegner points on modular abelian varieties with potentially ordinary (good or semistable) reduction at the primes above p . The formula is in terms of the cyclotomic derivative of a Rankin–Selberg p -adic L -function, which we construct. It generalises previous work of Perrin-Riou, Howard, and the author to the context of the work of Yuan–Zhang–Zhang on the archimedean Gross–Zagier formula and of Waldspurger on toric periods. We further construct analytic functions interpolating Heegner points in the anticyclotomic variables, and obtain a version of our formula for them. It is complemented, when the relevant root number is $+1$ rather than -1 , by an anticyclotomic version of the Waldspurger formula. When combined with work of Fouquet, the anticyclotomic Gross–Zagier formula implies one divisibility in a p -adic Birch and Swinnerton-Dyer conjecture in anticyclotomic families. Other applications described in the text will appear separately.

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1. Introduction

The main results of this paper are the general formula for the p -adic heights of Heegner points of Theorem B below, and its version in anticyclotomic families (contained in Theorem C). They are preceded by a flexible construction of the relevant p -adic L -function (Theorem A), and complemented by a version of the Waldspurger formula in anticyclotomic families (presented in Theorem C as well). In Theorem D, we give an application to a version of the p -adic Birch and Swinnerton-Dyer conjecture in anticyclotomic families. In Theorem E, we state a result on the generic non-vanishing of p -adic heights on CM abelian varieties, as a special case of a theorem to appear in joint work with Burungale.

Our theorems are key ingredients of a new Gross–Zagier formula for exceptional zeros [Dis16], and of a universal p -adic Gross–Zagier formula specialising to analogues of Theorem B in all

weights. These will be given in separate works. Here we would just like to mention that all of them, as well as Theorem E, make essential use of the new generality of the present work.

The rest of this introductory section contains the statements of our results, followed by an outline of their proofs. To avoid interrupting the flow of exposition, the discussion of previous and related works (notably by Perrin-Riou and Howard) has mostly been concentrated in § 1.6.

1.1 Heegner points and multiplicity one

Let A be a simple abelian variety of \mathbf{GL}_2 -type over a totally real field F ; recall that this means that $M := \text{End}^0(A)$ is a field of dimension equal to the dimension of A . One knows how to systematically construct points on A when A admits parametrisations by Shimura curves in the following sense. Let \mathbf{B} be a quaternion algebra over the adèle ring $\mathbf{A} = \mathbf{A}_F$ of F , and assume that \mathbf{B} is *incoherent*, i.e. that its ramification set $\Sigma_{\mathbf{B}}$ has odd cardinality. We further assume that $\Sigma_{\mathbf{B}}$ contains all the archimedean places of F . Under these conditions there is a tower of Shimura curves $\{X_U\}$ over F indexed by the open compact subgroups $U \subset \mathbf{B}^{\infty \times}$; let $X = X(\mathbf{B}) := \varprojlim_U X_U$. For each U , there is a canonical Hodge class $\xi_U \in \text{Pic}(X_U)_{\mathbf{Q}}$ having degree 1 in each connected component, inducing a compatible family $\iota_{\xi} = (\iota_{\xi,U})_U$ of quasi-embeddings¹ $\iota_{\xi,U} : X_U \hookrightarrow J_U := \text{Alb } X_U$. We write $J := \varprojlim_U J_U$. The M -vector space

$$\pi = \pi_A = \pi_A(\mathbf{B}) := \varinjlim_U \text{Hom}^0(J_U, A)$$

is either zero or a smooth irreducible admissible representation of $\mathbf{B}^{\infty \times}$. It comes with a natural stable lattice $\pi_{\mathbf{Z}} \subset \pi$, and its central character

$$\omega_A : F^{\times} \backslash \mathbf{A}^{\times} \rightarrow M^{\times}$$

corresponds, up to twist by the cyclotomic character, to the determinant of the Tate module under the class field theory isomorphism. When π_A is non-zero, A is said to be parametrised by $X(\mathbf{B})$. Under the conditions we are going to impose on A , the existence of such a parametrisation, for a suitable choice of \mathbf{B} (see below), is equivalent to the *modularity* conjecture. Recall that the latter asserts the existence of a unique M -rational (Definition 1.2.1 below) automorphic representation σ_A of weight 2 such that there is an equality of L -functions $L(A, s + 1/2) = L(s, \sigma_A)$. The conjecture is known to be true for ‘almost all’ elliptic curves A (see [LeH14]), and when $A_{\overline{F}}$ has complex multiplication.

Heegner points. Let A be parametrised by $X(\mathbf{B})$ and let E be a CM extension of F admitting an \mathbf{A}^{∞} -embedding $E_{\mathbf{A}^{\infty}} \hookrightarrow \mathbf{B}^{\infty}$, which we fix; we denote by η the associated quadratic character and by D_E its absolute discriminant. Then E^{\times} acts on X and by the theory of complex multiplication each closed point of the subscheme $X^{E^{\times}}$ is defined over E^{ab} , the maximal abelian extension of E . We fix one such CM point P . Let $L(\chi)$ be a field extension of M and let

$$\chi : E^{\times} \backslash E_{\mathbf{A}^{\infty}}^{\times} \rightarrow L(\chi)^{\times}$$

be a finite-order Hecke character such that

$$\omega_A \cdot \chi|_{\mathbf{A}^{\infty, \times}} = 1.$$

We can view χ as a character of $\mathcal{G}_E := \text{Gal}(\overline{E}/E)$ via the reciprocity map of class field theory (normalised, in this work, by sending uniformisers to geometric Frobenii). For each $f \in \pi_A$, we

¹ By ‘quasi-embedding’, we mean an element of $\text{Hom}(X_U, J_U) \otimes \mathbf{Q}$, a multiple of which is an embedding.

then have a *Heegner point*

$$P(f, \chi) = \int_{\text{Gal}(E^{\text{ab}}/E)} f(\iota_\xi(P)^\tau) \otimes \chi(\tau) d\tau \in A(\chi).$$

Here the integration uses the Haar measure of total volume 1, and

$$A(\chi) := (A(E^{\text{ab}}) \otimes_M L(\chi)_\chi)^{\text{Gal}(E^{\text{ab}}/E)},$$

where $L(\chi)_\chi$ denotes the one-dimensional Galois module $L(\chi)$ with action given by χ . The functional $f \mapsto P(f, \chi)$ defines an element of

$$\text{Hom}_{E_{\mathbf{A}^\infty}^\times}(\pi \otimes \chi, L(\chi)) \otimes_{L(\chi)} A(\chi).$$

A foundational local result of Tunnell and Saito [Tun83, Sai93] asserts that, for any irreducible representation π of \mathbf{B}^\times , the $L(\chi)$ -dimension of

$$\mathbf{H}(\pi, \chi) = \text{Hom}_{E_{\mathbf{A}^\infty}^\times}(\pi \otimes \chi L(\chi))$$

is either zero or one. It is one exactly when, for all places v of F , the local condition

$$\varepsilon(1/2, \pi_{E,v} \otimes \chi_v) = \chi_v(-1)\eta_v(-1)\varepsilon(\mathbf{B}_v) \tag{1.1.1}$$

holds, where π_E is the base-change of π to E , $\eta = \eta_{E/F}$ is the quadratic character of \mathbf{A}^\times associated to E , and $\varepsilon(\mathbf{B}_v) = +1$ if \mathbf{B}_v is split and -1 if \mathbf{B}_v is ramified. In this case, denoting by π^\vee the M -contragredient representation, there is an explicit generator

$$Q = \prod_{v \nmid \infty} Q_v \in \mathbf{H}(\pi, \chi) \otimes_{L(\chi)} \mathbf{H}(\pi^\vee, \chi^{-1})$$

defined by integration of local matrix coefficients

$$Q_v(f_{1,v}, f_{2,v}, \chi) = \frac{L(1, \eta_v)L(1, \pi_v, \text{ad})}{\zeta_{F,v}(2)L(1/2, \pi_{E,v} \otimes \chi_v)} \int_{E_v^\times/F_v^\times} \chi_v(t_v)(\pi(t_v)f_{1,v}, f_{2,v})_v dt_v \tag{1.1.2}$$

for a decomposition $(\cdot, \cdot) = \bigotimes_v (\cdot, \cdot)_v$ of the pairing $\pi \otimes_M \pi^\vee \rightarrow M$, and Haar measures dt_v assigning to $\mathcal{O}_{E,v}^\times/\mathcal{O}_{F_v}^\times$ the volume 1 if v is unramified in E and 2 if v ramifies in E . The normalisation is such that given f_1, f_2 , all but finitely many terms in the product are equal to 1. The pairings Q_v in fact depend on the choice of decomposition, which in general needs an extension of scalars; the global pairing is defined over M and independent of choices.

Note that the local root numbers are unchanged if one replaces π by its Jacquet–Langlands transfer to another quaternion algebra, and that when $\pi = \pi_A$ they equal the local root numbers $\varepsilon(A_{E,v}, \chi_v)$ of the motive $H_1(A \times_{\text{Spec } F} \text{Spec } E) \otimes_M \chi$ [Gro91]. In this way one can view the local conditions

$$\varepsilon(A_{E,v}, \chi_v) = \chi_v(-1)\eta_v(-1)\varepsilon(\mathbf{B}_v)$$

as determining a unique totally definite quaternion algebra $\mathbf{B} \supset E_{\mathbf{A}}$ over \mathbf{A} , which is incoherent precisely when the global root number $\varepsilon(A_E, \chi) = -1$. In this case, A is parametrised by $X(\mathbf{B})$ in the sense described above if and only if A is modular in the sense that the Galois representation afforded by its Tate module is attached to a cuspidal automorphic representation of $\mathbf{GL}_2(\mathbf{A}_F)$ of parallel weight 2. We assume this to be the case.

Gross–Zagier formulas. There is a natural identification $\pi^\vee = \pi_{A^\vee}$, where A^\vee is the dual abelian variety (explicitly, this is induced by the perfect $M = \text{End}^0(A)$ -valued pairing $f_{1,U} \otimes f_{2,U} \mapsto \text{vol}(X_U)^{-1} f_{1,U} \circ f_{2,U}^\vee$ using the canonical autoduality of J_U for any sufficiently small U ; the normalising factor $\text{vol}(X_U) \in \mathbf{Q}^\times$ is the hyperbolic volume of $X_U(\mathbf{C}_\tau)$ for any $\tau : F \hookrightarrow \mathbf{C}$; see [YZZ12, §1.2.2]). Similarly to the above, we have a Heegner point functional $P^\vee(\cdot, \chi^{-1}) \in \mathbf{H}(\pi^\vee, \chi^{-1}) \otimes_L A^\vee(\chi^{-1})$. Then the multiplicity-one result of Tunnell and Saito implies that for each bilinear pairing

$$\langle \cdot, \cdot \rangle : A(\chi) \otimes_{L(\chi)} A^\vee(\chi^{-1}) \rightarrow V$$

with values in an $L(\chi)$ -vector space V , there is an element $\mathcal{L} \in V$ such that

$$\langle P(f_1, \chi), P(f_2, \chi^{-1}) \rangle = \mathcal{L} \cdot Q(f_1, f_2, \chi)$$

for all $f_1 \in \pi$, $f_2 \in \pi^\vee$.

In this framework, we may call the ‘Gross–Zagier formula’ a formula for \mathcal{L} in terms of L -functions. When $\langle \cdot, \cdot \rangle$ is the Néron–Tate height pairing valued in $\mathbf{C} \xleftarrow{\iota} M$ for an archimedean place ι , the generalisation by Yuan–Zhang–Zhang [YZZ12] of the classical Gross–Zagier formula [GZ86, Zha01a, Zha01b, Zha04] yields

$$\mathcal{L} = \frac{c_E}{2} \cdot \frac{\pi^{2[F:\mathbf{Q}]} |D_F|^{1/2} L'(1/2, \sigma_{A,E}^\iota \otimes \chi^\iota)}{2L(1, \eta) L(1, \sigma_A^\iota, \text{ad})}, \tag{1.1.3}$$

where

$$c_E := \frac{\zeta_F(2)}{(\pi/2)^{[F:\mathbf{Q}]} |D_E|^{1/2} L(1, \eta)} \in \mathbf{Q}^\times \tag{1.1.4}$$

and, in the present Introduction, L -functions are as usual Euler products over all the *finite* places.² (However in the main body of the paper we will embrace the convention of [YZZ12] of including the archimedean factors.) The most important factor is the central derivative of the L -function $L(s, \sigma_{A,E}^\iota \otimes \chi)$.

When $\langle \cdot, \cdot \rangle$ is the product of the v -adic logarithms on $A(F_v)$ and $A^\vee(F_v)$, for a prime v of F which splits in E , the v -adic Waldspurger formula of Liu–Zhang–Zhang [LZZ15] (generalising [BDP13]) identifies \mathcal{L} with the special value of a v -adic Rankin–Selberg L -function obtained by interpolating the values $L(1/2, \sigma_{A,E} \otimes \chi'')$ at anticyclotomic Hecke characters χ'' of E of higher weight at v (in particular, the central value for the given character χ lies *outside* the range of interpolation).

The object of this paper is a formula for \mathcal{L} when $\langle \cdot, \cdot \rangle$ is a p -adic height pairing. In this case \mathcal{L} is given by the central derivative of a p -adic Rankin–Selberg L -function obtained by interpolation of $L(1/2, \sigma_{A,E} \otimes \chi')$ at finite-order Hecke characters of E , precisely up to the factor $c_E/2$ of (1.1.3). We describe in more detail the objects involved.

1.2 The p -adic L -function

We construct the relevant p -adic L -function as a function on a space of p -adic characters (which can be regarded as an abelian eigenvariety), characterised by an interpolation property at locally

² In [YZZ12], the formula has a slightly different appearance from (1.1.3), owing to the following conventions adopted there: the L - and zeta functions are complete including the archimedean factors; the functional Q includes archimedean factors $Q_v(f_{1,v}, f_{2,v}, \chi)$, which can be shown to be equal to $1/\pi$; and finally the product Haar measure on $E_{\mathbf{A}^\infty}^\times / \mathbf{A}^{\infty, \times}$ equals $|D_E|^{-1/2}$ times our measure (cf. [YZZ12, §1.6.1]). (When ‘ π ’ appears as a factor in a numerical formula, it denotes $\pi = 3.14\dots$; there should be no risk of confusion with the representation π_A .)

constant characters. It further depends on a choice of local models at p (in the present case, additive characters); this point is relevant for the study of fields of rationality and does not seem to have received much attention in the literature on p -adic L -functions.

DEFINITION 1.2.1. An M -rational³ cuspidal automorphic representation of \mathbf{GL}_2 of weight 2 is a representation σ^∞ of $\mathbf{GL}_2(\mathbf{A}^\infty)$ on a rational vector space V_{σ^∞} with $\text{End}_{\mathbf{GL}_2(\mathbf{A}^\infty)} \sigma^\infty = M$ (then V_{σ^∞} acquires the structure of an M -vector space), such that $\sigma^\infty \otimes_{\mathbf{Q}} \sigma_\infty^{(2)} = \bigoplus_{\iota: M \hookrightarrow \mathbf{C}} \sigma^\iota$ is a direct sum of irreducible cuspidal automorphic representations; here $\sigma_\infty^{(2)}$, a complex representation of $\mathbf{GL}_2(F_\infty) \cong \mathbf{GL}_2(\mathbf{R})^{[F:\mathbf{Q}]}$, is the product of a discrete series of parallel weight 2 and a trivial central character.

We fix from now on a rational prime p .

DEFINITION 1.2.2. Let F_v and L be finite extensions of \mathbf{Q}_p , let σ_v be a smooth irreducible representation of $\mathbf{GL}_2(F_v)$ on an L -vector space, and let $\alpha_v : F_v^\times \rightarrow \mathcal{O}_L^\times$ be a smooth character valued in the units of L . We say that σ_v is *nearly ordinary for weight 2* with unit character α_v if σ_v is an infinite-dimensional subrepresentation of the un-normalised principal series $\text{Ind}(|\cdot|_v \alpha_v, \beta_v)$ for some other character $\beta_v : F_v^\times \rightarrow L^\times$. (Concretely, σ_v is then either an irreducible principal series or special of the form $\text{St}(\alpha_v) := \text{St} \otimes (\alpha_v \circ \det)$, where St is the Steinberg representation.)

If M is a number field, \mathfrak{p} is a prime of M above p , and σ_v is a representation of $\mathbf{GL}_2(F_v)$ on an M -vector space, we say that σ_v is *nearly \mathfrak{p} -ordinary* for weight 2 if there is a finite extension L of $M_{\mathfrak{p}}$ such that $\sigma_v \otimes_M L$ is nearly \mathfrak{p} -ordinary for weight 2.

In the rest of this paper we omit the clause ‘for weight 2’.⁴

Fix an M -rational cuspidal automorphic representation σ^∞ of $\mathbf{GL}_2(\mathbf{A}^\infty)$ of weight 2; if there is no risk of confusion we will lighten the notation and write σ instead of σ^∞ . Let $\omega : F^\times \backslash \mathbf{A}^\times \rightarrow M^\times$ be the central character of σ , which is necessarily of finite order.

Fix moreover a prime \mathfrak{p} of M above p and assume that for all $v|p$ the local components σ_v of σ are nearly \mathfrak{p} -ordinary with respective characters α_v ; we write α to denote the collection $(\alpha_v)_{v|p}$. We replace L by its subfield $M_{\mathfrak{p}}(\alpha)$ generated by the values of all the α_v , and we similarly let $M(\alpha) \subset L$ be the finite extension of M generated by the values of all the α_v .

Spaces of p -adic and locally constant characters. Fix throughout this work an arbitrary compact open subgroup $V^p \subset \widehat{\mathcal{O}}_E^{p,\times} := \prod_{w \nmid p} \mathcal{O}_{E,w}^\times$. Let

$$\Gamma = E_{\mathbf{A}^\infty}^\times / \overline{E^\times V^p}, \quad \Gamma_F = \mathbf{A}^{\infty,\times} / \overline{F^\times \widehat{\mathcal{O}}_F^{p,\times}}.$$

Then we have rigid spaces $\mathcal{Y}' = \mathcal{Y}'_\omega(V^p)$, $\mathcal{Y} = \mathcal{Y}_\omega(V^p)$, \mathcal{Y}_F of respective dimensions $[F : \mathbf{Q}] + 1 + \delta$, $[F : \mathbf{Q}]$, $1 + \delta$ (where $\delta \geq 0$ is the Leopoldt defect of F , conjectured to be zero) representing the functors on L -affinoid algebras

$$\begin{aligned} \mathcal{Y}'_\omega(V^p)(A) &= \{\chi' : \Gamma \rightarrow A^\times : \omega \cdot \chi' |_{\widehat{\mathcal{O}}_F^{p,\times}} = 1\}, \\ \mathcal{Y}_\omega(V^p)(A) &= \{\chi : \Gamma \rightarrow A^\times : \omega \cdot \chi |_{\mathbf{A}^{\infty,\times}} = 1\}, \\ \mathcal{Y}_F(A) &= \{\chi_F : \Gamma_F \rightarrow A^\times\}, \end{aligned}$$

³ See [YZZ12, § 3.2.2] for more details on this notion.

⁴ Which we have introduced in order to avoid misleading the reader into thinking of ordinarity of an automorphic representation as a purely local notion (but see [Eme06] for how to approach it as such).

where the sets on the right-hand sides are intended to consist of continuous homomorphisms. The inclusion $\mathcal{Y} \subset \mathcal{Y}'$ sits in the Cartesian diagram

$$\begin{array}{ccc} \mathcal{Y} & \longrightarrow & \mathcal{Y}' \\ \downarrow & & \downarrow \\ \{\mathbf{1}\} & \longrightarrow & \mathcal{Y}_F \end{array} \tag{1.2.1}$$

where the vertical maps are given by $\chi' \mapsto \chi_F = \omega \cdot \chi'|_{\mathbf{A}^{\infty, \times}}$. When $\omega = \mathbf{1}$, \mathcal{Y}_1 is a group object (the ‘Cartier dual’ of Γ/Γ_F); in general, \mathcal{Y}_ω is a principal homogeneous space for \mathcal{Y}_1 under the action $\chi_0 \cdot \chi = \chi_0\chi$.

Let $\mu_{\mathbf{Q}}$ denote the ind-scheme over \mathbf{Q} of all roots of unity and μ_M its base-change to M . Then there are ind-schemes $\mathcal{Y}^{1.c.}$, $\mathcal{Y}^{l.c.}$, $\mathcal{Y}_F^{1.c.}$, ind-finite over M , representing the functors on M -algebras

$$\begin{aligned} \mathcal{Y}^{1.c.}(A) &= \{\chi' : \Gamma \rightarrow \mu_M(A) : \omega \cdot \chi'|_{\widehat{\mathcal{O}}_F^{\times}} = 1\}, \\ \mathcal{Y}^{l.c.}(A) &= \{\chi : \Gamma \rightarrow \mu_M(A) : \omega \cdot \chi|_{\mathbf{A}^{\infty, \times}} = 1\}, \\ \mathcal{Y}_F^{1.c.}(A) &= \{\chi_F : \Gamma_F \rightarrow \mu_M(A)\}, \end{aligned}$$

where the sets on the right-hand sides are intended to consist of locally constant (equivalently, finite-order) characters.

DEFINITION 1.2.3. Let $\mathcal{Y}^?$ be one of the above rigid spaces and $\mathcal{Y}^{?, l.c., \text{an}} \subset \mathcal{Y}^?$ be the (ind-)rigid space which is the analytification of $\mathcal{Y}_L^{?, l.c.} := \mathcal{Y}^{?, l.c.} \times_{\text{Spec } M} \text{Spec } L$. For any finite extension M' of M contained in L , there is a natural map of locally M' -ringed spaces $j_{M'} : \mathcal{Y}^{?, l.c., \text{an}} \rightarrow \mathcal{Y}_{M'}^{?, l.c.}$. Let M' be a finite extension of M contained in L . We say that a section G of the structure sheaf of $\mathcal{Y}^?$ is *algebraic on $\mathcal{Y}_{M'}^{?, l.c.}$* if its restriction to $\mathcal{Y}^{?, l.c., \text{an}}$ equals $j_{M'}^\# G'$ for a (necessarily unique) section G' of the structure sheaf of $\mathcal{Y}_{M'}^{?, l.c.}$.⁵

In the situation of the definition, we will abusively still denote by G the function G' on $\mathcal{Y}_{M'}^{?, l.c.}$.

Local additive character. Let v be a non-archimedean place of F , $p_v \subset \mathcal{O}_{F,v}$ the maximal ideal, and $d_v \subset \mathcal{O}_{F,v}$ the different. We define the space of additive characters of F_v of level 0 to be

$$\Psi_v := \text{Hom}(F_v/d_v^{-1}\mathcal{O}_{F,v}, \mu_{\mathbf{Q}}) - \text{Hom}(F_v/p_v^{-1}d_v^{-1}\mathcal{O}_{F,v}, \mu_{\mathbf{Q}}),$$

where we regard $\text{Hom}(F_v/p_v^n\mathcal{O}_{F,v}, \mu_{\mathbf{Q}})$ as a profinite group scheme over \mathbf{Q} .⁶ The scheme Ψ_v is a torsor for the action of $\mathcal{O}_{F,v}^\times$ (viewed as a constant profinite group scheme over \mathbf{Q}) by $a.\psi(x) := \psi(ax)$.

If $\omega'_v : \mathcal{O}_{F,v}^\times \rightarrow \mathcal{O}(\mathcal{X})^\times$ is a continuous character for a scheme or rigid space \mathcal{X} , we denote by $\mathcal{O}_{\mathcal{X} \times \Psi_v}(\omega'_v) \subset \mathcal{O}_{\mathcal{X} \times \Psi_v}$ the subsheaf of functions G satisfying $G(x, a.\psi) = \omega'_v(a)(x)G(x, \psi)$ for $a \in \mathcal{O}_{F,v}^\times$. By the defining property, we can identify $\mathcal{O}_{\mathcal{X} \times \Psi_v}(\omega'_v)$ with $p_{\mathcal{X}*}\mathcal{O}_{\mathcal{X} \times \Psi_v}(\omega'_v)$ (where

⁵ To avoid all confusions due to the clash of notation, $\mathcal{Y}_F^{1.c.}$ will always denote the M -scheme of locally constant characters of Γ_F introduced above, and *not* the ‘base-change of $\mathcal{Y}^{1.c.}$ to F ’ (which is not defined as M is not a subfield of F in the generality adopted here).

⁶ If $F_v = \mathbf{Q}_p$, then $\text{Hom}(F_v/\mathcal{O}_{F,v}, \mu_{\mathbf{Q}}) = T_p\mu_{\mathbf{Q}}$, the p -adic Tate module of roots of unity. One could also construct and use a scheme parametrising all non-trivial characters of F_v .

$p_{\mathcal{X}} : \mathcal{X} \times \Psi_v \rightarrow \mathcal{X}$ is the projection), a locally free rank-1 $\mathcal{O}_{\mathcal{X}}$ -module with action by $\mathcal{G}_{\mathbf{Q}} := \text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. Finally, we denote $\Psi_p := \prod_{v|p} \Psi_v$ and, if $\omega'_p = \prod_{v|p} \omega'_v : \mathcal{O}_{F,p}^{\times} \rightarrow \mathcal{O}(\mathcal{X})^{\times}$,

$$\mathcal{O}_{\mathcal{X} \times \Psi_p}(\omega'_p) = \bigotimes_{v|p} \mathcal{O}_{\mathcal{X} \times \Psi_v}(\omega'_v),$$

where the tensor product is in the category of $\mathcal{O}_{\mathcal{X}}$ -modules. Its space of global sections over \mathcal{X} will be denoted by $\mathcal{O}_{\mathcal{X} \times \Psi_p}(\mathcal{X} \times \Psi_p, \omega'_p)$ or simply $\mathcal{O}_{\mathcal{X} \times \Psi_p}(\mathcal{X}, \omega'_p)$.

These sheaves will appear in the next theorem with $\omega'_v = \omega_v \chi_{F,\text{univ},v}^{-1} : \mathcal{O}_{F,v}^{\times} \rightarrow \mathcal{O}(\mathcal{Y}')^{\times}$, where ω_v is the central character of σ_v and $\chi_{F,\text{univ},v} : \mathcal{O}_{F,v}^{\times} \rightarrow \mathcal{O}(\mathcal{Y}_F)^{\times} \rightarrow \mathcal{O}(\mathcal{Y}')^{\times}$ comes from the restriction of the universal $\mathcal{O}(\mathcal{Y}_F)^{\times}$ -valued character of Γ_F . As Ψ_p is a scheme over \mathbf{Q} and $\chi_{F,\text{univ}}$ is obviously algebraic on $\mathcal{Y}_F^{1.c.}$, the notion of Definition 1.2.3 extends to define $\mathcal{Y}'_{M'}^{1.c.}$ -algebraicity of sections of $\mathcal{O}_{\mathcal{Y}' \times \Psi_p}(\omega'_v)$ (and we use the terminology ‘algebraic on $\mathcal{Y}'_{M'}^{1.c.} \times \Psi_p(\omega'_v)$ ’).

As a last preliminary, we introduce notation for bounded functions: if \mathcal{X} is a rigid space, then $\mathcal{O}_{\mathcal{X}}(\mathcal{X})^b \subset \mathcal{O}_{\mathcal{X}}(\mathcal{X})$ is the space of global sections G such that $\sup_{x \in \mathcal{X}} |G(x)|$ is finite; similarly, in the above situation, we let

$$\mathcal{O}_{\mathcal{X} \times \Psi_p}(\mathcal{X}, \omega'_p)^b := \left\{ G \in \mathcal{O}_{\mathcal{X} \times \Psi_p}(\mathcal{X}, \omega'_p) : \sup_{x \in \mathcal{X}} |G(x, \psi)| \text{ is finite for some } \psi \in \Psi_p \right\}.$$

As ω'_p is continuous, $\omega'_p(a)$ is bounded in $a \in \mathcal{O}_{F,p}^{\times}$: we could then equivalently replace ‘is finite for some $\psi \in \Psi_p$ ’ with ‘is uniformly bounded for all $\psi \in \Psi_p$ ’.

THEOREM A. *There is a bounded analytic function*

$$L_{p,\alpha}(\sigma_E) \in \mathcal{O}_{\mathcal{Y}' \times \Psi_p}(\mathcal{Y}', \omega_p^{-1} \chi_{F,\text{univ},p})^b$$

uniquely determined by the following property: $L_{p,\alpha}(\sigma_E)$ is algebraic on $\mathcal{Y}'_{M(\alpha)}^{1.c.} \times \Psi_p$ and, for each \mathbf{C} -valued geometric point

$$(\chi', \psi_p) \in \mathcal{Y}'_{M(\alpha)}^{1.c.} \times \Psi_p(\mathbf{C}),$$

letting $\iota : M(\alpha) \hookrightarrow \mathbf{C}$ be the embedding induced by the composition $\text{Spec } \mathbf{C} \xrightarrow{\chi'} \mathcal{Y}'_{M(\alpha)}^{1.c.} \rightarrow \text{Spec } M(\alpha)$, we have

$$L_{p,\alpha}(\sigma_E)(\chi', \psi_p) = \prod_{v|p} Z_v^{\circ}(\chi'_v, \psi_v) \frac{\pi^{2[F:\mathbf{Q}]} |D_F|^{1/2} L(1/2, \sigma'_E \otimes \chi')}{2L(1, \eta) L(1, \sigma^t, \text{ad})}$$

in \mathbf{C} . The interpolation factor is explicitly

$$Z_v^{\circ}(\chi'_v, \psi_v) := \frac{\zeta_{F,v}(2) L(1, \eta_v)^2}{L(1/2, \sigma_{E,v} \otimes \chi'_v)} \prod_{w|v} Z_w(\chi'_w, \psi_v)$$

with

$$Z_w(\chi'_w, \psi_v) = \begin{cases} \alpha_v(\varpi_v)^{-v(D)} \chi'_w(\varpi_w)^{-v(D)} \frac{1 - \alpha_v(\varpi_v)^{-f_w} \chi'_w(\varpi_w)^{-1}}{1 - \alpha_v(\varpi_v)^{f_w} \chi'_w(\varpi_w)^{-f_w} q_{F,v}^{-f_w}} & \text{if } \chi'_w \cdot \alpha_v \circ q_w \text{ is unramified,} \\ \tau(\chi'_w \cdot \alpha_v \circ q, \psi_{E_w}) & \text{if } \chi'_w \cdot \alpha_v \circ q_w \text{ is ramified.} \end{cases}$$

Here $d, D \in \mathbf{A}^{\infty, \times}$ are generators of the different of F and the relative discriminant of E/F , respectively, q_w is the relative norm of E/F , f_w is the inertia degree of $w|v$, and $q_{F,v}$ is the cardinality of the residue field at v ; finally, for any character $\tilde{\chi}'_w$ of E_w^\times of conductor \mathfrak{f} ,

$$\tau(\tilde{\chi}'_w, \psi_{E_w}) := \int_{w(t)=-w(\mathfrak{f})} \tilde{\chi}'_w(t) \psi_{E,w}(t) dt$$

with dt the additive Haar measure on E_w giving $\text{vol}(\mathcal{O}_{E_w}, dt) = 1$, and $\psi_{E,w} = \psi_{F,v} \circ \text{Tr}_{E_w/F_v}$.

Remark 1.2.4. It follows from the description of Lemma A.1.1 that the interpolation factors Z_v°, Z_w are sections of $\mathcal{O}_{\mathcal{Y}_v^{1.c.} \times \Psi_v}(\omega_v \chi_{F,\text{univ},v}^{-1})$, where $\mathcal{Y}_v^{1.c.}$ is the ind-finite reduced ind-scheme over $M(\alpha)$ representing $\mu_{M(\alpha)}$ -valued characters of E_v^\times . (Later, we will also similarly denote by $\mathcal{Y}_v^{1.c.} \subset \mathcal{Y}_v^{1.c.}$ the subscheme of characters satisfying $\chi_v|_{F_v^\times} = \omega_v^{-1}$.)

In fact, we only construct $L_{p,\alpha}(\sigma_E)$ as a bounded section of $\mathcal{O}_{\mathcal{Y}' \times \Psi_p}(\omega_p^{-1} \chi_{F,\text{univ},p})(D)$, where D is a divisor on \mathcal{Y}' supported away from \mathcal{Y} (i.e. for any polynomial function G on \mathcal{Y}' with divisor of zeroes $\geq D$, the function $G \cdot L_{p,\alpha}(\sigma_E)$ is a bounded global section of $\mathcal{O}_{\mathcal{Y}' \times \Psi_p}(\omega_p^{-1} \chi_{F,\text{univ},p})$);⁷ see Theorem 3.7.1 together with Proposition A.2.2 for the precise statement. This is sufficient for our purposes and to determine $L_{p,\alpha}(\sigma_E)$ uniquely. One can then deduce that it is possible to take $D = 0$ by comparing our p -adic L -function to some other construction where this difficulty does not arise. One such construction has been announced by David Hansen.

1.3 p -adic Gross–Zagier formula

Let us go back to the situation in which A is a modular abelian variety of \mathbf{GL}_2 -type, associated with an automorphic representation σ_A of $\text{Res}_{F/\mathbf{Q}} \mathbf{GL}_2$ of character $\omega = \omega_A$.

p -adic heights. Several authors (notably Mazur–Tate, Schneider, Zarhin, Nekovář) have defined p -adic height pairings on $A(\overline{F}) \times A^\vee(\overline{F})$ for an abelian variety A . These pairings are analogous to the classical Néron–Tate height pairings: in particular, they admit a decomposition into a sum of local symbols indexed by the (*finite*) places of F ; for $v \nmid p$ such symbols can be calculated from intersections of zero-cycles and degree-zero divisors on the local integral models of A .

In the general context of Nekovář [Nek93], adopted in this paper and recalled in § 4.1, height pairings can be defined for any geometric Galois representation V over a p -adic field; we are interested in the case $V = V_p A \otimes_{M_p} L$, where $M = \text{End}^0 A$ and L is a finite extension of a p -adic completion M_p of M . Different from the Néron–Tate heights, p -adic heights are associated with the auxiliary choice of splittings of the Hodge filtration on $\mathbf{D}_{\text{dR}}(V|_{\mathcal{G}_{F_v}})$ for the primes $v|p$; in our case, $\mathbf{D}_{\text{dR}}(V|_{\mathcal{G}_{F_v}}) = H_{\text{dR}}^1(A^\vee/F_v) \otimes_{M_p} L$. When $V|_{\mathcal{G}_{F_v}}$ is potentially ordinary, meaning that it is reducible in the category of de Rham representations (see more precisely Definition 4.1.1),⁸ there is a canonical such choice. If A is modular corresponding to an M -rational cuspidal automorphic representation σ_A^∞ , it follows from [Car86, Théorème A], together with [Nek06, (proof of) Proposition 12.11.5(iv)], that the restriction of $V = V_p A \otimes L$ to \mathcal{G}_{F_v} is potentially ordinary if and only if $\sigma_{A,v} \otimes L$ is nearly \mathfrak{p} -ordinary.

We assume this to be the case for all $v|p$. One then has a canonical p -adic height pairing

$$\langle \cdot, \cdot \rangle : A(\overline{F})_{\mathbf{Q}} \otimes_M A^\vee(\overline{F})_{\mathbf{Q}} \rightarrow \Gamma_F \hat{\otimes} L, \tag{1.3.1}$$

⁷ A similar difficulty is encountered for example by Hida in [Hid91].

⁸ This is a \mathfrak{p} -partial version of the notion of A_{F_v} acquiring ordinary (good or semistable) reduction over a finite extension of F_v .

whose precise definition will be recalled at the end of §4.1. Its equivariance properties under the action of $\mathcal{G}_F = \text{Gal}(\overline{F}/F)$ allow us to deduce from it pairings

$$\langle \cdot, \cdot \rangle : A(\chi) \otimes_{L(\chi)} A^\vee(\chi^{-1}) \rightarrow \Gamma_F \hat{\otimes} L(\chi) \tag{1.3.2}$$

for any character $\chi \in \mathcal{Y}_L^{1.c.}$.

Remark 1.3.1. Suppose that $\ell : \Gamma_F \rightarrow L(\chi)$ is any continuous homomorphism such that, for all $v|p$, $\ell_v|_{\mathcal{O}_{F,v}^\times} \neq 0$; we then call ℓ a *ramified logarithm*. Then it is conjectured, but not known in general, that the pairings deduced from (1.3.2) by composition with ℓ are non-degenerate. See Theorem E for a new result in this direction.

Remark 1.3.2. If χ is not exceptional in the sense of the next definition, then (1.3.2) is known to coincide with the norm-adapted height pairings à la Schneider [Sch82, Nek93], by [Nek93], and with the Mazur–Tate [MT83] height pairings, by [IW03].

DEFINITION 1.3.3. A locally constant character χ_w of E_w^\times is said to be *not exceptional* if $Z_w(\chi_w) \neq 0$.⁹ A character $\chi \in \mathcal{Y}_{M(\alpha)}^{1.c.}$ is said to be *not exceptional* if for all $w|p$, χ_w is not exceptional.

The formula. Let $\mathcal{Y} = \mathcal{Y}_\omega \subset \mathcal{Y}' = \mathcal{Y}'_\omega$ be the rigid spaces defined above. Denote by $\mathcal{I}_\mathcal{Y} \subset \mathcal{O}_{\mathcal{Y}'}$ the ideal sheaf of \mathcal{Y} and by $\mathcal{N}_{\mathcal{Y}'|\mathcal{Y}}^* = (\mathcal{I}_\mathcal{Y} / \mathcal{I}_\mathcal{Y}^2)|_{\mathcal{Y}}$ the conormal sheaf. By (1.2.1), it is canonically trivial:

$$\mathcal{N}_{\mathcal{Y}'|\mathcal{Y}}^* \cong \mathcal{O}_\mathcal{Y} \otimes T_1^* \mathcal{Y}_F \cong \mathcal{O}_\mathcal{Y} \otimes (\Gamma_F \hat{\otimes} L).$$

For a section G of $\mathcal{I}_\mathcal{Y}$, denote by $d_F G \in \mathcal{N}_{\mathcal{Y}'|\mathcal{Y}}^*$ its image; it can be thought of as the differential in the $1 + \delta$ cyclotomic variable(s).

Let $\chi \in \mathcal{Y}^{1.c.,\text{an}}$ be a character such that $\varepsilon(A_E, \chi) = -1$; denote by $L(\chi)$ its residue field. By the interpolation property, the complex functional equation, and the constancy of local root numbers, the p -adic L -function $L_{p,\alpha}(\sigma_{A,E})$ is a section of $\mathcal{I}_\mathcal{Y}$ in the connected component of $\chi \in \mathcal{Y}'$ (see Lemma 10.2.2). Let \mathbf{B} be the incoherent quaternion algebra determined by (1.1.1) and let $\pi_A = \pi_A(\mathbf{B})$, $\pi_{A^\vee} = \pi_{A^\vee}(\mathbf{B})$.

THEOREM B. *Suppose that:*

- for all $v|p$, A/F_v has potentially \mathfrak{p} -ordinary good or semistable reduction;
- for all $v|p$, E_v/F_v is split;
- the sign $\varepsilon(A_E, \chi) = -1$ and χ is not exceptional (Definition 1.3.3).

Then for all $f_1 \in \pi_A$, $f_2 \in \pi_{A^\vee}$, we have

$$\langle P(f_1, \chi), P^\vee(f_2, \chi^{-1}) \rangle = \frac{c_E}{2} \cdot \prod_{v|p} Z_v^\circ(\chi_v)^{-1} \cdot d_F L_{p,\alpha}(\sigma_{A,E})(\chi) \cdot Q(f_1, f_2, \chi)$$

in $\mathcal{N}_{\mathcal{Y}'|\mathcal{Y}}^*|_\chi \cong \Gamma_F \hat{\otimes} L(\chi)$. Here c_E is as in (1.1.4).

In the right-hand side, we have considered Remark 1.2.4 and used the canonical isomorphism $\mathcal{O}_{\Psi_p}(\omega_p^{-1}) \otimes_M \mathcal{O}_{\Psi_p}(\omega_p) = M$.

⁹As a function on Ψ_v ; by Remark 1.2.4, this is equivalent to $Z_w(\chi_w, \psi_v) \neq 0$ for every $\psi_v \in \Psi_v$.

1.4 Anticyclotomic theory

Consider the setup of § 1.2. Recall that in the case $\varepsilon(1/2, \sigma_E, \chi) = +1$, the definite quaternion algebra \mathbf{B} defined by (1.1.1) is coherent, i.e. it arises as $\mathbf{B} = B \otimes_F \mathbf{A}_F$ for a quaternion algebra B over F ; we may assume that the embedding $E_{\mathbf{A}} \hookrightarrow \mathbf{B}$ arises from an embedding $i : E \hookrightarrow B$. Let π be the automorphic representation of \mathbf{B}^\times attached to σ by the Jacquet–Langlands correspondence; it is realised in the space of locally constant functions $B^\times \backslash \mathbf{B}^\times \rightarrow M$, and this gives a stable lattice $\pi_{\mathcal{O}_M} \subset \pi$. Then, given a character $\chi \in \mathcal{Y}^{1.c.}$, the formalism of § 1.1 applies to the period functional $p \in H(\pi, \chi)$ defined by

$$p(f, \chi) := \int_{E^\times \backslash E_{\mathbf{A}}^\times} f(i(t))\chi(t) dt \tag{1.4.1}$$

and to its dual $p^\vee(\cdot, \chi^{-1}) \in H(\pi^\vee, \chi^{-1})$. Here dt is the Haar measure of total volume 1.

The formula expressing the decomposition of their product was proved by Waldspurger (see [Wal85] or [YZZ12]): for all finite-order characters $\chi : E^\times \backslash E_{\mathbf{A}}^\times \rightarrow M(\chi)^\times$ valued in some extension $M(\chi) \supset M$, and for all $f_1 \in \pi$, $f_2 \in \pi^\vee$, we have

$$p(f_1, \chi)p^\vee(f_2, \chi^{-1}) = \frac{c_E}{4} \cdot \frac{\pi^{2[F:\mathbf{Q}]}|D_F|^{1/2}L(1/2, \sigma_E \otimes \chi)}{2L(1, \eta)L(1, \sigma, \text{ad})} \cdot Q(f_1, f_2, \chi) \tag{1.4.2}$$

in $M(\chi)$. Notice that here we could trivially modify the right-hand side to replace the complex L -function with the p -adic L -function, thanks to the interpolation property defining the latter.

The L -function terms of both the Waldspurger and the p -adic Gross–Zagier formulas thus admit an interpolation as analytic functions (or sections of a sheaf) on \mathcal{Y}_ω . We can show that the other terms do as well.

Let π be the M -rational representation of the (coherent or incoherent) quaternion algebra $\mathbf{B}^\times \supset E_{\mathbf{A}}^\times$ considered above, with central character ω . It will be convenient to denote $\pi^+ = \pi$, $\pi^- = \pi^\vee$, $p^+ = p$, $p^- = p^\vee$, $\mathcal{Y}_\pm = \mathcal{Y}_{\omega^{\pm 1}}$, and, in the incoherent case, $A^+ = A$, $A^- = A^\vee$, $P^+ = P$, $P^- = P^\vee$, $\sigma = \sigma_A$.

We have a natural isomorphism $\mathcal{Y}_+ \cong \mathcal{Y}_-$ given by inversion. If \mathcal{F} is a sheaf on \mathcal{Y}_- , we denote by \mathcal{F}^ι its pullback to a sheaf on \mathcal{Y}_+ ; the same notation is used to transfer sections of such sheaves.

Big Selmer groups and heights. Let $\chi_{\text{univ}}^\pm : \Gamma \rightarrow (\mathcal{O}(\mathcal{Y}_\pm)^b)^\times$ be the tautological character such that $\chi_{\text{univ}}^\pm(t)(\chi) = \chi(t)^{\pm 1}$ for all $\chi \in \mathcal{Y}_\pm$, and define an $\mathcal{O}(\mathcal{Y}_\pm)^b$ -module

$$\mathbf{S}_p(A_E^\pm, \chi_{\text{univ}}^\pm, \mathcal{Y}_\pm)^b := H_f^1(E, V_p A_E^\pm \otimes \mathcal{O}(\mathcal{Y}_\pm)^b(\chi_{\text{univ}}^\pm)),$$

where $\mathcal{O}(\mathcal{Y}_\pm)^b(\chi_{\text{univ}}^\pm)$ denotes the module of bounded global sections $\mathcal{O}(\mathcal{Y}_\pm)^b$ with \mathcal{G}_E -action by χ_{univ}^\pm . Here, for a topological $\mathbf{Q}_p[\mathcal{G}_E]$ -module V which is potentially ordinary at all $w|p$ in the sense of Definition 4.1.1 below, with exact sequences $0 \rightarrow V_w^+ \rightarrow V_w \rightarrow V_w^- \rightarrow 0$, the (Greenberg) Selmer group $H_f^1(E, V) \subset H^1(E, V) := H^1(\mathcal{G}_E, V)$ is the group of those continuous cohomology classes c which are unramified away from p and such that, for every $w|p$, the restriction of c to a decomposition group at a w is in the kernel of

$$H^1(E_w, V) \rightarrow H^1(E_w, V^-).$$

(In the case at hand, $V_w^- = V_p A_E^\pm|_{\mathcal{G}_{E,w}}^-$ is the maximal potentially unramified quotient of $V_p A_E^\pm|_{\mathcal{G}_{E,w}}$; cf. § 4.1.) For every *non-exceptional* $\chi^\pm \in \mathcal{Y}_\pm^{1.c.}$, the specialisation $\mathbf{S}_p(A_E^\pm, \chi_{\text{univ}}^\pm, \mathcal{Y}_\pm)^b \otimes L(\chi)$ is isomorphic to the target of the Kummer map

$$\kappa : A_E^\pm(\chi^\pm) \rightarrow H_f^1(E, V_p A_E \otimes L(\chi^\pm)_{\chi^\pm}). \tag{1.4.3}$$

The work of Nekovář [Nek06] explains the exceptional specialisations and provides a height pairing on the big Selmer groups. The key underlying object is the *Selmer complex*

$$\widetilde{\mathrm{R}\Gamma}_f(E, V_{\mathfrak{p}}A_E^{\pm} \otimes \mathcal{O}(\mathcal{Y}_{\pm}^{\circ})^b(\chi_{\mathrm{univ}}^{\pm})), \tag{1.4.4}$$

an object in the derived category of $\mathcal{O}(\mathcal{Y}_{\pm})^b$ -modules defined as in [Nek06, §0.8] taking $T = V_{\mathfrak{p}}A_E^{\pm} \otimes \mathcal{O}(\mathcal{Y}_{\pm}^{\circ})^b(\chi_{\mathrm{univ}}^{\pm})$ and $U_w^+ = V_{\mathfrak{p}}A_E^{\pm}|_{\mathcal{G}_{E_w}^+} \otimes \mathcal{O}(\mathcal{Y}_{\pm}^{\circ})^b(\chi_{\mathrm{univ}}^{\pm})$ in the notation of [Nek06]. Its first cohomology group

$$\widetilde{H}_f^1(E, V_{\mathfrak{p}}A_E^{\pm} \otimes \mathcal{O}(\mathcal{Y}_{\pm}^{\circ})^b(\chi_{\mathrm{univ}}^{\pm}))$$

satisfies the following property. For every L -algebra quotient R of $\mathcal{O}(\mathcal{Y}_{\pm}^{\circ})^b$, letting $\chi_R^{\pm} : \Gamma \rightarrow R^{\times}$ be the character deduced from $\chi_{\mathrm{univ}}^{\pm}$, there is an exact sequence [Nek06, (0.8.0.1)]

$$\begin{aligned} 0 &\rightarrow \bigoplus_{w|p} H^0(E_w, V_{\mathfrak{p}}A_E^{\pm}|_{\mathcal{G}_{E_w}^{\pm}} \otimes R(\chi_R^{\pm})) \\ &\rightarrow \widetilde{H}_f^1(E, V_{\mathfrak{p}}A_E^{\pm} \otimes \mathcal{O}(\mathcal{Y}_{\pm}^{\circ})^b(\chi_{\mathrm{univ}}^{\pm})) \otimes R \rightarrow H_f^1(E, V_{\mathfrak{p}}A_E^{\pm} \otimes R(\chi_R^{\pm})) \rightarrow 0. \end{aligned} \tag{1.4.5}$$

When $R = \mathcal{O}(\mathcal{Y}_{\pm})^b$ itself, each group $H^0(E_w, V_{\mathfrak{p}}A_E^{\pm}|_{\mathcal{G}_{E_w}^{\pm}} \otimes \mathcal{O}(\mathcal{Y}_{\pm})^b(\chi_{\mathrm{univ}}^{\pm}))$ vanishes as $\chi_{\mathrm{univ},w}$ is infinitely ramified; hence,

$$\widetilde{H}_f^1(E, V_{\mathfrak{p}}A_E^{\pm} \otimes \mathcal{O}(\mathcal{Y}_{\pm}^{\circ})^b(\chi_{\mathrm{univ}}^{\pm})) \cong \mathbf{S}_{\mathfrak{p}}(A_E^{\pm}, \chi_{\mathrm{univ}}^{\pm}, \mathcal{Y}_{\pm})^b.$$

When $R = L(\chi)$ with $\chi \in \mathcal{Y}^{1.c.}$, the group $H^0(E_w, V_{\mathfrak{p}}A_E^{\pm}|_{\mathcal{G}_{E_w}^{\pm}} \otimes L(\chi^{\pm})_{\chi^{\pm}})$ vanishes unless $\chi_w \cdot \alpha_v \circ q_w = \mathbf{1}$ on E_w^{\times} , that is, unless χ_w is exceptional.

Finally, by [Nek06, ch. 11], there is a big height pairing

$$\langle \cdot, \cdot \rangle : \mathbf{S}_{\mathfrak{p}}(A_E^+, \chi_{\mathrm{univ}}^+, \mathcal{Y}_+)^b \otimes_{\mathcal{O}_{\mathcal{Y}_+}} \mathbf{S}_{\mathfrak{p}}(A_E^-, \chi_{\mathrm{univ}}^-, \mathcal{Y}_-)^{b,\iota} \rightarrow \mathcal{N}_{\mathcal{Y}_+/\mathcal{Y}_+}^*(\mathcal{Y}_+)^b \tag{1.4.6}$$

interpolating the height pairings on $H_f^1(E, V_{\mathfrak{p}}A \otimes L(\chi^{\pm})_{\chi^{\pm}})$ for non-exceptional $\chi \in \mathcal{Y}^{1.c.}$ (and more generally certain ‘extended’ pairings on $\widetilde{H}_f^1(E, V_{\mathfrak{p}}A \otimes L(\chi^{\pm})_{\chi^{\pm}})$ for all $\chi \in \mathcal{Y}^{1.c.}$; these will play no role here).

Heegner–theta elements and anticyclotomic formulas. Keep the assumptions that for all $v|p$, E_v/F_v is split and $\pi_v \cong \sigma_v$ is \mathfrak{p} -nearly ordinary with unit character α_v . Then, after tensoring with $\mathcal{O}_{\Psi_p}(\Psi_p)$ (in order to use Kirillov models at p), we will have a decomposition $\pi^{\pm} \cong \pi^{\pm,p} \otimes \pi_p^{\pm}$, which is an isometry with respect to pairings $(\cdot, \cdot)^p, (\cdot, \cdot)_p$ on each of the factors. By (1.1.2), for each $\chi = \chi^p \chi_p \in \mathcal{Y}_M^{1.c.}$ we can then define a toric period

$$Q^p(f^{+,p}, f^{-,p}, \chi) \in M(\chi) \otimes \mathcal{O}_{\Psi_p}(\omega_p^{-1}). \tag{1.4.7}$$

Given $f^{\pm,p} \in \pi^{\pm,p}$, we will construct an explicit pair of elements

$$f_{\alpha}^{\pm} = (f_{\alpha, V_p}^{\pm}) = (f^{\pm,p} \otimes f_{\alpha, p, V_p}^{\pm})_{V_p} \in \pi_{M(\alpha)}^{\pm,p} \otimes \varprojlim_{V_p} \pi_p^{\pm, V_p}, \tag{1.4.8}$$

where the inverse system is indexed by compact open subgroups $V_p \subset E_p^{\times} \subset \mathbf{B}_p^{\times}$ containing $\mathrm{Ker}(\omega_p)$, with transition maps being given by averages under their π_p^{\pm} -action, and f_{α, p, V_p}^{\pm} are suitable elements of π_p^{\pm, V_p} . We compute in Lemma 10.1.2 that we have

$$Q_p(f_{\alpha, p}^+, f_{\alpha, p}^-) = \zeta_{F,p}(2)^{-1} \prod_{v|p} Z_v^{\circ}$$

as sections of $\bigotimes_{v|p} \mathcal{O}_{\mathcal{Y}_v^{1.c.} \times \Psi_v}(\omega_v)$, where the left-hand side in the above expression is computed, for each $\chi_p \in \prod_{v|p} \mathcal{Y}_v^{1.c.}$, as the limit of $Q_p(f_{\alpha,p,V_p}^+, f_{\alpha,p,V_p}^-)$ as $V_p \rightarrow \text{Ker}(\omega_p)$.

For the following theorem, note that all the local signs in (1.1.1) extend to locally constant functions of \mathcal{Y}_+ (this is a simple special case of [PX14, Proposition 3.3.4]); the quaternion algebra over \mathbf{A} determined by (1.1.1) is then also constant along the connected components of \mathcal{Y}_+ . We will say that a connected component $\mathcal{Y}_+^\circ \subset \mathcal{Y}_+$ is of type $\varepsilon \in \{\pm 1\}$ if $\varepsilon(1/2, \sigma_E, \chi) = \varepsilon$ along \mathcal{Y}_+° .

THEOREM C. *Let $\mathcal{Y}_+^\circ \subset \mathcal{Y}_+$ be a connected component of type ε , let \mathbf{B} be the quaternion algebra determined by (1.1.1), and let π^\pm be the representations of \mathbf{B}^\times constructed above. Finally, let $\mathcal{Y}_-^\circ \subset \mathcal{Y}_-$ be the image of \mathcal{Y}_+° under the inversion map.*

- (1) (Heegner–theta elements.) *For each $f^{\pm,p} \in \pi^{\pm,p}$, there are elements*

$$\begin{aligned} \Theta_\alpha^\pm(f^{\pm,p}) &\in \mathcal{O}_{\mathcal{Y}_+}(\mathcal{Y}_+^\circ)^b \quad \text{if } \varepsilon = +1, \\ \mathcal{P}_\alpha^\pm(f^{\pm,p}) &\in \mathbf{S}_p(A_E^\pm, \chi_{\text{univ}}^\pm, \mathcal{Y}_\pm^\circ)^b \quad \text{if } \varepsilon = -1 \end{aligned}$$

uniquely determined by the property that, for any compact open subgroup $V_p \subset E_p^\times$ and any V_p -invariant character $\chi^\pm \in \mathcal{Y}_\pm^\circ$, we have

$$\begin{aligned} \Theta_\alpha^\pm(f^{\pm,p})(\chi^\pm) &= p(f_{\alpha,V_p}^\pm, \chi^\pm), \\ \mathcal{P}_\alpha^\pm(f^{\pm,p})(\chi^\pm) &= \kappa(P(f_{\alpha,V_p}^\pm, \chi^\pm)), \end{aligned}$$

where f_α^\pm is the element (1.4.8), $p(\cdot)$ is the period integral (1.4.1), and κ is the Kummer map (1.4.3).

- (2) *There is an element*

$$\mathcal{Q} = \zeta_{F,p}(2)^{-1} \prod_{v|p} \mathcal{Q}_v \in \text{Hom}_{\mathcal{O}(\mathcal{Y}_+^\circ)^b[E_{\mathbf{A}^p}^\times]}(\pi^{+,p} \otimes \pi^{-,p} \otimes \mathcal{O}(\mathcal{Y}_+^\circ)^b, \mathcal{O}(\mathcal{Y}_+^\circ)^b \otimes \mathcal{O}_{\Psi_p}(\omega_p^{-1}))$$

uniquely determined by the property that, for all $f^{\pm,p} \in \pi^{\pm,p}$ and all $\chi \in \mathcal{Y}_+^{\circ,1.c.}$, we have

$$\mathcal{Q}(f^{+,p}, f^{-,p})(\chi) = \zeta_{F,p}(2)^{-1} \cdot Q^p(f^{+,p}, f^{-,p}, \chi^p).$$

- (3) (Anticyclotomic Waldspurger formula.) *If $\varepsilon = +1$, we have*

$$\Theta_\alpha^+(f^{+,p}) \cdot \Theta_\alpha^-(f^{-,p})^\iota = \frac{c_E}{4} \cdot L_{p,\alpha}(\sigma_E) \cdot \mathcal{Q}(f^{+,p}, f^{-,p})$$

in $\mathcal{O}(\mathcal{Y}_+^\circ)^b$.

- (4) (Anticyclotomic Gross–Zagier formula.) *If $\varepsilon = -1$ and A has potentially \mathfrak{p} -ordinary reduction at all $v|p$, we have*

$$\langle \mathcal{P}_\alpha^+(f^{+,p}), \mathcal{P}_\alpha^-(f^{-,p})^\iota \rangle = \frac{c_E}{2} \cdot d_F L_{p,\alpha}(\sigma_E) \cdot \mathcal{Q}(f^{+,p}, f^{-,p})$$

in $\mathcal{N}_{\mathcal{Y}_+/\mathcal{Y}_+}^*(\mathcal{Y}_+^\circ)^b$.

In parts (3) and (4), we have used the canonical isomorphism $\mathcal{O}_{\Psi_p}(\omega_p) \otimes \mathcal{O}_{\Psi_p}(\omega_p^{-1}) \cong M$. The height pairing of part (4) is (1.4.6).

Remark 1.4.1. Theorem C(4) specialises to $0 = 0$ at any exceptional character $\chi \in \mathcal{Y}_+^{1.c.}$, and in fact by the archimedean Gross–Zagier formula of [YZZ12] it follows that the ‘pair of points’ $\mathcal{P}_\alpha^+(f^{+,p}) \otimes \mathcal{P}_\alpha^-(f^{-,p})^\iota$ itself vanishes there. The leading term of $L_{p,\alpha}$ at exceptional characters is studied in [Dis16].

1.5 Applications

Theorem B has by now standard applications to the p -adic and the classical Birch and Swinnerton-Dyer conjectures; the interested reader will have no difficulty in obtaining them as in [Per87, Dis15]. We obtain in particular one p -divisibility in the classical Birch and Swinnerton-Dyer conjecture for a p -ordinary CM elliptic curve A over a totally real field as in [Dis15, Theorem D] without the spurious assumptions of [Dis15] on the behaviour of p in F . In the rest of this subsection, we describe two other applications.

On the p -adic Birch and Swinnerton-Dyer conjecture in anticyclotomic families. The next theorem, which can be thought of as a case of the p -adic Birch and Swinnerton-Dyer conjecture in anticyclotomic families, combines Theorem C(4) with work of Fouquet [Fou13] to generalise a result of Howard [How05] towards a conjecture of Perrin-Riou [Per87]. We first introduce some notation: let $\Lambda := \mathcal{O}(\mathcal{Y}_+^\circ)^b$, and let the *anticyclotomic height regulator*

$$\mathcal{R} \subset \Lambda \hat{\otimes} \text{Sym}^r \Gamma_F \tag{1.5.1}$$

be the discriminant of (1.4.6) on the Λ -module

$$\mathbf{S}_p(A_E^+, \chi_{\text{univ}}^+, \mathcal{Y}_+^\circ)^b \otimes_\Lambda \mathbf{S}_p(A_E^-, \chi_{\text{univ}}^-, \mathcal{Y}_-^\circ)^{b,\iota},$$

where the integer r in (1.5.1) is the generic rank of the finite-type Λ -module $\mathbf{S}_p(A_E^+, \chi_{\text{univ}}^+, \mathcal{Y}_+^\circ)^b$. Recall that this module is the first cohomology of the Selmer complex $\widetilde{\text{R}}\Gamma_f(E, V_p A^+ \otimes \mathcal{O}(\mathcal{Y}_+^\circ)^b(\chi_{\text{univ}}^+))$ of (1.4.4). Let

$$\widetilde{H}_f^2(E, V_p A^+ \otimes \mathcal{O}(\mathcal{Y})^b(\chi_{\text{univ}}))_{\text{tors}}$$

be the torsion part of the second cohomology group. Its characteristic ideal in Λ can roughly be thought of as interpolating the p -parts of the rational terms (order of the Tate–Shafarevich group, Tamagawa numbers) appearing on the algebraic side of the Birch and Swinnerton-Dyer conjecture for $A(\chi)$.

THEOREM D. *In the situation of Theorem C(4), assume furthermore that:*

- $p \geq 5$;
- $V_p A$ is potentially crystalline as a \mathcal{G}_{F_v} -representation for all $v|p$;
- the character ω is trivial and \mathcal{Y}° is the connected component of $\mathbf{1} \in \mathcal{Y}$;
- the residual representation $\bar{\rho} : \mathcal{G}_F \rightarrow \text{Aut}_{\mathbf{F}_p}(T_p A \otimes \mathbf{F}_p)$ is irreducible (where \mathbf{F}_p is the residue field of M_p), and it remains irreducible when restricted to the Galois group of the Hilbert class field of E ;
- for all $v|p$, the image of $\rho|_{\mathcal{G}_{F,v}}$ is not scalar.

Then

$$\mathbf{S}_p(A_E, \chi_{\text{univ}}, \mathcal{Y}^\circ)^b, \quad \mathbf{S}_p(A_E, \chi_{\text{univ}}^{-1}, \mathcal{Y}^\circ)^{b\iota}$$

both have generic rank 1 over Λ , a non-torsion element of their tensor product over Λ is given by any $\mathcal{P}_\alpha^+(f^{+p}) \otimes \mathcal{P}_\alpha^-(f^{-p})^\iota$ such that $\mathcal{L}(f^{+p}, f^{-p}) \neq 0$, and

$$(\text{d}_F L_{p,\alpha}(\sigma_E)|_{\mathcal{Y}^\circ}) \subset \mathcal{R} \cdot \text{char}_\Lambda \widetilde{H}_f^2(E, V_p A \otimes \Lambda(\chi_{\text{univ}}))_{\text{tors}} \tag{1.5.2}$$

as Λ -submodules of $\Lambda \hat{\otimes} \Gamma_F$.

The ‘potentially crystalline’ assumption for $V_p A$, which is satisfied if A has potentially good reduction at all $v|p$, is imposed in order for $V_p A \otimes \mathcal{O}(\mathcal{Y}^\circ)^b$ to be ‘non-exceptional’ in the sense of [Fou13] (which is more restrictive than ours); the assumption on ω allows us to invoke the results of [CV05, AN10] on the non-vanishing of anticyclotomic Heegner points, and to write $A = A^+ = A^-$, $\mathcal{Y} = \mathcal{Y}_+ = \mathcal{Y}_- = \mathcal{Y}_1$. See [Fou13, Theorem B(ii)] for the exact assumptions needed, which are slightly weaker.

The proof of Theorem D will be given in § 10.3.

Remark 1.5.1. When $F = \mathbf{Q}$, the converse divisibility to (1.5.2) was recently proved by Wan [Wan14] under some assumptions.

Generic non-vanishing of p -adic heights on CM abelian varieties. The non-vanishing of (cyclotomic) p -adic heights is in general, as we have mentioned, a deep conjecture (or a ‘strong suspicion’) of Schneider [Sch85]. The following result provides some new evidence towards it. It is a corollary of Theorem C(4) together with the non-vanishing results for Katz p -adic L -functions of Hida [Hid10], Hsieh [Hsi14], and Burungale [Bur15] (via a factorisation of the p -adic L -function). The result is a special case of a finer one to appear in forthcoming joint work with Burungale. For CM elliptic curves over \mathbf{Q} , it was known as a consequence of different non-vanishing results of Bertrand [Ber83] and Rohrlich [Roh84] (see [AH06, Appendix A, by K. Rubin]).

THEOREM E. *In the situation of Theorem C(4), suppose that A_E has complex multiplication¹⁰ and that $p \nmid 2D_F h_E^-$, where $h_E^- = h_E/h_F$ is the relative class number. Let $\langle \cdot, \cdot \rangle_{\text{cyc}}$ be the pairing deduced from (1.4.6) by the map $\mathcal{N}_{\mathcal{Y}/\mathcal{Y}'}(\mathcal{Y}^\circ)^b \cong \mathcal{O}(\mathcal{Y}^\circ)^b \hat{\otimes} \Gamma_F \rightarrow \mathcal{O}(\mathcal{Y}^\circ)^b \hat{\otimes} \Gamma_{\text{cyc}}$, where $\Gamma_{\text{cyc}} = \Gamma_{\mathbf{Q}}$ viewed as a quotient of Γ_F via the adèlic norm map.*

Then, for any $f^{\pm p}$ such that $\mathcal{Q}(f^{+p}, f^{-p}) \neq 0$ in $\mathcal{O}(\mathcal{Y}^\circ)^b$, we have

$$\langle \mathcal{P}_\alpha^+(f^{+p}), \mathcal{P}_\alpha^-(f^{-p}) \rangle_{\text{cyc}} \neq 0 \quad \text{in } \mathcal{O}(\mathcal{Y}^\circ)^b \hat{\otimes} \Gamma_{\text{cyc}}.$$

1.6 History and related work

We briefly discuss previous work towards our main theorems, and some related works. We will loosely term the ‘classical context’ the following specialisation of the setting of our main results: A is an elliptic curve over \mathbf{Q} with conductor N and *good* ordinary reduction at p ; p is odd; the quadratic imaginary field E has discriminant coprime to N and it satisfies the Heegner condition: all primes dividing N split in E (this implies that \mathbf{B}^∞ is split); the parametrisation $f : J \rightarrow A$ factors through the Jacobian of the modular curve $X_0(N)$; the character χ is unramified everywhere, or unramified away from p .

Ancestors. In the classical context, Theorems A and B were proved by Perrin-Riou [Per87]; intermediate steps towards the present generality were taken in [Dis15, Ma16]. When E/F is split above p , Theorem A can essentially be deduced from a general theorem of Hida [Hid91] (cf. [Wan15, § 7.3]), except for the location of the possible poles. Theorems C(4) and D in the classical context are due to Howard [How05] (in fact, Theorems B and C(4) were first envisioned by Mazur [Maz83] in that context, whereas Perrin-Riou [Per87] had conjectured the equality in (1.5.2)). Theorem C(3) is hardly new and has many antecedents in the literature: see e.g. [Van12] and references therein.

¹⁰ In the strict sense that the algebra $\text{End}^0(A_E)$ of endomorphisms *defined over* E is a CM field.

Relatives. Some analogues of Theorem B were proven in situations which differ from the classical context in directions which are orthogonal to those of the present work: Nekovář [Nek95] and Shnidman [Shn16] dealt with the case of higher weights; Kobayashi [Kob13] dealt with the case of elliptic curves with supersingular reduction.

Friends. We have already mentioned two other fully general Gross–Zagier formulas in the sense of § 1.1, namely the original archimedean one of [YZZ12] generalising [GZ86], and a different p -adic formula proved in [LZZ15] generalising [BDP13]. The panorama of existing formulas of this type is complemented by a handful of results, mostly in the classical context, valid in the presence of an exceptional zero (the case excluded in Theorem B). We refer the reader to [Dis16], where we prove a new such formula for p -adic heights and review other ones due to Bertolini–Darmon. It is to be expected that all of those results should be generalisable to the framework of § 1.1.

Children. Finally, explicit versions of any Gross–Zagier formula in the framework of § 1.1 can be obtained by the explicit computation of the local integrals Q_v . This is carried out in [CST14], where it is applied to the cases of the archimedean Gross–Zagier formula and of the Waldspurger formula; the application to an explicit version of Theorem B can be obtained in exactly the same manner. An explicit version of the anticyclotomic formulas of Theorem C can also be obtained as a consequence: see [Dis16] for a special case.

1.7 Outline of proofs and organisation of the paper

Let us briefly explain the main arguments and at the same time the organisation of the paper.

For the sake of simplicity, the notation used in this introductory discussion slightly differs from that of the body text, and we ignore powers of π , square roots of discriminants, and choices of additive character.

Construction of the p -adic L -function (§ 3). It is crucial for us to have a flexible construction which does not depend on choices of newforms. The starting point is Waldspurger’s [Wal85] Rankin–Selberg integral

$$\frac{(\varphi, I(\phi, \chi'))_{\text{Pet}}}{2L(1, \sigma, \text{ad})/\zeta_F(2)} = \frac{L(1/2, \sigma_E \otimes \chi')}{L(1, \eta)} \prod_v R_v^{\natural}(\varphi_v, \phi_v, \chi'_v), \tag{1.7.1}$$

where $\varphi \in \sigma$, $I(\phi, \chi')$ is a mixed theta-Eisenstein series depending on a choice of an adèlic Schwartz function ϕ , and $R_v^{\natural}(\varphi_v, \phi_v, \chi'_v)$ are normalised local integrals (almost all of which are equal to 1). Then, after dividing both sides by the period $2L(1, \sigma, \text{ad})$, we can:

- interpolate the kernel $\chi' \mapsto I(0, \phi, \chi')$ to a \mathcal{Y}' -family $\mathcal{I}(\phi^{p^\infty}; \chi')$ of p -adic modular forms for *any* choice of the components ϕ^{p^∞} , and a *well-chosen* ϕ_{p^∞} (we will set $\phi_v(x, u)$ to be ‘standard’ at $v|\infty$, and close to a delta function in x at $v|p$);
- interpolate the functional ‘Petersson product with φ ’ to a functional $\ell_{\varphi^p, \alpha}$ on p -adic modular forms, for *any* $\varphi \in \sigma$ which is a ‘ U_v -eigenvector of eigenvalue α_v ’ at the places $v|p$, and is antiholomorphic at infinity;
- interpolate the normalised local integrals $\chi' \mapsto R_v^{\natural}(\varphi_v, \phi_v, \chi'_v)$ to functions $\mathcal{R}_v^{\natural}(\varphi_v, \phi_v)$ for all $v \nmid p^\infty$ and any φ_v, ϕ_v .

To conclude, we cover \mathcal{Y}' by finitely many open subsets \mathcal{U}_i ; for each i , we choose appropriate φ^p , ϕ^p and we define¹¹

$$L_{p,\alpha}(\sigma_E)|_{\mathcal{U}_i} = \frac{\ell_{\varphi^p,\alpha}(\mathcal{I}(\varphi^{p\infty}))}{\prod_{v \nmid p\infty} \mathcal{R}_v^{\mathfrak{a}}(\varphi_v, \phi_v)}.$$

The explicit computation of the local integrals at p (in the Appendix) and at infinity yields the interpolation factor.

Proof of the Gross–Zagier formula and its anticyclotomic version. We outline the main arguments of our proof, with an emphasis on the reduction steps.

Multiplicity one. We borrow or adapt many ideas (and calculations) from [YZZ12], in particular the systematic use of the multiplicity-one principle of § 1.1. As both sides of the formula are functionals in the same one-dimensional vector space, it is enough to prove the result for one pair f_1, f_2 with $Q(f_1, f_2, \chi) \neq 0$; finding such $f_1 \otimes f_2$ is a local problem. It is equivalent to choosing functions $\varphi_v \otimes \phi'_v = \theta^{-1}(f_1 \otimes f_2)$ as just above, by the Shimizu lift θ realising the Jacquet–Langlands correspondence (§ 5.1). For $v|p$, we thus have an explicit choice of such, corresponding to the one made above.¹² For $v \nmid p$, we can introduce several restrictions on (φ_v, ϕ_v) as in [YZZ12], with the effect of simplifying many calculations of local heights (§ 6).

Arithmetic theta lifting and kernel identity (§ 5). In [YZZ12], the authors introduce an arithmetic–geometric analogue of the Shimizu lift, by means of which they are able to write also the Heegner-points side of their formula as a Petersson product with φ of a certain geometric kernel. We can adapt without difficulty their results to reduce our formula to the assertion¹³ that

$$d_F \mathcal{I}(\phi^{p\infty}; \chi) - 2L_{(p)}(1, \eta) \tilde{Z}(\phi^\infty, \chi)$$

is killed by the p -adic Petersson product $\ell_{\varphi^p,\alpha}$. Here $\tilde{Z}(\phi^\infty, \chi)$ is a modular form depending on ϕ encoding the height pairings of CM points on Shimura curves and their Hecke translates; it generalises the classical generating series $\sum_m \langle \iota_\xi(P)[\chi], T(m)\iota_\xi(P) \rangle \mathbf{q}^m$.

Decomposition and comparison (§§ 7–8). Both terms in the kernel identity are sums of local terms indexed by the finite places of F . For $v \nmid p$, we compute both sides and show that the difference essentially coincides with the one computed in [YZZ12]: it is either zero or, at bad places, a modular form orthogonal to all forms in σ . In fact, we can show this only for a certain restricted set of q -expansion coefficients; as the global kernels are p -adic modular forms, this will suffice by a simple approximation argument (Lemma 2.1.2).

p -adic Arakelov theory or analytic continuation. The argument just sketched relies on calculations of arithmetic intersections of CM points; this in general does not suffice, as we need to consider the contribution of the Hodge classes in the generating series too. It will turn out that such contribution vanishes; two approaches can be followed to show this. The first one, in analogy with [Zha01a, YZZ12] and already used in a simpler context in [Dis15], is to make use of Besser’s p -adic Arakelov theory¹⁴ [Bes05] in order to separate such contribution.

¹¹ This is the point which possibly produces poles.

¹² The notation ϕ' refers to the application to ϕ of a local operator at $v|p$ appearing in the interpolation of the Petersson product.

¹³ Together with a comparison of local terms at p described below.

¹⁴ Recall that an Arakelov theory is an arithmetic intersection theory which allows us to pair cycles of any degree, recovering the height pairing for cycles of degree zero.

We will follow an alternative approach (see Proposition 10.2.3), which exploits the generality of our context and the existence of extra variables in the p -adic world. Once having constructed the Heegner–theta element \mathcal{P}^\pm , the anticyclotomic formula of Theorem C(4) is essentially a corollary of Theorem B for all finite-order characters χ : we only need to check the compatibility $Q_v(f_{\alpha,v}^+, f_{\alpha,v}^-, \chi) = \zeta_{F,v}(2)^{-1} \cdot Z_v(\chi_v)$ for all $v|p$ by explicit computation. Conversely, thanks to the multiplicity-one result, it is also true that Theorem B for any χ is obtained as a corollary of Theorem C(4) by specialisation. We make use of both of these observations: we first prove Theorem B for all but finitely many finite-order characters χ ; this suffices to deduce Theorem C(4) by an analytic continuation argument, which finally yields Theorem B for the remaining characters χ as well. The initially excluded characters are those (such as the trivial character when it is contemplated) for which the contribution of the Hodge classes is not already annihilated by χ -averaging; for all other characters the Arakelov-theoretic arguments just mentioned are then unnecessary.

Annihilation of p -adic heights (§ 9). We are left to deal with the contribution of the places $v|p$. We can show quite easily that this is zero for the analytic kernel. As in the original work of Perrin-Riou [Per87], the vanishing of the contribution of the geometric kernel is the heart of the argument. We establish it via an elaboration of a method of Nekovář [Nek95] and Shnidman [Shn16]. The key new ingredient in adapting it to our semistable case is a simple integrality criterion for local heights in terms of intersections, introduced in § 4.3, after a review of the theory of heights.

Local toric period. Finally, in the Appendix we compute the local toric period $Q(\theta(\varphi_v \otimes \phi'_v), \chi_v)$ for $v|p$ and compare it to the interpolation factor of the p -adic L -function. Both are highly ramified local integrals, and they turn out to differ by the multiplicative constant $L(1, \eta_v)$; this completes the comparison between the kernel identity and Theorem B.

1.8 Notation

We largely follow the notation and conventions of [YZZ12, § 1.6].

L -functions. In the rest of the paper (and *unlike* in the Introduction, where we adhere to the more standard convention), *all complex L - and zeta functions are complete including the Γ -factors at the infinite places.* (This is to facilitate referring to the results and calculations of [YZZ12], where this convention is adopted.)

Fields and adèles. The fields E and F will be as fixed in the Introduction unless otherwise noted. The adèle ring of F will be denoted \mathbf{A}_F or simply \mathbf{A} ; it contains the ring \mathbf{A}^∞ of finite adèles. We let D_F and D_E be the absolute discriminants of F and E , respectively. We also choose an idèle $d \in \mathbf{A}^{\infty, \times}$ generating the different of F/\mathbf{Q} , and an idèle $D \in \mathbf{A}^{\infty, \times}$ generating the relative discriminant of E/F .

We use standard notation to restrict adèlic objects (groups, L -functions, and so on) away from a finite set of places S , e.g. $\mathbf{A}^S := \prod'_{v \notin S} F_v$, whereas $F_S := \prod_{v \in S} F_v$. When S is the set of places above p (respectively ∞), we use this notation with ‘ S ’ replaced by ‘ p ’ (respectively ‘ ∞ ’).

We denote by $F_\infty^+ \subset F_\infty$ the group of $(x_\tau)_{\tau| \infty}$ with $x_\tau > 0$ for all τ , and we let $\mathbf{A}_+^\times := \mathbf{A}^{\infty, \times} F_\infty^+$, $F_+^\times := F^\times \cap F_\infty^+$.

For a non-archimedean prime v of a number field F , we denote by $q_{F,v}$ the cardinality of the residue field and by ϖ_v a uniformiser.

Subgroups of \mathbf{GL}_2 . We consider \mathbf{GL}_2 as an algebraic group over F . We denote by P , respectively P^1 , the subgroup of \mathbf{GL}_2 , respectively \mathbf{SL}_2 , consisting of upper-triangular matrices;

by $A \subset P \subset \mathbf{GL}_2$ the diagonal torus; and by $N \subset P \subset \mathbf{GL}_2$ the unipotent radical of P . We let $n(x) := \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}$ and

$$w := \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}.$$

Quadratic torus. We let $T := \text{Res}_{E/F} \mathbf{G}_m$; the embedding $T(\mathbf{A}^\infty) \subset \mathbf{B}^\times$ is fixed. We let $Z := \mathbf{G}_{m,F}$, and view it both as a subgroup of T and as the centre of \mathbf{GL}_2 .

Automorphic quotients. If G is a reductive group over the totally real field F , we denote

$$[G] := G(F) \backslash G(\mathbf{A}) / Z(\mathbf{A}).$$

Measures. We choose local and global Haar measures as in [YZZ12]. In particular, we have

$$\text{vol}(\mathbf{GL}_2(\mathcal{O}_{F,v})) = |d_v^2 \zeta_{F,v}(2)|^{-1}$$

for all non-archimedean v .

We denote by dt the local and global measures on T/Z of [YZZ12], which give $\text{vol}([T], dt) = 2L(1, \eta)$. The global measure

$$d^\circ t := |D_F|^{1/2} |D_{E/F}|^{1/2} dt$$

gives $\text{vol}([T], d^\circ t) \in \mathbf{Q}^\times$.

Regularised averages and integration. We borrow some notation from [YZZ12, § 1.6.7]. If G is a topological group with a left Haar measure dg with finite volume, we define

$$\int_G f dg := \frac{1}{\text{vol}(G)} \int_G f(g) dg.$$

(This reduces to the usual average when G is a finite group.)

If F is a totally real field and f is a function on $F^\times \backslash \mathbf{A}^\times$ invariant under F_∞^\times , we denote

$$\int_{\mathbf{A}^\times} f(z) dz := \int_{F^\times \backslash \mathbf{A}^\times / F_\tau^\times} f(z) dz,$$

where τ is any archimedean place of F . If f is further invariant under a compact open subgroup U , this reduces to the average over $F^\times \backslash \mathbf{A}^\times / F_\infty^\times U$.

Finally, let G be a reductive group over F with an embedding of $\mathbf{G}_{m/F}$ into the centre G , and assume that dg is a left Haar measure giving finite volume to $[G] = G(F) \backslash G(\mathbf{A}) / Z(\mathbf{A})$. Let f be a function on $G(F) \backslash G(\mathbf{A}) / Z(F_\infty)$; then we define

$$\int_{[G]}^* f(g) dg := \int_{[G]} \int_{Z(\mathbf{A})} f(zg) dz dg$$

and

$$\int_{[G]} f(g) dg := \frac{1}{\text{vol}([G])} \int_{[G]} \int_{Z(\mathbf{A})} f(zg) dz dg.$$

Note in particular that for functions which factor through a compact quotient of $G(F) \backslash G(\mathbf{A})$ and are locally constant there, the regularised integration reduces to a finite sum and, when using \mathbf{Q} -valued measures such as the measure $d^\circ t$ on T , it makes sense for functions taking p -adic values as well.

Multi-indices. If S is a set and $r \in \mathbf{Z}^S$, $p \in G^S$ for some group G , we often write $p^r := \prod_{v \in S} p_v^{r_v}$. This will typically be applied in the following situation: $S = S_p$ is the set of places of F above p , G is the (semi)group of ideals of \mathcal{O}_F , and p_v is the ideal corresponding to v .

Functions of p -adic characters. When $\mathcal{Y}^?$ is one of the rigid spaces introduced above and $G(A) \in \mathcal{O}(\mathcal{Y}^?)$ is a function on $\mathcal{Y}^?$ depending on other ‘parameters’ A (e.g. a p -adic L -function), we write $G(A; \chi)$ for the evaluation $G(A)(\chi)$.

2. p -adic modular forms

2.1 Modular forms and their q -expansions

Let $K \subset \mathbf{GL}_2(\widehat{\mathcal{O}}_F)$ be an open compact subgroup. Recall that a Hilbert automorphic form of level K is a smooth function of moderate growth

$$\varphi : \mathbf{GL}_2(F) \backslash \mathbf{GL}_2(\mathbf{A}) / K \rightarrow \mathbf{C}.$$

Let $k \in \mathbf{Z}^{\text{Hom}(F, \mathbf{R})}$. Then an automorphic form is said to be of weight k if it satisfies

$$\varphi(gr_\theta) = \varphi(g)\psi_\infty(k \cdot \theta)$$

for all $r_\theta = \left(\begin{pmatrix} \cos 2\pi\theta_v & \sin 2\pi\theta_v \\ -\sin 2\pi\theta_v & \cos 2\pi\theta_v \end{pmatrix} \right)_{v|\infty} \in \mathbf{SO}_2(F_\infty)$. It is said to be holomorphic of weight k if for all $g \in \mathbf{GL}_2(\mathbf{A}^\infty)$, the function of $z_\infty = (x_v + iy_v)_{v|\infty} \in \mathfrak{h}^{\text{Hom}(F, \mathbf{R})}$,

$$z_\infty \mapsto |y_\infty|_\infty^{-k/2} \varphi \left(g \begin{pmatrix} y_\infty & x_\infty \\ 1 & 1 \end{pmatrix} \right),$$

is holomorphic. Holomorphic Hilbert automorphic forms will be simply called *modular* forms.

Let $\omega : F^\times \backslash \mathbf{A}^\times \rightarrow \mathbf{C}^\times$ be a finite-order character. Then φ is said to be of character ω if it satisfies $\varphi(zg) = \omega(z)\varphi(g)$ for all $z \in Z(\mathbf{A}) \cong \mathbf{A}^\times$. We denote by $M_k(K, \mathbf{C})$ the space of modular forms of level K and weight k , and by $S_k(K, \mathbf{C})$ its subspace of cuspforms. We further denote by $M_k(K, \omega, \mathbf{C})$, $S_k(K, \omega, \mathbf{C})$ the subspaces of forms of character ω . We identify a scalar weight $k \in \mathbf{Z}_{\geq 0}$ with the corresponding parallel weight $(k, \dots, k) \in \mathbf{Z}_{\geq 0}^{\text{Hom}(F, \mathbf{R})}$.

For v a finite place of F and N an ideal of $\mathcal{O}_{F,v}$, we define subgroups of $\mathbf{GL}_2(\mathcal{O}_{F,v})$ by

$$\begin{aligned} K_0(N)_v &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid c \equiv 0 \pmod{N} \right\}, \\ K_1(N)_v &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid c, d - 1 \equiv 0 \pmod{N} \right\}, \\ K^1(N)_v &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid c, a - 1 \equiv 0 \pmod{N} \right\}, \\ K_1^1(N)_v &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid c, a - 1, d - 1 \equiv 0 \pmod{N} \right\}, \\ K(N)_v &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid b, c, a - 1, d - 1 \equiv 0 \pmod{N} \right\}. \end{aligned}$$

If N is an ideal of \mathcal{O}_F and $* \in \{0, 1, \frac{1}{2}, \emptyset\}$, we define subgroups $K^*(N)$ of $\mathbf{GL}_2(\widehat{\mathcal{O}}_F)$ by $K^*(N) = \prod_v K^*(N)_v$. If p is a rational prime and $r \in \mathbf{Z}_{\geq 0}^{\{v|p\}}$, we further define $K^*(p^r)_p = \prod_v K^*(\varpi_v^{r_v})_v \subset \mathbf{GL}_2(\mathcal{O}_{F,p})$.

Fix a non-trivial character $\psi : \mathbf{A}/F \rightarrow \mathbf{C}^\times$. Any automorphic form φ admits a Fourier–Whittaker expansion $\varphi(g) = \sum_{a \in F} W_a(g)$, where $W_a(g) = W_{\varphi, \psi, a}(g)$ satisfies $W_a(n(x)g) = \psi(ax)W_a(g)$ for all $x \in \mathbf{A}$. If φ is holomorphic of weight k , we can further write $W_a(g) = W_{a^\infty}(g^\infty)W_{a, \infty}(g_\infty)$ with $W_{a, \infty}(g) = \prod_{v|\infty} W_{a, v}(g)$, where $W_{a, v} = W_{a, v}^{(k_v)}$ is the *standard holomorphic Whittaker function* of weight k given by (suppressing the subscripts and using the Iwasawa decomposition)

$$W_a^{(k)}\left(\begin{pmatrix} z & \\ & z \end{pmatrix} \begin{pmatrix} y & x \\ & 1 \end{pmatrix} r_\theta\right) = \begin{cases} |y|^{k/2} \psi(a(x + iy)) \psi(k\theta) \mathbf{1}_{\mathbf{R}_+}(ay) & \text{if } a \neq 0, \\ |y|^{k/2} \psi(k\theta) \mathbf{1}_{\mathbf{R}_+}(y) & \text{if } a = 0. \end{cases} \tag{2.1.1}$$

(Similarly, we have a description in terms of the *standard antiholomorphic Whittaker function*

$$W_a^{(-k)}\left(\begin{pmatrix} z & \\ & z \end{pmatrix} \begin{pmatrix} y & x \\ & 1 \end{pmatrix} r_\theta\right) = \begin{cases} |y|^{k/2} \psi(a(x + iy)) \psi(-k\theta) \mathbf{1}_{\mathbf{R}_+}(-ay) & \text{if } a \neq 0, \\ |y|^{k/2} \psi(-k\theta) \mathbf{1}_{\mathbf{R}_+}(-y) & \text{if } a = 0, \end{cases} \tag{2.1.2}$$

for antiholomorphic forms of weight $-k < 0$.)

In this case we have an expansion

$$\varphi\left(\begin{pmatrix} y & x \\ & 1 \end{pmatrix}\right) = |y|_\infty^{k/2} \sum_{a \in F_{\geq 0}} W_a^\infty\left(\begin{pmatrix} y^\infty & \\ & 1 \end{pmatrix}\right) \psi_\infty(iay_\infty) \psi(ax)$$

for all $y \in \mathbf{A}_+^\times, x \in \mathbf{A}$; here $F_{\geq 0}$ denotes the set of $a \in F$ satisfying $\tau(a) \geq 0$ for all $\tau : F \hookrightarrow \mathbf{R}$.

For a field L , let the *space of formal q -expansions* $C^\infty(\mathbf{A}^{\infty, \times}, L)[[\mathbf{q}^{F_{\geq 0}}]]^\circ$ be the set of those formal sums $W = \sum_{a \in F_{\geq 0}} W_a \mathbf{q}^a$ with coefficients $W_a \in C^\infty(\mathbf{A}^{\infty, \times}, L)$ such that, for some compact subset $A_W \subset \mathbf{A}^\infty$, we have $W_a(y) = 0$ unless $ay \in A_W$.

Let φ be a holomorphic automorphic form. The expression

$$\mathfrak{q}\varphi(y) := \sum_{a \in F} W_{\varphi, a}^\infty\left(\begin{pmatrix} y & \\ & 1 \end{pmatrix}\right) \mathbf{q}^a, \quad y \in \mathbf{A}^{\infty, \times} \tag{2.1.3}$$

belongs to $C^\infty(\mathbf{A}^{\infty, \times}, \mathbf{C})[[\mathbf{q}^{F_{\geq 0}}]]^\circ$ and it is called the *formal q -expansion* of φ . The space of formal q -expansions is an algebra in the obvious way, compatibly with the algebra structure on automorphic forms.

PROPOSITION 2.1.1 (*q -expansion principle*). *Let $K \subset \mathbf{GL}_2(\widehat{\mathcal{O}}_F)$ be an open compact subgroup and let $k \in \mathbf{Z}_{\geq 0}^{\text{Hom}(F, \mathbf{R})}$. The q -expansion map defined by (2.1.3)*

$$\begin{aligned} M_k(K, \mathbf{C}) &\rightarrow C^\infty(\mathbf{A}^{\infty, \times}, \mathbf{C})[[\mathbf{q}^{F_{\geq 0}}]]^\circ \\ \varphi &\mapsto \mathfrak{q}\varphi \end{aligned}$$

is injective.

We say that a formal q -expansion is *modular* if it belongs to the image of the q -expansion map.

Proof. This is (a weak form) of the q -expansion principle of [Rap78, Théorème 6.7(i)]. In fact, our modular forms φ are identified with tuples $(\varphi_c)_{c \in \text{Cl}(F)^+}$ of Hilbert modular forms in the sense of [Rap78]. Then the non-vanishing of $\mathfrak{q}\varphi$ for $\varphi \neq 0$ is obtained by applying the result of [Rap78] to each φ_c . See [Rap78, Lemme 6.12] for the comparison between various notions of Hilbert modular forms used there. \square

The spaces of formal q -expansions introduced so far will often be convenient for us in terms of notation, but they are redundant: if $k \in \mathbf{Z}_{\geq 0}$, $\varphi \in M_k(K, \mathbf{C})$, we have $W_a((\begin{smallmatrix} y & \\ & 1 \end{smallmatrix})) = W_1((\begin{smallmatrix} ay & \\ & 1 \end{smallmatrix}))$ for all $a \in F^\times$, $y \in \mathbf{A}_+^\times$. Moreover, if $K \subset K(N)$, then $|y^\infty|^{-k/2}W_0((\begin{smallmatrix} y^\infty & \\ & 1 \end{smallmatrix}))$ and $|y^\infty|^{-k/2}W_1^\infty((\begin{smallmatrix} y^\infty & \\ & 1 \end{smallmatrix}))$ are further invariant under the action of $U_F(N) = \{u \in \widehat{\mathcal{O}}_F^\times \mid u \equiv 1 \pmod N\}$ by multiplication on y (see [Hid91, Theorem 1.1]). We term *reducible of weight k* those formal q -expansions satisfying these conditions for some N .

Define the space of reduced q -expansions (of level N) with values in a ring A to be

$$M'(K(N), L) := C(\mathbf{A}^{\infty, \times} / F_+^\times U_F(N), L) \times L^{\mathbf{A}^{\infty, \times} / U_F(N)},$$

if K is any compact open subgroup, we define $M'(K, L) := M'(K(N), L)$ for the largest subgroup $K(N) \subset K$. Let $M'(L) := \bigcup_N M'(K(N), L)$ and $M'(K^p, L) := \bigcup_r M'(K^p K_p(p^r), L)$.

Given a reducible q -expansion W of weight k , we can then define the associated *reduced q -expansion* $(W_0^\natural(y), (W_a^\natural)_{a \in \mathbf{A}^{\infty, \times}}) \in M'(L)$ by

$$W_0^\natural(y) := |y|^{-k/2}W_0^\infty((\begin{smallmatrix} y & \\ & 1 \end{smallmatrix})), \quad W_a^\natural := |a|^{-k/2}W_1^\infty((\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}))).$$

If $A \subset \mathbf{C}$ is a subring, we denote by $M_k(K, A) \subset M_k(K, \mathbf{C})$, $S_k(K, A) \subset S_k(K, \mathbf{C})$ the subspaces of forms with reduced q -expansion coefficients in A . If A is any \mathbf{Q} -algebra, we let $M_k(K, A) = M_k(K, \mathbf{Q}) \otimes A$, $S_k(K, A) = S_k(K, \mathbf{Q}) \otimes A$. Then it makes sense to talk about the q -expansion of an element of those spaces.

If φ is a modular form, we still denote by ${}^{\mathfrak{q}}\varphi$ its reduced q -expansion; in cases where the distinction is significant, the precise meaning of the expression ${}^{\mathfrak{q}}\varphi$ will be clear from its context.

p -adic modular forms. Let $N \subset \mathcal{O}_F$ be a non-zero ideal prime to p , $U_F(Np^\infty) = \bigcap_{r \geq 0} U_F(Np^r)$. We endow the quotient $\mathbf{A}^{\infty, \times} / F_+^\times U_F(Np^\infty)$ with the profinite topology. Let L be a complete Banach ring with norm $|\cdot|$. We define the space of p -adic reduced q -expansions with values in L to be

$$M'(K^p(N), L) := C(\mathbf{A}^{\infty, \times} / F_+^\times U_F(Np^\infty), L) \times L^{\mathbf{A}^{\infty, \times} / U_F(Np^\infty)}.$$

If $K^p \subset \mathbf{GL}_2(\mathbf{A}^{p^\infty})$ is a compact open subgroup in general, we define $\mathbf{M}'(K^p, L) := \mathbf{M}'(K^p(N), L)$ for the largest subgroup $K^p(N) \subset K^p$.

Define a ‘norm’ (possibly taking the value ∞) $\|\cdot\|$ on $M'(K^p(N), L)$ by

$$\|(W_0^\natural, (W_a^\natural)_{a \in \mathbf{A}^{\infty, \times} / U_F(Np^\infty)})\| := \sup_{(y, a)} \{|W_0^\natural(y)|, |W_a^\natural|\}. \tag{2.1.4}$$

It induces a ‘norm’ on the (isomorphic) space of reducible q -expansions with values in L . Let $M'(K^p, L)^\circ \subset M'(K^p, L)$ be the set of elements on which $\|\cdot\|$ is finite. We define the Banach space of *p -adic reduced q -expansions*

$$\mathbf{M}'(K^p, L)$$

to be the completion of $M'(K^p, L)^\circ$ with respect to the norm $\|\cdot\|$. We denote by $\mathbf{S}'(L) \subset \mathbf{M}'(L)$ the space of reduced q -expansions with vanishing constant coefficients; when there is no risk of confusion we shall omit L from the notation.

Suppose that L is a field extension of \mathbf{Q}_p . The space of p -adic modular forms of tame level $K^p \subset \mathbf{GL}_2(\widehat{\mathcal{O}}_F^p)$ with coefficients in L , denoted by $\mathbf{M}(K^p, L)$, is defined to be the closure in $\mathbf{M}'(K^p, L)$ of the subspace generated by the reduced q -expansions of elements of $M_2(K^p K^1(p^\infty)_p, L) = \bigcup_{r \geq 0} M_2(K^p K^1(p^r)_p, L)$. Tame levels and coefficient rings will be omitted

from the notation when they are understood from context. We denote by $\mathbf{S} := \mathbf{M} \cap \mathbf{S}'$ the space of p -adic modular cuspforms.

Approximation. The q -expansion principle of Proposition 2.1.1 is complemented by the following (obvious) result to provide a p -adic replacement for the approximation argument in [YZZ12].

LEMMA 2.1.2 (Approximation). *Let S be a finite set of finite places of F , not containing any place v above p . Let φ be a p -adic modular cuspform all of whose reduced q -expansion coefficients $W_{a,\varphi}^{\natural}$ are zero for all $a \in F^{\times} \mathbf{A}^{S_{\infty,\times}}$. Then $\varphi = 0$.*

Proof. The form φ has some tame level K^p ; then its coefficients are invariant under the action of some compact open $U_F^p \subset \mathbf{A}^{\infty,\times}$ on the indices a . Since $F^{\times} \mathbf{A}^{S_{\infty,\times}} U_F^p = \mathbf{A}^{\infty,\times}$, the lemma follows. \square

Let $\overline{\mathbf{S}}'$ be the quotient of \mathbf{S}' by the subspace of reduced q -expansions which are zero at all $a \in F^{\times} \mathbf{A}^{S_{\infty,\times}}$, and let $\overline{\mathbf{S}}$ be the image of \mathbf{S} in $\overline{\mathbf{S}}'$ (these notions depend on the set S , which in our uses will be clear from the context). Then the lemma says that in

$$\mathbf{S} \rightarrow \overline{\mathbf{S}} \hookrightarrow \overline{\mathbf{S}}', \tag{2.1.5}$$

the first map is an isomorphism and the composition is an injection. We use the notation $\overline{\mathbf{S}}_S(K^p)$, $\overline{\mathbf{S}}'_S(K^p)$ if we want to specify the set of places S and the tame level K^p .

Families. Let $\mathcal{Y}^?$ be one of the rigid spaces defined in the Introduction, and $K^p \subset \mathbf{GL}_2(\mathbf{A}^{p^\infty})$ be a compact open subgroup.

DEFINITION 2.1.3. A $\mathcal{Y}^?$ -family of q -expansions of modular forms of tame level K^p is a reduced q -expansion $\underline{\varphi}$ with values in $\mathcal{O}(\mathcal{Y}^?)$, whose coefficients are algebraic on $\mathcal{Y}^{?1.c.}$, and such that for every point $\chi \in \mathcal{Y}^{?1.c.}$, $\varphi(\chi)$ is the reduced q -expansion of a classical modular form $\varphi(\chi)$ of level $K^p K^1(p^\infty)_p$ with coefficients in $M(\chi)$. We say that $\underline{\varphi}$ is *bounded* if it is bounded for the norm (2.1.4).

Twisted modular forms. It will be convenient to consider the following relaxation of the notion of modular forms.

DEFINITION 2.1.4. A *twisted* Hilbert automorphic form of weight $k \in \mathbf{Z}_{\geq 0}^{\text{Hom}(F,\mathbf{R})}$ and level $K \subset \mathbf{GL}_2(\widehat{\mathcal{O}}_F)$ is a smooth function

$$\tilde{\varphi} : \mathbf{GL}_2(\mathbf{A})/K \times \mathbf{A}^{\times} \rightarrow \mathbf{C}$$

satisfying:

- for all $\gamma \in \mathbf{GL}_2(F)$, $r_\theta \in \mathbf{SO}_2(F_\infty)$,

$$\tilde{\varphi}(\gamma g r_\theta, u) = \tilde{\varphi}(g, \det(\gamma)^{-1} u) \psi_\infty(k \cdot \theta);$$

- $\tilde{\varphi}$ is of moderate growth in the variable $g \in \mathbf{GL}_2(\mathbf{A})$ and, for all $g \in \mathbf{GL}_2(\mathbf{A})$, $u = u_\infty u^\infty \mapsto \tilde{\varphi}(g, u)$ is the product of a function of the variable u^∞ and of the function $\mathbf{1}_{F_\infty^+}(u_\infty)$ of the variable u_∞ ;
- there exists a compact open subgroup $U_F \subset \mathbf{A}^{\infty,\times}$ such that for all g , $\tilde{\varphi}(g, \cdot)$ is invariant under U_F ;

- for each $g \in \mathbf{GL}_2(\mathbf{A})$, there is an open compact subset $K_g \subset \mathbf{A}^{\infty, \times}$ such that $\tilde{\varphi}(g, \cdot)$ is supported in $K_g F_{\infty}^+$.

Let $\omega : F^{\times} \backslash \mathbf{A}^{\times} \rightarrow \mathbf{C}^{\times}$ be a finite-order character. We say that a twisted automorphic form $\tilde{\varphi}$ has central character ω if it satisfies

$$\tilde{\varphi}(zg, u) = \omega(z)\tilde{\varphi}(g, z^{-2}u)$$

for all $z \in Z(\mathbf{A}) \cong \mathbf{A}^{\times}$. We say that it is holomorphic (of weight k) or simply a *twisted modular form* if $z_{\infty} \mapsto |y_{\infty}|_{\infty}^{-k/2} \tilde{\varphi}(g \begin{pmatrix} y_{\infty} & x_{\infty} \\ 1 & 1 \end{pmatrix}, u)$ is holomorphic in $z_{\infty} = (x_v + iy_v)_{v|\infty} \in \mathfrak{h}^{\text{Hom}(F, \mathbf{R})}$ for all $u \in \mathbf{A}^{\times}$.

We let $M_k^{\text{tw}}(K, \mathbf{C})$ denote the space of twisted modular forms of weight k and $M_k^{\text{tw}}(K, \omega, \mathbf{C})$ its subspace of forms with central character ω . We omit the K from the notation if we do not wish to specify the level.

If $\tilde{\varphi}$ is a twisted modular form, then, for each g, u , the function $x \mapsto \varphi(n(x)g, u)$ descends to $F \backslash \mathbf{A}$ and therefore it admits a Fourier–Whittaker expansion in the usual way. To the restriction of $\tilde{\varphi}$ to $\mathbf{GL}_2(\mathbf{A}) \times F^{\times}$ we then attach a *twisted formal q -expansion*

$$\sum_{a \in F} |y|_{\infty}^{k/2} W_a^{\infty} \left(\begin{pmatrix} y & \\ & 1 \end{pmatrix}, u \right) \mathbf{q}^a \in C^{\infty}(\mathbf{A}^{\infty, \times} \times F^{\times}, \mathbf{C})[[\mathbf{q}^{F_{\geq 0}}]]^{\circ}$$

such that

$$\tilde{\varphi} \left(\begin{pmatrix} y & x \\ & 1 \end{pmatrix}, u \right) = |y|_{\infty}^{k/2} \sum_{a \in F_{>0}} W_a^{\infty} \left(\begin{pmatrix} y & \\ & 1 \end{pmatrix}, u \right) \psi_{\infty}(iay_{\infty}) \psi(ax)$$

for all $y \in \mathbf{A}_+^{\times}, x \in \mathbf{A}, u \in F^{\times}$. Here the space $C^{\infty}(\mathbf{A}^{\infty, \times} \times F^{\times}, \mathbf{C})[[\mathbf{q}^{F_{\geq 0}}]]^{\circ}$ consists of q -expansions W whose coefficients $W_a(y, u)$ vanish for ay outside of some compact open subset $A_W \subset \mathbf{A}^{\infty}$.

Let $\tilde{\varphi}$ be a twisted modular form, let $U_F \subset \mathbf{A}^{\infty, \times}$ be a compact open subgroup satisfying the condition of the previous definition, and let $\mu_{U_F} = F^{\times} \cap U_F$. Then the sum

$$\varphi(g) := \sum_{u \in \mu_{U_F}^2 \backslash F^{\times}} \tilde{\varphi}(g, u)$$

is finite for each g (if $K_g \subset \mathbf{A}^{\infty, \times}$ is a compact subset such that $K_g F_{\infty}^{\times}$ contains the support of $\tilde{\varphi}(g, \cdot)$, the sum is supported on $\mu_{U_F}^2 \backslash (F^{\times} \cap K_g)$, which is commensurable with the finite group $\mu_{U_F}^2 \backslash \mathcal{O}_F^{\times}$). It defines a modular form in the usual sense, with formal q -expansion

$$\mathbf{q}\varphi(y) = \sum_{u \in \mu_{U_F}^2 \backslash F^{\times}} \mathbf{q}\tilde{\varphi}(y, u).$$

One can, similarly to the above, define a norm on the space of twisted formal q -expansion coefficients of a fixed parallel weight k with values in a Banach ring L , namely $\|W\| := \sup_{(a, y_{\infty}, u)} |y_{\infty}|^{-k/2} |W_a^{\infty}(y, u)|$. The p -adic completion $\mathbf{M}_k^{\text{tw}}(K^p, L)$ of the subspace of q -expansions of twisted modular forms (of some tame level K^p) is called the space of p -adic twisted modular forms (of tame level K^p). Finally, there is a notion of a \mathscr{Y} -family of q -expansions of twisted modular forms.

2.2 Hecke algebra and operators U_v

Let L be a field and let

$$\mathcal{H}(L) = C_c^\infty(\mathbf{GL}_2(\mathbf{A}^\infty), L)$$

be the Hecke algebra of smooth compactly supported functions with the convolution operation (denoted by $*$) and, for any finite set of non-archimedean places S , let $\mathcal{H}^S(L) = C_c^\infty(\mathbf{GL}_2(\mathbf{A}^{S\infty}), L)$, $\mathcal{H}_S(L) = C_c^\infty(\mathbf{GL}_2(F_S), L)$. When $L = \mathbf{Q}$ it will be omitted from the notation.

The group $\mathbf{GL}_2(\mathbf{A}^\infty)$ has a natural left action on automorphic forms by right multiplication. This action is extended to elements $f \in \mathcal{H} \otimes \mathbf{C}$ by

$$T(f)\varphi(g) = \int_{\mathbf{GL}_2(\mathbf{A}^\infty)} f(h)\varphi(gh) dh,$$

where $dh = \prod dh_v$ with dh_v the Haar measure on $\mathbf{GL}_2(F_v)$ assigning volume 1 to $\mathbf{GL}_2(\mathcal{O}_{F,v})$. If $K \subset \mathbf{GL}_2(\mathbf{A}^\infty)$ is a compact open subgroup, we define $e_K = T(\text{vol}(K)^{-1}\mathbf{1}_K) \in \mathcal{H}$. It acts as a projector on K -invariant forms. If $g \in \mathbf{GL}_2(\mathbf{A}^\infty)$ and $K, K' \subset \mathbf{GL}_2(\widehat{\mathcal{O}}_F)$ are open compact subgroups, we define the operator $[KgK'] := T(\mathbf{1}_{KgK'})$.

By the strong multiplicity-one theorem, for each level K , each M -rational automorphic representation σ which is a discrete series of weight 2 at all infinite places, and each finite set of non-archimedean places S such that K is maximal away from S , there are spherical (that is, $K(1)^S$ -bi-invariant) elements $T(\sigma) \in \mathcal{H}^S(M)$ whose action on $M_2(K, M)$ is given by the idempotent projection e_σ onto $\sigma^K \subset M_2(K, M)$.

On the space $\mathbf{M}(K^p, L)$ of p -adic modular forms, with $K^p \supset K(N)^p$, there is a continuous action of $Z(Np^\infty) := \mathbf{A}^{\infty, \times} / \overline{F^\times U_F(Np^\infty)}$, extending the central action $z.\varphi(g) = \varphi(gz)$ on modular forms. For a continuous character $\omega : Z(Np^\infty) \rightarrow L^\times$, we denote by $\mathbf{M}(K^p, \omega, L)$ the set of p -adic modular forms φ satisfying $z.\varphi = \omega(z)\varphi$, and by $\mathbf{S}(K^p, \omega, L)$ its subspace of cuspidal forms. If ω is the restriction of a finite-order character of $Z(\mathbf{A}^\infty)/U_F(Np^\infty)$, then we have $M_2(K^p K(p^\infty)_p, \omega, L) \subset \mathbf{M}(K^p, \omega, L)$.

The action of $\mathcal{H}^{S^p} = C_c^\infty(\mathbf{GL}_2(\mathbf{A}^{S^p\infty}), \mathbf{Q})$ extends continuously to the space $\mathbf{S}(K^p, \omega, L)$ if K^p is maximal away from S ; explicitly, if φ is the q -expansion with reduced coefficients $W_{a,\varphi}^{\natural}$ and $h(x) = \mathbf{1}_{K(1)^{Np}(\varpi_v^{-1})K(1)^{Np}}$, we have

$$W_{a,T(h)\varphi}^{\natural} = W_{a\varpi_v,\varphi}^{\natural} + \omega^{-1}(\varpi_v)W_{a/\varpi_v,\varphi}^{\natural}. \tag{2.2.1}$$

Moreover, if S' is another set of finite places not containing those above p and $S'' = S \cup S'$, the action of $\mathcal{H}^{S''p}$ extends in the same way to the space $\overline{\mathbf{S}}' = \overline{\mathbf{S}}'_{S'}(K^p)$ defined after Lemma 2.1.2.

Operators U_v . Let v be a finite place of F , $\varpi_v \in F_v$ a uniformiser, and $K^v \subset \mathbf{GL}_2(\widehat{\mathcal{O}}_F^v)$ a compact open subgroup. For each $r \geq 1$, we define Hecke operators

$$\begin{aligned} U_{v,r}^* &= [K^v K_1^1(\varpi_v^r)_v(\varpi_v^{-1})K^v K_1^1(\varpi_v^r)_v], \\ U_{v,*,r} &= [K^v K_1^1(\varpi_v^r)_v(\varpi_v^{-1})K^v K_1^1(\varpi_v^r)_v]. \end{aligned}$$

They depend on the choice of uniformisers ϖ_v , although a sufficiently high (depending on r) integer power of them does not. They are compatible with changing r in the sense that $U_{v,*,r}e_{K_1^1(\varpi_v^{r'})_v} = U_{v,*,r'}$ for $r' \leq r$ and similarly for $U_{v,r}^*$; we will hence omit the r from the notation. If $\varphi \in S_2(K^p K_1^1(p^r)_p, \omega)$ has reduced q -expansion coefficients $W_{\varphi,a}^{\natural}$ for $a \in \mathbf{A}^{\infty, \times}$, then $U_{v,*}\varphi$ has reduced q -expansion coefficients $W_{U_{v,*}\varphi,a}^{\natural} = \omega^{-1}(\varpi_v)W_{\varphi,a\varpi_v}^{\natural}$. By this formula we can

extend $U_{v,*}$ to a continuous operator on p -adic reduced q -expansions, and in particular on p -adic modular forms.

Atkin–Lehner operators. Let v be a finite place and fix the same uniformiser ϖ_v as in the previous paragraph. Then we define elements

$$w_{r,v} := \begin{pmatrix} & 1 \\ -\varpi_v^r & \end{pmatrix} \in \mathbf{GL}_2(F_v) \subset \mathbf{GL}_2(\mathbf{A})$$

for $r \geq 0$, and denote by the same names the operators they induce on automorphic forms by right multiplication. We have $w_{r,v}^{-1}K_1^1(\varpi^s)_v w_{r,v} = K_1^1(\varpi^s)_v$.

If $r = (r_v)_{v|p}$, we define $w_r = (w_{r_v,v})_{v|p} \in \mathbf{GL}_2(F_p) = \prod_{v|p} \mathbf{GL}_2(F_v)$, and similarly w_r^{-1} .

2.3 Universal Kirillov and Whittaker models

Let F_v be a non-archimedean local field, and recall the space Ψ_v of abstract additive characters of level 0 of F_v defined in § 1.2. Let $\psi_{\text{univ},v} : F_v \rightarrow \mathcal{O}(\Psi_v)^\times$ be the tautological character, which we identify with an action of the unipotent subgroup $N = N(F_v) \cong F_v \subset \mathbf{GL}_2(F_v)$ on the sheaf \mathcal{O}_{Ψ_v} . Let σ_v be an infinite-dimensional representation of $\mathbf{GL}_2(F_v)$ on a vector space over a number field M . A *Whittaker model* over $M \otimes \mathcal{O}_{\Psi_v}$ for $\sigma_v \otimes_{\mathbf{Q}} \mathcal{O}_{\Psi_v}$ is a non-trivial $\mathbf{GL}_2(F_v)$ -equivariant map $\sigma_v \otimes \mathcal{O}_{\Psi_v} \rightarrow M \otimes \text{Ind}_N^{\mathbf{GL}_2(F_v)} \psi_{\text{univ},v}$ of free sheaves over $M \otimes \mathcal{O}_{\Psi_v}$. We will often identify this map with its image.

Let $P_0 \subset \mathbf{GL}_2(F_v)$ be the mirabolic group of matrices $\begin{pmatrix} a & b \\ & 1 \end{pmatrix}$. A *Kirillov model* over $M \otimes \mathcal{O}_{\Psi_v}$ for $\sigma_v \otimes_{\mathbf{Q}} \mathcal{O}_{\Psi_v}$ is a non-trivial P_0 -equivariant map $\sigma_v \otimes \mathcal{O}_{\Psi_v} \rightarrow M \otimes \text{Ind}_N^{P_0} \psi_{\text{univ},v}$. We will often identify this map with its image and the image with a subsheaf of $C^\infty(F_v^\times, M) \otimes \mathcal{O}_{\Psi_v}$ by restricting functions from P_0 to $\{ \begin{pmatrix} a & \\ & 1 \end{pmatrix} \mid a \in F_v^\times \} \cong F_v^\times$.

LEMMA 2.3.1. *Let σ_v be an irreducible admissible infinite-dimensional representation of $\mathbf{GL}_2(F_v)$ on a rational vector space, $M = \text{End}(\sigma_v)$. Then $\sigma_v \otimes_{\mathbf{Q}} \mathcal{O}_{\Psi_v}$ admits a Whittaker model $\mathcal{W}(\sigma_v, \psi_{\text{univ},v})$ (respectively, a Kirillov model $\mathcal{K}(\sigma_v, \psi_{\text{univ},v})$) over $M \otimes \mathcal{O}_{\Psi_v}$, unique up to $(M \otimes \mathcal{O}_{\Psi_v})^\times$, whose specialisation at every closed point $\psi_v \in \Psi_v$ is the unique Whittaker model $\mathcal{W}(\sigma_v, \psi_v)$ (respectively, the unique Kirillov model $\mathcal{K}(\sigma_v, \psi_v)$) of $\sigma_v \otimes_{\mathbf{Q}} \mathbf{Q}(\psi_v)$.*

If we view $\mathcal{W}(\sigma_v, \psi_{\text{univ},v})$ (respectively, $\mathcal{K}(\sigma_v, \psi_{\text{univ},v})$) as a subsheaf of $C^\infty(\mathbf{GL}_2(F_v), M) \otimes \mathcal{O}_{\Psi_v}$ (respectively, as a subsheaf of $C^\infty(F_v^\times, M) \otimes \mathcal{O}_{\Psi_v}$), then the restriction map $W \mapsto f$, $f(y) := W\left(\begin{pmatrix} y & \\ & 1 \end{pmatrix}\right)$, induces an isomorphism $\mathcal{W}(\sigma_v, \psi_{\text{univ},v}) \rightarrow \mathcal{K}(\sigma_v, \psi_{\text{univ},v})$.

We call $\mathcal{W}(\sigma_v, \psi_{\text{univ},v})$ (respectively, $\mathcal{K}(\sigma_v, \psi_{\text{univ},v})$) the *universal Whittaker model* (respectively, the *universal Kirillov model*) for σ_v . The universal Kirillov model admits a natural M -structure, that is, an M -vector space¹⁵

$$\mathcal{K}(\sigma_v, \psi_{\text{univ},v})_M \subset C^\infty(F_v^\times, M)$$

such that $\mathcal{K}(\sigma_v, \psi_{\text{univ},v})_M \otimes \mathcal{O}_{\Psi_v} = \mathcal{K}(\sigma_v, \psi_{\text{univ},v})$.

Proof. The proof of existence and uniqueness of Whittaker models given e.g. in [Bum97, § 4.4] carries over to our context after replacing \mathbf{C} by $M \otimes \mathcal{O}_{\Psi_v}$ and the fixed \mathbf{C}^\times -valued character ψ_v of [Bum97] with $\psi_{\text{univ},v}$. The analogous result for Kirillov models, together with the isomorphism $\mathcal{W}(\sigma_v, \psi_{\text{univ},v}) \rightarrow \mathcal{K}(\sigma_v, \psi_{\text{univ},v})$, follows formally from Frobenius reciprocity as in [BH06, Corollary 36.2]. We prove the assertion on the M -structure for $\mathcal{K}(\sigma_v, \psi_{\text{univ},v})$, after dropping subscripts v .

¹⁵ Which is *not* stable under the action of $\mathbf{GL}_2(F_v)$.

As in the classical case, the space of Schwartz functions $\mathcal{S}(F^\times, M) \otimes \mathcal{O}_\Psi$ is an irreducible P_0 -representation (see [BH06, Corollary 8.2]) and hence contained in $\mathcal{K} := \mathcal{K}(\sigma^\iota, \psi_{\text{univ}}) \subset C^\infty(F^\times, M) \otimes \mathcal{O}_\Psi$. Moreover, $\overline{\mathcal{K}} := \mathcal{K}/\mathcal{S}(F^\times, M) \otimes \mathcal{O}_\Psi$ is a free sheaf over $M \otimes \mathcal{O}_\Psi$ of rank $d \leq 2$ depending on the type of σ (as can be checked on the points of Ψ by the classical theory). Since the space $\mathcal{S}(F^\times, M) \otimes \mathcal{O}_\Psi$ has the obvious M -structure $\mathcal{S}(F^\times, M)$, it suffices to describe d generators for $\overline{\mathcal{K}}$ represented by functions in $C^\infty(F^\times, M)$.

If σ is supercuspidal, then $d = 0$ and there is nothing to prove. If $\sigma = \text{St}(\mu|\cdot|^{-1})$ is special with M^\times -valued central character $\mu^2|\cdot|^{-2}$, then $d = 1$ and a generator for $\overline{\mathcal{K}}$ is $f_\mu(y) := \mu(y)\mathbf{1}_{\mathcal{O}_F - \{0\}}(y)$. If σ is an irreducible principal series $\text{Ind}(\mu_1, \mu_2|\cdot|^{-1})$ (plain un-normalised induction) with M^\times -valued characters μ_1, μ_2 , then $d = 2$; if $\mu_1 \neq \mu_2$, a pair of generators for $\overline{\mathcal{K}}$ is $\{f_{\mu_1}, f_{\mu_2}\}$. If $\mu_1 = \mu_2 = \mu$, a pair of generators is $\{f_\mu, f'_\mu\}$ with $f'_\mu(y) := v(y)\mu(y)\mathbf{1}_{\mathcal{O}_F - \{0\}}(y)$. \square

We will often slightly abusively identify Whittaker and Kirillov models by $W \mapsto f, f(y) = W\left(\begin{smallmatrix} y & \\ & 1 \end{smallmatrix}\right)$.

If σ^∞ is an M -rational automorphic representation of weight 2, then after choosing any embedding $\iota : M \hookrightarrow \mathbf{C}$ and any non-trivial character $\psi : \mathbf{A}/F \rightarrow \mathbf{C}^\times$, the q -expansion coefficients of any $\varphi \in \sigma^\infty$ can be identified with the product of the local Kirillov restrictions f_v of the Whittaker function $W = W_v$ of φ^ι (when W is indeed factorisable). Equivalently, the f_v belong to the M -rational subspaces and are therefore independent of the choice of additive character.

LEMMA 2.3.2. *In the situation of the previous lemma, there is a pairing*

$$(\cdot, \cdot)_v : \mathcal{K}(\sigma_v, \psi_{\text{univ},v}) \otimes_M \mathcal{K}(\sigma_v^\vee, \psi_{\text{univ},v}^{-1}) \rightarrow M \otimes \mathcal{O}_{\Psi_v}$$

such that for any f_1, f_2 in the M -rational subspaces $\mathcal{K}(\sigma_v, \psi_{\text{univ},v})_M$, respectively, $\mathcal{K}(\sigma_v^\vee, \psi_{\text{univ},v}^{-1})_M$, the pairing $(f_1, f_2)_v \in M$, and for any $\iota : M \hookrightarrow \mathbf{C}$, we have

$$\iota(f_1, f_2)_v = \frac{\zeta_{F,v}(2)}{L(1, \sigma_v^\iota \times \sigma_v^{\iota\vee})} \int_{F_v^\times} \iota f_1(y) \iota f_2(y) \frac{d^\times y}{|d|_v^{1/2}}. \tag{2.3.1}$$

The right-hand side is understood in the sense of analytic continuation to $s = 0$ for the function of s defined, for $\Re(s)$ sufficiently large, by the normalised convergent integral

$$\frac{\zeta_{F,v}(2)}{L(1 + s, \sigma_v^\iota \times \sigma_v^{\iota\vee})} \int_{F^\times} \iota f_1(y) \iota f_2(y) |y|^s \frac{d^\times y}{|d|_v^{1/2}}.$$

The normalisation is such that the pairing equals 1 when σ_v is an unramified principal series and the f_i are normalised new vectors.

Proof. We use the notation of the proof of Lemma 2.3.1, dropping all subscripts v . We simply need to show that the given expression belongs to ιM if f_1, f_2 belong to the M -rational subspace of \mathcal{K} and that any pole of the integral $I_s(f_1, f_2) := \int_{F^\times} \iota f_1(y) \iota f_2(y) |y|^s (d^\times y/|d|^{1/2})$ is cancelled by a pole of $L(1 + s, \sigma^\iota \times \sigma^{\iota\vee})$. If either of $f_i \in \mathcal{S}(F^\times, M)$, the integral is just a finite sum of elements in ιM . Then we only need to compute the integral when f_1, f_2 are among the M -rational generators of $\overline{\mathcal{K}}$, which is a standard calculation.

In our application there will be no poles by the Weil conjectures, so we limit ourselves to proving the statement in the case where $\sigma = \text{Ind}(\mu_1, \mu_2|\cdot|^{-1})$ is a principal series with $\mu_1 \neq \mu_2$. (The other cases are similar; cf. also the proof of Proposition 3.6.1.) Then $\sigma^\vee = \text{Ind}(\mu'_1, \mu'_2|\cdot|^{-1})$ with $\mu'_1 = \mu_1^{-1}|\cdot|, \mu'_2 = \mu_2^{-1}|\cdot|$, and (dropping also the ι from the notation)

$L(1 + s, \sigma \times \sigma^\vee) = (1 - q_F^{-1-s})^{-2}(1 - \mu_1\mu_2'(v)q_F^{-s})^{-1}(1 - \mu_1'\mu_2(v)q_F^{-s})^{-1}$, where $\mu(v) := \mu(\varpi_v)$ if μ is an unramified character and $\mu(v) := 0$ otherwise.

Assume that $f_1 = f_{\mu_1}$ (the case $f_1 = f_{\mu_2}$ is similar). If $f_2 = f_{\mu_1'}$, then $I_s(f_1, f_2) = (1 - q_F^{-1-s})^{-1}$ has no pole at $s = 0$. If $f_2 = f_{\mu_2'}$, then $I_s(f_1, f_2) = (1 - \mu_1\mu_2'(v)q_F^{-s})^{-1}$, whose inverse is a factor of $L(1 + s, \sigma_v \times \sigma_v^\vee)^{-1}$ in $M[q_F^{-s}]$. □

2.4 p -critical forms and the p -adic Petersson product

As in [Dis15], we introduce the following notion.

DEFINITION 2.4.1. Let $W = (0, (W_a)) \in \mathbf{S}'(L)$ be a reduced q -expansion without constant term, with values in a p -adic field L , and let $v|p$. We say that W is v -critical if for some integer r , the following condition is satisfied: there is $c \in \mathbf{Z}$ such that, for each $a \in \mathbf{A}^{\infty, \times}$ with $v(a) = r$ and $s \in \mathbf{N}$,

$$W_{a\varpi_v^s} \in q_{F,v}^{s-c} \mathcal{O}_L.$$

We say that W is p -critical if it is a sum of v -critical q -expansions for $v|p$.

For each $v|p$, we define ordinary projectors e_v and e on $\mathbf{M}(K^p, \omega, L)$ by

$$e_v(\varphi') = \lim_{n \rightarrow \infty} U_{v,*}^{n!} \varphi', \quad e := \prod_v e_v.$$

They are independent of the choice of uniformisers. The image of e_v is contained in $M_2(K^p K_1^1(p^\infty), \omega, L)$. It is clear that v -critical forms belong to the kernel of e_v .

p -adic Petersson product. Let M be a number field, and let σ^∞ be an M -rational cuspidal automorphic representation of \mathbf{GL}_2 of weight 2 as in Definition 1.2.1, with central character $\omega : F^\times \backslash \mathbf{A} \rightarrow M^\times$.

Following Hida, we will define a p -adic analogue of the Petersson inner product with a form φ in σ^∞ when σ_p is \mathfrak{p} -ordinary for a prime $\mathfrak{p}|p$ of M . First we define an algebraic version of the Petersson product, which requires no ordinariness assumption. If $\iota : M \hookrightarrow \mathbf{C}$, let $\varphi^\iota := \iota\varphi \otimes \varphi_\infty \in \sigma^\iota$ be the automorphic form whose Whittaker function at infinity is antiholomorphic of smallest K_∞ -type.

LEMMA 2.4.2. *There is a unique pairing*

$$(\cdot, \cdot)_{\sigma^\infty} : \sigma^\infty \otimes_M M_2(\omega^{-1}, M) \rightarrow M$$

such that for any $\varphi_1 \in \sigma^\infty$, $\varphi_2 \in M_2(\omega^{-1}, M)$, and $\iota : M \hookrightarrow \mathbf{C}$, we have

$$(\varphi_1, \varphi_2)_\sigma = \frac{|D_F|^{1/2} \zeta_F(2)}{L(1, \sigma^\iota, \text{ad})} (\varphi_1^\iota, \iota\varphi_2),$$

where

$$(\varphi_1', \varphi_2') := \int_{\mathbf{GL}_2(F)Z(\mathbf{A}) \backslash \mathbf{GL}_2(\mathbf{A})} \varphi_1'(g) \varphi_2'(g) dg$$

is the usual Petersson product on complex automorphic forms with respect to the Tamagawa measure dg .

Proof. Note first that if such a pairing exists, it annihilates forms on the right-hand side which are orthogonal (under the complex Petersson product in any embedding) to forms in σ . Then we just need to use a well-known formula for the adjoint L -value in terms of Petersson product; we quote it in the version given in [TYZ14, p. 55]: for an antiholomorphic form φ'_1 in the space of σ^ι and a holomorphic form φ'_2 in the space of $\sigma^{\vee,\iota}$, both rational over ιM , with factorisable Whittaker functions W_i^ι , we have

$$\frac{|D_F|^{1/2} \zeta_F(2)(\varphi'_1, \varphi'_2)}{2L(1, \sigma^\iota, \text{ad})} = \prod_v (W_{1,v}^\iota, W_{2,v}^\iota)_v \tag{2.4.1}$$

where for all v the local pairings are given by the right-hand side of (2.3.1) and do not depend on the choice of additive characters. Each local factor in the product is rational over ιM and almost all of them are equal to 1. \square

Remark 2.4.3. If $\varphi_2 \in M_2(M)$ does not have central character ω^{-1} , we can still define $(\varphi_1, \varphi_2)_{\sigma^\infty} := (\varphi_1, \varphi_{2,\omega^{-1}})_{\sigma^\infty}$, where

$$\varphi_{2,\omega^{-1}}(g) := \int_{Z(F)\backslash Z(\mathbf{A})} \varphi_2(zg)\omega(z) dz.$$

Now fix a prime $\mathfrak{p}|p$ of M and a finite extension L of $M_{\mathfrak{p}}$, and assume that for all $v|p$, $\sigma_v \otimes L$ is nearly \mathfrak{p} -ordinary with unit character $\alpha_v : F_v^\times \rightarrow \mathcal{O}_L^\times$ in the sense of Definition 1.2.2. Fix a Whittaker functional $\mathscr{W}_p = \prod_v \mathscr{W}_v$ at p and let $\varphi \in \sigma^\infty \otimes_M M(\alpha)$ be a form in the space of σ^∞ whose image under \mathscr{W}_v is the function (viewed in the $M(\alpha)$ -rational part of any Kirillov model)

$$W_v(y) = \mathbf{1}_{\mathcal{O}_F - \{0\}}(y)|y|\alpha_v(y). \tag{2.4.2}$$

Note that W_v , viewed in a Kirillov model associated to an additive character of level 0, satisfies $U_v^* W_v = \alpha_v(\varpi_v) W_v$.

In the next proposition, we use the notation $\alpha(\varpi)^r := \prod_{v|p} \alpha_v(\varpi_v)^{r_v}$.

PROPOSITION 2.4.4. *There exists a unique bounded linear functional*

$$\ell_{\varphi^p, \alpha} : \mathbf{M}(K^p, \omega^{-1}, L) \rightarrow L$$

satisfying the following.

- (1) Let $r = (r_v)_{v|p} \in \mathbf{Z}_{\geq 1}^{\{v|p\}}$. The restriction of $\ell_{\varphi^p, \alpha}$ to $M_2(K^p K^1(p^r)_p, M(\alpha))$ is given by

$$\ell_{\varphi^p, \alpha}(\varphi') = \alpha(\varpi)^{-r} (w_r \varphi, \varphi')_{\sigma^\infty} = \alpha(\varpi)^{-r} (\varphi, w_r^{-1} \varphi')_{\sigma^\infty} \in M(\alpha) \tag{2.4.3}$$

for any choice of uniformisers ϖ_v in the definitions of $U_{v,*}$, U_v^* , w_r .

- (2) We have

$$\ell_{\varphi^p, \alpha}(U_{v,*} \varphi') = \alpha_v(\varpi_v) \ell_{\varphi^p, \alpha}(\varphi')$$

for all $v|p$ and all φ' .

- (3) $\ell_{\varphi^p, \alpha}$ vanishes on p -critical forms.
- (4) Let $T(\sigma^\vee) \in \mathscr{H}^S(M)$ (where S is any sufficiently large set of finite places containing those dividing p) be any element whose image $T(\sigma)^\iota \in \mathscr{H}^S(M) \otimes_{M,\iota} \mathbf{C}$ acts on $S_2(K^p K^1(p^r)_p, \mathbf{C})$ as the idempotent projector onto $(\sigma^{\vee,\iota})^{K^p K^1(p^r)_p}$ for any $\iota : M \hookrightarrow \mathbf{C}$ and $r \geq 1$. Let $T_{\iota_p}(\sigma^\vee)$ be the image of $T(\sigma^\vee)$ in $\mathscr{H}^S(M) \otimes_{M,\iota_p} L$. Then

$$\ell_{\varphi^p, \alpha} \circ T_{\iota_p}(\sigma^\vee) = \ell_{\varphi^p, \alpha}.$$

Proof. By property (2), for each v we must have

$$\ell_{\varphi^p, \alpha}(e_v \varphi') = \lim_{n \rightarrow \infty} \ell_{\varphi^p, \alpha}(U_{v, * }^{n!} \varphi') = \lim_{n \rightarrow \infty} \alpha_v(\varpi_v)^{n!} \ell_{\varphi^p, \alpha}(\varphi') = \ell_{\varphi^p, \alpha}(\varphi') \tag{2.4.4}$$

as $\alpha_v(\varpi_v)$ is a p -adic unit; note that this expression does not depend on the choice of uniformisers. It follows that $\ell_{\varphi^p, \alpha}$ must factor through the ordinary projection

$$e : \mathbf{M}(K^p, \omega^{-1}, L) \rightarrow M_2(K^p K^1(p^\infty)_p, \omega^{-1}, L),$$

which implies property (3). On the image of e , $\ell_{\varphi^p, \alpha}$ must be defined defined by (2.4.3), which makes uniqueness and property (4) clear.

It remains to show the existence (that is, that (2.4.3) is compatible with changing r) and that the first equality in (2.4.3) holds for all r for the functional $\ell_{\varphi^p, \alpha}$ just defined (the second one is trivial). For the latter, we have

$$\begin{aligned} (w_r \varphi, U_{v, * } \varphi') &= (w_r \varphi, K^1(\varpi_v^{r \cdot v})_v ({}^1_{\varpi_v^{-1}}) \varphi') = (({}^1_{\varpi_v}) K^1(\varpi_v^{r \cdot v})_v w_r \varphi, \varphi') \\ &= (w_r K_1(\varpi_v^{r \cdot v})_v ({}^{\varpi_v}_{1}) \varphi, \varphi') = (w_r U_{v, * } \varphi, \varphi') = \alpha_v(\varpi_v)(w_r \varphi, \varphi'). \end{aligned} \tag{2.4.5}$$

The compatibility with change of r can be seen by a similar calculation. □

We still use the notation $\ell_{\varphi^p, \alpha}$ for the linear form deduced from $\ell_{\varphi^p, \alpha}$ by extending scalars to some L -algebra. The analogous remark will apply to $(,)_{\sigma_\infty}$.

3. The p -adic L -function

3.1 Weil representation

We start by recalling from [Wal85, YZZ12] the definition of the Weil representation for groups of similitudes.

Local case. Let $V = (V, q)$ be a quadratic space of even dimension over a local field F of characteristic not 2. Fix a non-trivial additive character ψ of F . For simplicity, we assume that V has even dimension. For $u \in F^\times$, we denote by V_u the quadratic space (V, uq) . We let $\mathbf{GL}_2(F) \times \mathbf{GO}(V)$ act on the usual space of Schwartz functions¹⁶ $\mathcal{S}(V \times F^\times)$ as follows (here $\nu : \mathbf{GO}(V) \rightarrow \mathbf{G}_m$ denotes the similitude character):

- $r(h)\phi(x, u) = \phi(h^{-1}x, \nu(h)u)$ for $h \in \mathbf{GO}(V)$;
- $r(n(b))\phi(x, u) = \psi(buq(x))\phi(x, u)$ for $n(b) \in N(F) \subset \mathbf{GL}_2(F)$;
- $r(\begin{pmatrix} a & \\ & d \end{pmatrix})\phi(x, u) = \chi_{V_u}(a)|a/d|^{\dim V/4}\phi(at, d^{-1}a^{-1}u)$;
- $r(w)\phi(x, u) = \gamma(V_u)\hat{\phi}(x, u)$ for $w = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$.

Here $\chi_V = \chi_{(V, q)}$ is the quadratic character attached to V , $\gamma(V, q)$ is a fourth root of unity, and $\hat{\phi}$ denotes the Fourier transform in the first variable with respect to the self-dual measure for the character $\psi_u(x) = \psi(ux)$. We will need to note the following facts (see e.g. [JL70]): χ_V is trivial if V is a quaternion algebra over F or $V = F \oplus F$, and $\chi_V = \eta$ if V is a separable quadratic extension E of F with associated character η ; and $\gamma(V) = +1$ if V is the space of 2×2 matrices or $V = F \oplus F$, $\gamma(V) = -1$ if V is a non-split quaternion algebra.

We state here a lemma which will be useful later.

¹⁶ The notation is only provisional for the archimedean places; see below.

LEMMA 3.1.1. *Let F be a non-archimedean local field and $\phi \in \mathcal{S}(V \times F^\times)$ a Schwartz function with support contained in*

$$\{(x, u) \in V \times F^\times : uq(x) \in \mathcal{O}_F\}.$$

Suppose that the character ψ used to construct the Weil representation has level 0. Then ϕ is invariant under $K_1^1(\varpi^r) \subset \mathbf{GL}_2(\mathcal{O}_F)$ for sufficiently large r . If moreover $\phi(x, u)$ depends only on x and on the valuation $v(u)$, then ϕ is invariant under $K^1(\varpi^r)$.

Proof. By continuity of the Weil representation, for the first assertion it suffices to show the invariance under $N(\mathcal{O}_F)$. This follows from the observation that under our assumption, in the formula

$$r(n(b))\phi(x, u) = \psi(ubq(x))\phi(x, u),$$

the multiplier $\psi(ubq(x)) = 1$ whenever (x, u) is in the support of ϕ . The second assertion is then equivalent to the invariance of ϕ under the subgroup $(\begin{smallmatrix} 1 & \\ & \mathcal{O}_F^\times \end{smallmatrix}) \subset \mathbf{GL}_2(\mathcal{O}_F)$, which is clear. \square

Fock model and reduced Fock model. Assume that $F = \mathbf{R}$ and V is positive definite. Then we will prefer to consider a modified version of the previous setting. Let the *Fock model* $\mathcal{S}(V \times \mathbf{R}^\times, \mathbf{C})$ be the space of functions spanned by those of the form

$$H(u)P(x)e^{-2\pi|u|q(x)},$$

where H is a compactly supported smooth function on \mathbf{R}^\times and P is a complex polynomial function on V . This space is not stable under the action of $\mathbf{GL}_2(\mathbf{R})$, but it is so under the restriction of the induced $(\mathfrak{gl}_{2,\mathbf{R}}, \mathbf{O}_2(\mathbf{R}))$ -action on the usual Schwartz space (see [YZZ12, § 2.1.2]).

We will also need to consider the *reduced Fock space* $\overline{\mathcal{S}}(V \times \mathbf{R}^\times)$ spanned by functions of the form

$$\phi(x, u) = (P_1(uq(x)) + \operatorname{sgn}(u)P_2(uq(x)))e^{-2\pi|u|q(x)},$$

where P_1, P_2 are polynomial functions with rational coefficients. It contains the *standard Schwartz function*

$$\phi(x, u) = \mathbf{1}_{\mathbf{R}_+}(u)e^{-2\pi|u|q(x)},$$

which for $x \neq 0$ satisfies

$$r(g)\phi(x, u) = W_{uq(x)}^{(d)}(g) \tag{3.1.1}$$

if V has dimension $2d$ and $W^{(d)}$ is the standard holomorphic Whittaker function (2.1.1) (see [YZZ12, § 4.1.1]).

By [YZZ12, §§ 4.4.1 and 3.4.1], there is a surjective quotient map

$$\begin{aligned} \mathcal{S}(V \times \mathbf{R}^\times, \mathbf{C}) &\rightarrow \overline{\mathcal{S}}(V \times \mathbf{R}^\times) \otimes_{\mathbf{Q}} \mathbf{C} \\ \Phi &\mapsto \phi(x, u) = \overline{\Phi}(x, u) = \int_{\mathbf{R}^\times} \int_{\mathbf{O}(V)} r(ch)\Phi(x, u) dh dc. \end{aligned} \tag{3.1.2}$$

We let $\mathcal{S}(V \times \mathbf{R}^\times) \subset \mathcal{S}(V \times \mathbf{R}^\times, \mathbf{C})$ be the preimage of $\overline{\mathcal{S}}(V \times \mathbf{R}^\times)$. For the sake of uniformity, when F is non-archimedean we set $\overline{\mathcal{S}}(V \times F^\times) := \mathcal{S}(V \times F^\times)$.

Global case. Let (\mathbf{V}, q) be an even-dimensional quadratic space over the adèles $\mathbf{A} = \mathbf{A}_F$ of a totally real number field F , and suppose that \mathbf{V}_∞ is positive definite; we say that \mathbf{V} is *coherent* if it has a model over F and *incoherent* otherwise. Given an $\widehat{\mathcal{O}}_F$ -lattice $\mathcal{V} \subset \mathbf{V}$, we define the

space $\mathcal{S}(\mathbf{V} \times \mathbf{A}^\times)$ as the restricted tensor product of the corresponding local spaces, with respect to the spherical elements

$$\phi_v(x, u) = \mathbf{1}_{\mathcal{V}_v}(x) \mathbf{1}_{\varpi_v^{n_v}}(u),$$

if ψ_v has level n_v . We call such ϕ_v the *standard Schwartz function* at a non-archimedean place v . We define similarly the reduced space $\overline{\mathcal{S}}(\mathbf{V} \times \mathbf{A}^\times)$, which admits a quotient map

$$\mathcal{S}(\mathbf{V} \times \mathbf{A}^\times) \rightarrow \overline{\mathcal{S}}(\mathbf{V} \times \mathbf{A}^\times) \tag{3.1.3}$$

defined by the product of the maps (3.1.2) at the infinite places and of the identity at the finite places. The Weil representation of $\mathbf{GL}_2(\mathbf{A}^\infty) \times \mathbf{GO}(\mathbf{V}^\infty) \times (\mathfrak{gl}_{2, F_\infty}, \mathbf{O}(\mathbf{V}_\infty))$ is the restricted tensor product of the local representations.

3.2 Eisenstein series

Let \mathbf{V}_2 be a two-dimensional quadratic space over \mathbf{A}_F , totally definite at the archimedean places. Consider the Eisenstein series

$$E_r(g, u, \phi_2, \chi_F) = \sum_{\gamma \in P^1(F) \backslash \mathrm{SL}_2(F)} \delta_{\chi_F, r}(\gamma g w_r) r(\gamma g) \phi_2(0, u),$$

where

$$\delta_{\chi_F, r}(g) = \begin{cases} \chi_F(d)^{-1} & \text{if } g = \begin{pmatrix} a & b \\ & d \end{pmatrix} k \text{ with } k \in K_1^1(p^r), \\ 0 & \text{if } g \notin PK_0(p^r) \end{cases}$$

and $\phi_2 \in \overline{\mathcal{S}}(\mathbf{V}_2 \times \mathbf{A}^\times)$. (The defining sum is in fact not absolutely convergent, so it must be interpreted in the sense of analytic continuation at $s = 0$ from the series obtained by replacing $\delta_{\chi_F, r}$ with $\delta_{\chi_F, r} \delta_s$, where $\delta_s(g) = |a/d|^s$ if $g = \begin{pmatrix} a & b \\ & d \end{pmatrix} k$, $k \in K_0(1)$.) It belongs to the space $M_1^{\mathrm{tw}}(\eta \chi_F^{-1}, \mathbf{C})$ of twisted modular forms of parallel weight 1 and central character $\eta \chi_F^{-1}$.

After a suitable modification, we study its Fourier–Whittaker expansion and show that it interpolates to a \mathcal{Y}_F -family of q -expansions of twisted modular forms.

PROPOSITION 3.2.1. *We have*

$$L^{(p)}(1, \eta \chi_F) E_r\left(\begin{pmatrix} y & x \\ & 1 \end{pmatrix}, u, \phi_2, \chi_F\right) = \sum_{a \in F} W_{a, r}\left(\begin{pmatrix} y & \\ & 1 \end{pmatrix}, u, \phi_2, \chi_F\right) \psi(ax),$$

where

$$W_{a, r}(g, u, \phi_2, \chi_F) = \prod_v W_{a, r, v}(g, u, \phi_{2, v}, \chi_{F, v}),$$

with, for each v and $a \in F_v$,

$$W_{a, r, v}(g, u, \phi_{2, v}, \chi_{F, v}) = L^{(p)}(1, \eta_v \chi_{F, v}) \int_{F_v} \delta_{\chi_{F, v}, r}(w_n(b) g w_r) r(w_n(b) g) \phi_{2, v}(0, u) \psi_v(-ab) db.$$

Here $L^{(p)}(s, \xi_v) := L(s, \xi_v)$ if $v \nmid p$ and $L^{(p)}(s, \xi_v) := 1$ if $v|p$, and we use the convention that $r_v = 0$ if $v \nmid p$.

Proof. The standard expansion of Eisenstein series reads

$$E_r(g w_r, u, \phi_2, \chi_F) = \delta_{\chi_F, r}(g w_r) r(g) \phi_2(0, u) + \sum_{a \in F} W_{a, r}^*(g, u, \phi_2, \chi_F) \psi(ax),$$

where $W_{a, r}^* = L^{(p)}(1, \eta \chi_F)^{-1} W_{a, r}$; but it is easy to check that $\delta_{\chi_F, r}\left(\begin{pmatrix} y & x \\ & 1 \end{pmatrix} w_r\right) = 0$. □

We choose convenient normalisations for the local Whittaker functions: let $\gamma_{u,v} = \gamma(\mathbf{V}_{2,v}, uq)$ be the Weil index and, for $a \in F_v^\times$, set

$$W_{a,r,v}^\circ(g, u, \phi_{2,v}, \chi_{F,v}) := \gamma_{u,v}^{-1} W_{a,r,v}(g, u, \phi_{2,v}, \chi_{F,v}).$$

For the constant term, set

$$W_{0,r,v}^\circ(g, u, \phi_{2,v}, \chi_{F,v}) := \frac{\gamma_{u,v}^{-1}}{L^{(p)}(0, \eta_v \chi_{F,v})} W_{0,r,v}(g, u, \phi_{2,v}, \chi_{F,v}).$$

Then for the global Whittaker functions we have

$$W_{a,r}(g, u, \phi_2, \chi_F) = -\varepsilon(\mathbf{V}_2) \prod_v W_{a,r,v}^\circ(g, u, \phi_{2,v}, \chi_{F,v}) \tag{3.2.1}$$

if $a \in F^\times$, where $\varepsilon(\mathbf{V}_2) = \prod_v \gamma_{u,v}$ equals -1 if \mathbf{V}_2 is coherent or $+1$ if \mathbf{V}_2 is incoherent; and

$$W_{0,r}(g, u, \phi_2, \chi_F) = -\varepsilon(\mathbf{V}_2) L^{(p)}(0, \eta \chi_F) \prod_v W_{0,r,v}^\circ(g, u, \phi_{2,v}, \chi_{F,v}). \tag{3.2.2}$$

We sometimes drop ϕ_2 from the notation in what follows.

LEMMA 3.2.2. *For each finite place v and $y \in F_v^\times$, $x \in F_v$, $u \in F_v^\times$, we have*

$$W_{a,v}\left(\begin{pmatrix} y & x \\ & 1 \end{pmatrix}, u\right) = \psi_v(ax) \chi_F(y)^{-1} |y|^{1/2} W_{ay,v}(1, y^{-1}u).$$

The proof is an easy calculation.

PROPOSITION 3.2.3. *The local Whittaker functions satisfy the following.*

- (1) *If $v \nmid p\infty$, then $W_{a,v,r}^\circ = W_{a,v}^\circ$ does not depend on r and, for all $a \in F_v$,*

$$W_{a,v}^\circ(1, u, \chi_F) = |d_v|^{1/2} L(1, \eta_v \chi_{F,v}) (1 - \chi_{F,v}(\varpi_v)) \sum_{n=0}^\infty \chi_{F,v}(\varpi_v)^n q_{F,v}^n \int_{D_n(a)} \phi_{2,v}(x_2, u) d_u x_2,$$

where $d_u x_2$ is the self-dual measure on $(\mathbf{V}_{2,v}, uq)$ and

$$D_n(a) = \{x_2 \in \mathbf{V}_{2,v} \mid uq(x_2) \in a + p_v^n d_v^{-1}\}.$$

(When the sum is infinite, it is to be understood in the sense of analytic continuation from characters $\chi_F | \cdot |^s$ with $s > 0$; cf. the proof of Lemma 3.3.1 below.)

- (2) *If $v|p$ and $\phi_{2,v}$ is the standard Schwartz function, then*

$$W_{a,r,v}^\circ(1, u, \chi_F) = \begin{cases} |d_v|^{3/2} |D_v|^{1/2} \chi_{F,v}(-1) & \text{if } v(a) \geq -v(d_v) \text{ and } v(u) = -v(d_v), \\ 0 & \text{otherwise.} \end{cases}$$

- (3) *If $v|\infty$ and $\phi_{2,v}$ is the standard Schwartz function, then*

$$W_{a,v}(1, u) = \begin{cases} 2e^{-2\pi a} & \text{if } ua > 0, \\ 1 & \text{if } a = 0, \\ 0 & \text{if } ua < 0. \end{cases}$$

Proof. Part (1) is proved similarly to [YZZ12, Proposition 6.10(1)], whose Whittaker function $W_{a,v}^\circ(s, 1, u)$ equals our $L(1, \eta_v | \cdot |_v^s)^{-1} W_{a,v}^\circ(1, u, | \cdot |_v^s)$. The proof of Part 2 is similar to that of [Dis15, Proposition 3.2.1, places $|M/\delta|$]. Part (3) is also well known; see e.g. [YZZ12, Proposition 2.11], whose normalisation differs from ours by a factor of $\gamma_v L(1, \eta_v)^{-1} = \pi i$. \square

LEMMA 3.2.4. *Let $a \in F$. For all finite places v , $|d|_v^{-3/2} |D_v|^{-1/2} W_{a,v}^\circ(1, u, \chi_F) \in \mathbf{Q}[\chi_F, \phi_v]$ and, for almost all v , we have*

$$|d|_v^{-3/2} |D_v|^{-1/2} W_{a,v}^\circ(1, u, \chi_F) = \begin{cases} 1 & \text{if } v(a) \geq -v(d_v) \text{ and } v(u) = -v(d_v), \\ 0 & \text{otherwise.} \end{cases}$$

Proof. This follows from Proposition 3.2.3(1) by an explicit computation which is neither difficult nor unpleasant: we leave it to the reader. \square

3.3 Eisenstein family

Recall from § 1.2 the profinite groups Γ and Γ_F and the associated rigid spaces \mathcal{Y}' , \mathcal{Y} , \mathcal{Y}_F (only the latter is relevant for this subsection). For each finite place $v \nmid p$ of F , there are local versions

$$\mathcal{Y}'_v, \quad \mathcal{Y}_v, \quad \mathcal{Y}_{F,v}, \tag{3.3.1}$$

which are schemes over M representing the corresponding spaces of $\mathbf{G}_{m,M}$ -valued homomorphisms with domain $E_v^\times / (V^p \cap E_v^\times)$ (for $\mathcal{Y}'_v, \mathcal{Y}_v$, where $V^p \subset E_{\mathbf{A}^{p\infty}}^\times$ is the subgroup fixed in the Introduction) or F_v^\times (for $\mathcal{Y}_{F,v}$).¹⁷ Letting \otimes' denote the restricted tensor product with respect to the constant function 1, and the symbol $\mathcal{Y}^?$ stand for any of the symbols \mathcal{Y}' , \mathcal{Y} , \mathcal{Y}_F , we let

$$\mathcal{O}_{\mathcal{Y}^?}(\mathcal{Y}^?)^f \subset \mathcal{O}_{\mathcal{Y}^?}(\mathcal{Y}^?)^b$$

denote the image of $\otimes'_{v \nmid p} \mathcal{O}(\mathcal{Y}_v^?) \otimes_M L \rightarrow \mathcal{O}_{\mathcal{Y}^?}(\mathcal{Y}^?)$.

LEMMA 3.3.1. *For each $a \in F$, $y \in \mathbf{A}^{\infty, \times}$, and rational Schwartz function $\phi_2^{p\infty}$, there are:*

- (1) for each $v \nmid p\infty$:
 - (a) a Schwartz function $\phi_{2,v}(\cdot) \in \mathcal{S}(V_{2,v}, \mathcal{O}(\mathcal{Y}_{F,v}))$ such that $\phi_{2,v}(\mathbf{1}) = \phi_{2,v}$ and $\phi_{2,v}(\cdot)$ is identically equal to $\phi_{2,v}$ if $\phi_{2,v}$ is standard;
 - (b) a function

$$W_{a,v}^\circ(y_v, u, \phi_{2,v}) \in \mathcal{O}_{\mathcal{Y}_{F,v}}(\mathcal{Y}_{F,v})$$

satisfying

$$\mathcal{W}_{a,v}^\circ(y_v, u, \phi_{2,v}; \chi_F) = |d_v|^{-3/2} |D_v|^{-1/2} W_{a,r,v}^\circ\left(\begin{pmatrix} y_v & \\ & 1 \end{pmatrix}, u, \phi_{2,v}(\chi_{F,v}), \chi_{F,v}\right)$$

for all $\chi_{F,v} \in \mathcal{Y}_{F,v}(\mathbf{C})$;

- (2) a global function

$$\mathcal{W}_a(y, u, \phi_2^{p\infty}) \in \mathcal{O}_{\mathcal{Y}_F}(\mathcal{Y}_F)^b,$$

which is algebraic on $\mathcal{Y}_F^{1.c}$ and satisfies

$$\mathcal{W}_a(y, u, \phi_2^{p\infty}; \chi_F) = |D_F|^{1/2} |D_E|^{1/2} W_{a,r}^\infty\left(\begin{pmatrix} y & \\ & 1 \end{pmatrix}, u, \phi_2(\chi_F), \chi_F\right)$$

¹⁷ Concretely, they are closed subschemes of split tori over M ; cf. the proof of Proposition 3.6.1.

for each $\chi_F \in \mathcal{Y}_F^{1,c}(\mathbf{C})$; here $\phi_2(\chi_F) = \prod_{v|p\infty} \phi_{2,v}(\chi_{F,v})_2 \phi_{2,p\infty}$ with $\phi_{2,v}$ the standard Schwartz function for each $v|p\infty$. The function $|y|^{-1/2} \mathcal{W}_a(y, u, \phi_2^{p\infty})$ is bounded solely in terms of $\max |\phi_2^{p\infty}|$ and, if $a \neq 0$, then $\mathcal{W}_a \in \mathcal{O}(\mathcal{Y}_F)^f$.

Proof. If $a \neq 0$, by Lemma 3.2.4 and Proposition 3.2.3(2), we can deduce the existence of the global function in part (2) from the local result of part (1). If $a = 0$, then by (3.2.2) the same is true thanks to the well-known existence [DR80] of a bounded analytic function on \mathcal{Y}_F interpolating $\chi_F \mapsto L^{(p)}(0, \eta\chi_F)$.

It thus suffices to prove part (1), and moreover we may restrict to $y = 1$ in view of Lemma 3.2.2. We can uniquely write $\phi_{2,v} = c\phi_{2,v}^\circ + \phi'_{2,v}$, where $\phi_{2,v}^\circ$ is the standard Schwartz function and $c = \phi_{2,v}(0)$. Then we set

$$\phi_{2,v}(\chi_{F,v}) := c\phi_{2,v}^\circ + \frac{L(1, \eta_v)}{L(1, \eta_v\chi_{F,v})} \phi'_{2,v}. \tag{3.3.2}$$

We need to show that, upon substituting it in the expression for the local Whittaker functions given in Proposition 3.2.3(1), we obtain a Laurent polynomial in $\chi_{F,v}(\varpi_v)$ (which gives the canonical coordinate on $\mathcal{Y}_{F,v} \cong \mathbf{G}_{m,M}$). By linearity and Lemma 3.2.4, it suffices to show this for the summand $(L(1, \eta_v)/L(1, \eta_v\chi_{F,v}))\phi'_{2,v}$, whose coefficient is designed to cancel the factor $L(1, \eta_v, \chi_v)$ appearing in that expression. The only source of possible poles is the infinite sum. For n sufficiently large, if a is not in the image of uq , then $D_n(a)$ is empty and therefore the sum is actually finite. On the other hand if $a = uq(x_a)$, then for n large the function $\phi_{2,v}$ is constant and equal to $\phi_{2,v}(x_a)$ on $D_n(a)$; it follows that

$$\int_{D_n(a)} \phi_{2,v}(x_2, u) d_u x_2 = c' q_{F,v}^{-n}$$

for some constant c' independent of n and $\chi_{F,v}$. Then the tail of the sum is

$$\sum_{n \geq n_0} c' \chi_{F,v}(\varpi_v)^n = c' \frac{\chi_{F,v}(\varpi_v)^{n_0}}{1 - \chi_{F,v}(\varpi_v)};$$

its product with the factor $1 - \chi_{F,v}(\varpi_v)$ appearing in front of it is then also a polynomial in $\chi_{F,v}(\varpi_v)$.

Finally, the last two statements of part (2) follow by the construction and Lemma 3.2.2. \square

PROPOSITION 3.3.2. *There is a bounded \mathcal{Y}_F -family of q -expansions of twisted modular forms of parallel weight 1*

$$\mathcal{E}(u, \phi_2^{p\infty})$$

such that for any $\chi_F \in \mathcal{Y}_F^{1,c}(\mathbf{C})$ and any $r = (r_v)_{v|p}$ satisfying $c(\chi_F)|p^r$, we have

$$\mathcal{E}(u, \phi_2^{p\infty}; \chi_F) = |D_F| \frac{L^{(p)}(1, \eta\chi_F)}{L^{(p)}(1, \eta)} \mathbf{q} E_r(u, \phi_2, \chi_F),$$

where $\phi_2 = \phi_2^{p\infty}(\chi_F)\phi_{2,p\infty}$ with $\phi_{2,v}$ the standard Schwartz function for $v|p\infty$.

Proof. This follows from Lemma 3.3.1 and Proposition 3.2.3(3): we take the q -expansion with coefficients $(2^{[F:\mathbf{Q}]}|D_F|^{1/2}/|D_E|^{1/2}|L^{(p)}(1, \eta))\mathcal{W}_a(y, u, \phi_2^{p\infty})$. \square

3.4 Analytic kernel

We first construct certain bounded $\mathcal{Y}^?$ -families of q -expansions of modular forms for $\mathcal{Y}^? = \mathcal{Y}_F$ or \mathcal{Y}' . In general, if $\mathcal{Y}^?$ is the space of p -adic characters of a profinite group $\Gamma^?$, then it is equivalent to giving a compatible system, for each extension L' of L , of bounded functionals $\mathcal{C}(\Gamma^?, L') \rightarrow \mathbf{M}(K^p, L')$, where the source is the space of L' -valued continuous functions on $\Gamma^?$. This can be applied to the case of \mathcal{Y}_F (with Γ_F), and to the case of \mathcal{Y}' with the variation that \mathcal{Y}' -families correspond to bounded functionals on the space $\mathcal{C}(\Gamma, \omega, L')$ of functions f on Γ satisfying $f(zt) = \omega^{-1}(z)f(t)$ for all $z \in \mathbf{A}^{\infty, \times}$.

Let \mathbf{B} be a (coherent or incoherent) totally definite quaternion algebra over $\mathbf{A} = \mathbf{A}_F$ and let E be a totally imaginary quadratic extension of F with an embedding $E_{\mathbf{A}} \hookrightarrow \mathbf{B}$ which we fix. Let \mathbf{V} be the orthogonal space \mathbf{B} with reduced norm q . We have an orthogonal decomposition

$$\mathbf{V} = \mathbf{V}_1 \oplus \mathbf{V}_2,$$

where

$$\mathbf{V}_1 = E_{\mathbf{A}}, \quad \mathbf{V}_2 = E_{\mathbf{A}}j, \quad j \notin E_{\mathbf{A}}, \quad j^2 \in \mathbf{A}^{\times}.$$

The restriction of q to \mathbf{V}_1 is the adelisation of the norm of E/F .

We have an embedding (cf. [YZZ12, p. 36])

$$\mathbf{A}^{\times} \backslash \mathbf{B}^{\times} \times \mathbf{B}^{\times} \hookrightarrow \mathbf{GO}(\mathbf{V}),$$

where $\mathbf{B}^{\times} \times \mathbf{B}^{\times}$ acts on \mathbf{V} by $(h_1, h_2)x = h_1 x h_2^{-1}$.

Let $\phi^{p\infty} \in \mathcal{S}(\mathbf{V}^{p\infty} \times \mathbf{A}^{p\infty, \times})$ be a Schwartz function and let $U^p \subset \mathbf{B}^{\infty \times}$ be a compact open subgroup fixing $\phi^{p\infty}$. For $\phi_1 \in \mathcal{S}(\mathbf{V}_1 \times \mathbf{A}^{\times})$ a Schwartz function such that $\phi_{1, \infty}$ is standard, let $\theta(u, \phi_1)$ be the twisted modular form

$$\theta(g, u, \phi_1) := \sum_{x_1 \in E} r(g)\phi_1(x_1, u).$$

We define the modular form

$$I_{F,r}(\phi_1 \otimes \phi_2, \chi_F) = \frac{c_{U^p}}{|D_F|^{1/2}} \cdot \frac{L^{(p)}(1, \eta\chi_F)}{L^{(p)}(1, \eta)} \sum_{u \in \mu_{U^p}^2 \backslash F^{\times}} \theta(u, \phi_1) E_r(u, \phi_2, \chi_F) \tag{3.4.1}$$

for sufficiently large $r = (r_v)_{v|p}$, and the \mathcal{Y}_F -family of q -expansions of weight-2 modular forms

$$\mathcal{I}_F(\phi_1^{\infty} \otimes \phi_2^{p\infty}; \chi_F) = c_{U^p} \sum_{u \in \mu_{U^p}^2 \backslash F^{\times}} \mathfrak{a}\theta(u, \phi_1) \mathcal{E}(u, \phi_2^{p\infty}; \chi_F), \tag{3.4.2}$$

where, letting $\mu_{U^p} = F^{\times} \cap U^p \mathcal{O}_{\mathbf{B}, p}^{\times}$, we set

$$c_{U^p} := \frac{2^{[F:\mathbf{Q}]-1} h_F}{[\mathcal{O}_F^{\times} : \mu_{U^p}^2]} \tag{3.4.3}$$

and $\phi(x_1, x_2, u) = \phi_1(x_1, u)\phi_2(x_2, u)$ with $\phi_i = \phi_i^{p\infty}$ for $\phi_{i,v}$ the standard Schwartz function if $v|\infty$ or $i = 2$ and $v|p$. The definition is independent of the choice of U^p (cf. [YZZ12, (5.1.3)]).

The action of the subgroup $T(\mathbf{A}) \times T(\mathbf{A}) \subset \mathbf{B}^{\times} \times \mathbf{B}^{\times}$ on $\mathcal{S}(\mathbf{V} \times \mathbf{A}^{\times}) = \mathcal{S}(\mathbf{V}_1 \times \mathbf{A}^{\times}) \otimes \mathcal{S}(\mathbf{V}_2 \times \mathbf{A}^{\times})$ preserves this tensor product decomposition and thus it can be written as $r = r_1 \otimes r_2$

for the actions r_1, r_2 on each of the two factors. We obtain an action of $T(\mathbf{A}^\infty) \times T(\mathbf{A}^\infty)$ on the forms $I_{F,r}$ and the families \mathcal{I}_F with orbits

$$I_{F,r}((t_1, t_2), \phi_1 \otimes \phi_2, \chi_F) := \frac{c_{Up}}{|D_F|^{1/2}} \frac{L^{(p)}(1, \eta\chi_F)}{L^{(p)}(1, \eta)} \sum_{u \in \mu_{Up}^2 \setminus F^\times} \theta(u, r_1(t_1, t_2), \phi_1) E_r(u, \phi_2, \chi_F^t),$$

$$\mathcal{I}_F((t_1, t_2), \phi_1^\infty \otimes \phi_2^{p\infty}; \chi_F) := c_{Up} \sum_{u \in \mu_{Up}^2 \setminus F^\times} \mathfrak{q}\theta(q(t)u, r_1(t_1, t_2)\phi_1) \mathcal{E}(q(t)u, \phi_2^{p\infty}; \chi_F).$$

It is a bounded action in the sense that the orbit $\{\mathcal{I}_F((t_1, t_2), \phi_1^\infty \otimes \phi_2^{p\infty}) \mid t_1, t_2 \in T(\mathbf{A}^\infty)\}$ is a bounded subset of the space of \mathcal{Y}_F -families of q -expansions, as both \mathcal{E} and $\mathfrak{q}\theta$ are bounded in terms of $\max|\phi^{p\infty}|$.

Define, for the fixed finite-order character $\omega : F^\times \setminus \mathbf{A}^\times \rightarrow M^\times$,

$$\mathcal{I}_{F,\omega^{-1}}((t_1, t_2), \phi_1^\infty \otimes \phi_2^{p\infty}; \chi_F) := \int_{\mathbf{A}^\times} \omega^{-1}(z)\chi_F(z)\mathcal{I}_F((zt_1, t_2), \phi_1^\infty \otimes \phi_2^{p\infty}; \chi_F) dz, \tag{3.4.4}$$

a bounded \mathcal{Y}_F -family of q -expansions of forms of central character ω^{-1} , corresponding to a bounded functional on $\mathcal{C}(\Gamma_F, L)$ valued in $\mathbf{M}(K^p, \omega^{-1}, L)$ for a suitable K^p .

We further obtain a bounded functional \mathcal{I} on $\mathcal{C}(\Gamma, \omega, L)$, valued in $\mathbf{M}(K^p, \omega^{-1}, L)$, which is defined on the set (generating a dense subalgebra) of finite-order characters $\chi' \in \mathcal{C}(\Gamma, \omega, L)$ by

$$\mathcal{I}(\phi^{p\infty}; \chi') := \int_{[T]} \chi'(t)\mathcal{I}_{F,\omega^{-1}}((t, 1), \phi_1^p\phi_{1,p} \otimes \phi_2^p; \omega \cdot \chi'|_{\mathbf{A}^\times}) d^\circ t$$

if $\phi^{p\infty} = \phi_1^{p\infty} \otimes \phi_2^{p\infty}$. Here $\phi_1^\infty = \phi_1^{p\infty}\phi_{1,p}$ with

$$\phi_{1,v}(x_1, u) = \delta_{1,U_{T,v}}(x_1)\mathbf{1}_{\mathcal{O}_{F,v}^\times}(u) \tag{3.4.5}$$

if $v|p$, where $U_{T,p} \subset \mathcal{O}_{E,p}^\times$ is a compact open subgroup small enough that $\chi'_p|_{U_{T,p}} = 1$ and

$$\delta_{1,U_{T,v}}(x_1) = \frac{\text{vol}(\mathcal{O}_{E,v}, dx)}{\text{vol}(U_{T,v}, dx)} \mathbf{1}_{U_v \cap \mathcal{O}_E}(x_1).$$

(The notation is meant to suggest a Dirac delta at 1 in the variable x_1 , to which this is the finest $U_v \times U_v$ -invariant approximation. Both volumes are taken with respect to a Haar measure on E_v .)

By construction, the induced rigid analytic function on $\mathcal{Y}' = \mathcal{Y}'_\omega$, still denoted by \mathcal{I} , satisfies the following.

PROPOSITION 3.4.1. *There is a bounded \mathcal{Y}' -family of q -expansions of modular forms $\mathcal{I}(\phi^{p\infty})$ such that for each $\chi' \in \mathcal{Y}'^{\text{l.c.}}(\mathbf{C})$, we have*

$$\mathcal{I}(\phi^{p\infty}; \chi') = |D_E|^{1/2}|D_F| \mathfrak{q}I_r(\phi, \chi'^t),$$

where

$$I_r(\phi, \chi') := \int_{[T]}^* \chi'(t)I_{F,r}((t, 1), \phi, \chi_F) dt$$

with $I_{F,r}(\phi)$ as in (3.4.1), with $\phi_{p\infty}$ chosen as above.

3.5 Waldspurger’s Rankin–Selberg integral

Let $\chi' \in \mathcal{Y}'^{1.c.}_{M(\alpha)}(\mathbf{C})$ be a character and $\iota : M(\alpha) \hookrightarrow \mathbf{C}$ be the induced embedding. Let $\psi : \mathbf{A}/F \rightarrow \mathbf{C}^\times$ be an additive character and let $r = r_\psi$ be the associated Weil representation.

PROPOSITION 3.5.1. *Let $\varphi \in \sigma^\iota$ be a form with factorisable Whittaker function, and let $\phi = \bigotimes_v \phi_v \in \overline{\mathcal{S}}(\mathbf{V} \times \mathbf{A}^\times)$. For sufficiently large $r = (r_v)_{v|p}$, we have*

$$\prod_{v|p} \iota\alpha_v(\varpi_v)^{-r_v} \cdot (\varphi, w_r^{-1}I_r(\phi, \chi')) = \prod_v R_{r,v}^\circ(W_v, \phi_v, \chi'_v, \psi_v), \tag{3.5.1}$$

where

$$R_{r,v}^\circ(W_v, \phi_v, \chi'_v, \psi_v) = \iota\alpha_v(\varpi_v)^{-r_v} \frac{L^{(p)}(1, \eta_v \chi_{F,v})}{L^{(p)}(1, \eta_v)} R_{r,v}$$

with

$$R_{r,v} = \int_{Z(F_v)N(F_v)\backslash \mathbf{GL}_2(F_v)} W_{-1,v}(g)\delta_{\chi_{F,r,v}}(g) \int_{T(F_v)} \chi'_v(t)r(gw_r^{-1})\Phi_v(t^{-1}, q(t)) dt dg.$$

Here $\Phi_v = \phi_v$ if v is non-archimedean and Φ_v is a preimage of ϕ_v under (3.1.3) if v is archimedean, $W_{-1,v}$ is the local Whittaker function of φ for the character $\overline{\psi}_v$, and we use the convention that $r_v = 0$, $w_{r,v} = 1$, $\iota\alpha_v(\varpi_v)^{-r_v} = 1$ if $v \nmid p$.

Note that the integral $R_{r,v}$ does not depend on $r \geq 1$ unless $v|p$ and it does not depend on χ' if $v|\infty$; we will accordingly simplify the notation in these cases.

Proof. This is shown similarly to [YZZ12, Proposition 2.5]; see [YZZ12, (5.1.3)] for the equality between the kernel functions denoted there by $I(s, \chi, \phi)$ (similar to our $c_{U^p}^{-1}I_r(\phi, \chi')$) and $I(s, \chi, \Phi)$ (which intervenes in the analogue in [YZZ12] of the left-hand side of (3.5.1)). \square

We will sometimes lighten a bit the notation for R_v° by omitting ψ_v from it.

LEMMA 3.5.2. *When everything is unramified, we have*

$$R_v^\circ(W_v, \phi_v, \chi'_v) = \frac{L(1/2, \sigma_{E,v} \otimes \chi'_v)}{\zeta_{F,v}(2)L(1, \eta_v)}.$$

Proof. With a slightly different setup,¹⁸ Waldspurger [Wal85, Lemmes 2 and 3] showed that

$$R_v(W_v, \phi_v, \chi'_v) = \frac{L(1/2, \sigma_{E,v} \otimes \chi'_v)}{\zeta_{F,v}(2)L(1, \eta_v \chi_{F,v})}$$

when $\chi_{F,v} = |\cdot|^s$, but his calculation goes through for any unramified character $\chi_{F,v}$. \square

Define

$$R_{r,v}^\natural(W_v, \phi_v \chi'_v, \psi_v) := |d_v|^{-2}|D_v|^{-1/2} \frac{\zeta_{F,v}(2)L(1, \eta_v)}{L(1/2, \sigma_{E,v} \otimes \chi'_v)} R_{r,v}^\circ(W_v, \phi_v \chi'_v, \psi_v). \tag{3.5.2}$$

Then the previous lemma combined with Proposition 3.5.1 gives the following result.

¹⁸ Notably, the local measures in [Wal85] are normalised by $\text{vol}(\mathbf{GL}_2(\mathcal{O}_{F,v})) = 1$ for almost all finite places v , whereas we have $\text{vol}(\mathbf{GL}_2(\mathcal{O}_{F,v})) = \zeta_{F,v}(2)^{-1}|d_v|^2$ (cf. [YZZ12, p. 23]; the second displayed formula of [YZZ12, p. 42] neglects this discrepancy).

PROPOSITION 3.5.3. *We have*

$$\begin{aligned} & \iota\alpha^{-r}(\varphi, w_r^{-1}I(\phi, \chi')) \\ &= |D_F|^{-1}|D_E|^{-1/2} \frac{L^{(\infty)}(1/2, \sigma_E \otimes \chi')}{\zeta_F^{(\infty)}(2)L^{(\infty)}(1, \eta)} \prod_{v \nmid \infty} R_{r,v}^{\natural}(W_v, \phi_v, \chi'_v) \prod_{v|\infty} R_v^{\circ}(W_v, \Phi_v, \chi'_v), \end{aligned}$$

where all but finitely many of the factors in the infinite product are equal to 1.

Archimedean zeta integral. We compute the local integral R_v when $v|\infty$.

LEMMA 3.5.4. *If $v|\infty$, ϕ_v is standard, and $W_{-1,v}$ is the standard antiholomorphic Whittaker function of weight 2 of (2.1.2), then*

$$R_v^{\circ}(W_v, \phi_v, \chi'_v) = R_v(W_v, \phi_v, \chi'_v) = 1/2.$$

Proof. By the Iwasawa decomposition, we can uniquely write any $g \in \mathbf{GL}_2(\mathbf{R})$ as

$$g = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} z & \\ & z \end{pmatrix} \begin{pmatrix} y & \\ & 1 \end{pmatrix} \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$$

with $x \in \mathbf{R}$, $z \in \mathbf{R}^{\times}$, $y \in \mathbf{R}^{\times}$, $\theta \in [0, 2\pi)$; the local Tamagawa measure is then $dg = dx d^{\times}z (d^{\times}y/|y|) (d\theta/2)$. The integral in $Z(\mathbf{R}) \subset T(\mathbf{R})$ realises the map $\Phi \rightarrow \phi$; and it is easy to verify that $r(g)\phi(1, 1)$ is the standard holomorphic Whittaker function of weight 2.

We then have, dropping subscripts v ,

$$\begin{aligned} R_v(\varphi, \phi) &= \int_{T(\mathbf{R})/Z(\mathbf{R})} \int_0^{2\pi} \int_{\mathbf{R}^{\times}} (|y|e^{-2\pi y})^2 \frac{d^{\times}y}{|y|} \frac{d\theta}{2} dt \\ &= 2 \cdot (4\pi)^{-1} \pi = 1/2, \end{aligned}$$

where $(4\pi)^{-1}$ comes from a change of variable, $2 = \text{vol}(T(\mathbf{R})/Z(\mathbf{R}))$, and π comes from the integration in $d\theta$. □

3.6 Interpolation of local zeta integrals

When $v \nmid p$, the normalised local zeta integrals admit an interpolation as well. Recall from § 1.2 that Ψ_v denotes the scheme of all local additive characters of level 0.

PROPOSITION 3.6.1. *Let $v \nmid p$ be a finite place, and let $\mathcal{K}(\sigma_v, \psi_{\text{univ},v})$ be the universal Kirillov model of σ_v . Then, for any $\phi_v \in \mathcal{S}(V_v \times F_v^{\times})$, $W_v \in \mathcal{K}(\sigma_v, \psi_{\text{univ},v})$, there exists a function*

$$\mathcal{R}^{\natural}(W_v, \phi_v) \in L(1, \eta_v \chi_{F,v}) \mathcal{O}_{\mathcal{Y}'_v \times \Psi_v}(\mathcal{Y}'_v, \omega_v \chi_{F,v,\text{univ}}^{-1})$$

such that for all $\chi'_v \in \mathcal{Y}'_v(\mathbf{C})$, $\psi_v \in \Psi_v(\mathbf{C})$, we have

$$\mathcal{R}^{\natural}(W_v, \phi_v; \chi'_v, \psi_v) = R_v^{\natural}(W_v^{\iota}, \phi_v(\chi'_v), \chi'_v, \psi_v),$$

where $\phi_v(\chi'_v) = \phi_{1,v} \phi_{2,v}(\chi_{F,v})$ with $\phi_{2,v}(\chi_{F,v})$ is as in (3.3.2).

In the statement, we consider $L(1, \eta_v \chi_{F,v})^{-1}$ as an element of $\mathcal{O}(\mathcal{Y}'_v)$ (coming by pullback from $\mathcal{Y}_{F,v}$). Note that it equals the non-zero constant $L(1, \eta_v)^{-1}$ along $\mathcal{Y}_v \subset \mathcal{Y}'_v$.

Proof. Note that the assertion on the subsheaf of $\mathcal{O}_{\mathcal{Y}_v \times \Psi_v}$ of which \mathcal{R}_v^\natural is a section simply encodes the dependence of R_v^\natural on the additive character, which is easy to ascertain by a change of variables. By the definitions, it suffices to show that

$$L(1/2, \sigma_{E,v} \otimes \chi'_v)^{-1} R_v(W_v, \phi_v, \chi'_v) \tag{3.6.1}$$

extends to a regular function on \mathcal{Y}'_v . We will more precisely show that $L(1/2, \sigma_{E,v} \otimes \chi'_v)^{-1}$ is a product of various factors all of which extend to regular functions on \mathcal{Y}'_v , and that the product of some of those factors and $R_v(W_v, \phi_v, \chi'_v)$ also extends to a regular function on \mathcal{Y}'_v . Concretely, if $A \subset E_v^\times / (E_v^\times \cap V^p)$ is any finite set, then the evaluations $\chi'_v \mapsto (\chi'_v(a))_{a \in A}$ define a morphism $\text{ev}_A : \mathcal{Y}'_v \rightarrow \mathbf{G}_{m,M}^A$,¹⁹ so that finite sums of evaluations of characters are regular functions on \mathcal{Y}'_v obtained by pullback along ev_A .

Interpolation of R_v . Within the expression for R_v , we can use the Iwasawa decomposition and note that integration over $K = \mathbf{GL}_2(\mathcal{O}_{F,v})$ yields a finite sum of integrals of the form (dropping subscripts v)²⁰

$$\int_{F^\times} f'(y) \int_{T(F)} \chi'(t) \phi'(yt^{-1}, y^{-1}q(t)) dt dy$$

for some Schwartz functions ϕ' and elements f' of the Kirillov model of σ : namely, the translates of W_{-1} and of ϕ_v by the action of K . (More precisely, taking into account the dependence on χ' of ϕ , also products of the above integrals and of $L(1, \eta_{\chi_{F,v}})^{-1}$ can occur; the factor $L(1, \eta_{\chi_{F,v}})^{-1}$ clearly interpolates to a regular function on \mathcal{Y}'_v .)

It is easy to see that the integral reduces to a finite sum if either W is compactly supported or $\phi'_1(\cdot, u)$ is supported away from $0 \in E$. It thus suffices to study the case where $\phi'_1(x_1, u) = \mathbf{1}_{\mathcal{O}_E}(x_1) \phi'_F(u)$, and f' belongs to the basis of the quotient space $\overline{\mathcal{K}}$ introduced in the proof of Lemma 2.3.1. Moreover, up to simple manipulations, we may assume that $\phi_F(u)$ is close to a delta function supported at $u = 1$. We distinguish three different cases.

σ_v is supercuspidal. In this case $\overline{\mathcal{K}} = 0$ and there is nothing to prove.

σ_v is a special representation $\text{St}(\mu | \cdot |^{-1})$. In this case $\overline{\mathcal{K}}$ is spanned by $f_\mu = \mu \cdot \mathbf{1}_{\mathcal{O}_F - \{0\}}$. We find that the integral is essentially²¹ 0 if there is a place w of E above v such that, for $\chi'_w := \chi'_v|_{E_w^\times}$, the character $\chi'_w \cdot \mu \circ q$ of E_w^\times is ramified; and it essentially equals

$$\prod_{w|v} (1 - \chi'_w(\varpi_w) \mu(q(\varpi_w)) q_{E,w}^{-1})^{-1} \tag{3.6.2}$$

otherwise.²² In the latter case, $L(1/2, \sigma_{E,v} \otimes \chi'_w)$ is also equal to (3.6.2). We conclude that (3.6.1) extends to a regular function on \mathcal{Y}'_v .

σ_v is an irreducible principal series $\text{Ind}(\mu, \mu' | \cdot |^{-1})$.²³ The space $\overline{\mathcal{K}}$ has dimension 2 and f_μ as above provides a non-zero element. Again the corresponding integral yields either 0 or (3.6.2),

¹⁹ Moreover, if A is sufficiently large, the morphism ev_A is a closed embedding.

²⁰ See Proposition A.2.2 and Lemma A.1.1 for some more detailed calculations similar to the ones of the present proof.

²¹ Here we use this adverb with the precise meaning: up to addition of and multiplication by finite combination of evaluations of χ' .

²² In the last expression, q is the norm of E_w/F_v , whereas $q_{E,w}$ is the cardinality of the residue field of E_w . We apologise for the near-clash of notation.

²³ Here Ind is plain (un-normalised) induction.

the latter happening precisely when (3.6.2) is a factor of $L(1/2, \sigma_E \otimes \chi)$. If $\mu' \neq \mu$, then a second basis element is $f_{\mu'}$, for which the same discussion applies. If $\mu' = \mu$, then a second basis element is $f'_\mu(y) := v(y)\mu(y)\mathbf{1}_{\mathcal{O}_F - \{0\}}(y)$. The integral is essentially 0 if some $\chi'_w \cdot \mu \circ q$ is ramified, and

$$\prod_{w|v} (1 - \chi'_w(\varpi_w)\mu(q(\varpi_w))q_{E,w}^{-1})^{-2} \tag{3.6.3}$$

otherwise. In the latter case, $L(1/2, \sigma_{E,v} \otimes \chi'_w)$ equals (3.6.3) as well.

Interpolation of $L(1/2, \sigma_{E,v} \otimes \chi'_v)^{-1}$. Depending only on σ_v , as recalled above, for each place $w|v$ of E , there exist at most two characters ν_{w,i_w} of E_w^\times such that for all $\chi'_v \in \mathcal{Y}'_v(\mathbf{C})$, we can write $L(1/2, \sigma_{E,v} \otimes \chi'_v)^{-1} = \prod'_{w,i_w} (1 - \nu_w \chi'_w(\varpi_w))$, where the product \prod' extends over those pairs (w, i_w) such that $\nu_{w,i_w} \chi'_w$ is unramified. We can replace the partial product by a genuine product and each of the factors by

$$1 - \left(\sum'_{x \in \mathcal{O}_{E,w}^\times / (V^p \cap E_w^\times)} \nu_{w,i_w}(x) \chi'_w(x) \right) \cdot \nu_{w,i_w} \chi'_w(\varpi_w),$$

where \sum' denotes average. This expression is the value at χ'_v of an element of $\mathcal{O}(\mathcal{Y}'_v)$, as desired. □

3.7 Definition and interpolation property

Let $\mathcal{M}_{\mathcal{Y}'-\mathcal{Y}}$ be the multiplicative part of $\mathcal{O}(\mathcal{Y}')^f$ consisting of functions whose restriction to \mathcal{Y} is invertible. (Recall that $\mathcal{O}(\mathcal{Y}')^f \subset \mathcal{O}(\mathcal{Y}')^b$ is the image of $\bigotimes_{v \nmid p\infty} \mathcal{O}(\mathcal{Y}'_v)$.)

THEOREM 3.7.1. *There exists a unique function*

$$L_{p,\alpha}(\sigma_E) \in \mathcal{O}_{\mathcal{Y}' \times \Psi_p}(\mathcal{Y}', \omega_p \chi_{F,\text{univ},p}^{-1})^b [\mathcal{M}_{\mathcal{Y}'-\mathcal{Y}}^{-1}]$$

which is algebraic on $\mathcal{Y}'_{M(\alpha)}{}^{1.c.} \times \Psi_p$ and satisfies

$$L_{p,\alpha}(\sigma_E)(\chi', \psi_p) = \frac{\pi^{2[F:\mathbf{Q}]} |D_F|^{1/2} L^{(\infty)}(1/2, \sigma_E^\iota, \chi'^\iota)}{2L^{(\infty)}(1, \eta) L^{(\infty)}(1, \sigma^\iota, \text{ad})} \prod_{v|p} Z_v^\circ(\chi'_v, \psi_v)$$

for every $\chi' \in \mathcal{Y}'_{M(\alpha)}(\mathbf{C})$ inducing an embedding $\iota : M(\alpha) \hookrightarrow \mathbf{C}$. Here Z_v° is as in Theorem A.

Let $\mathcal{Y}'^\circ \subset \mathcal{Y}'$ be any connected component, $\mathcal{Y}^\circ := \mathcal{Y} \cap \mathcal{Y}'^\circ$ the corresponding connected component of \mathcal{Y} , and let \mathbf{B} be the quaternion algebra over \mathbf{A}^∞ determined by (1.1.1) for any (equivalently, all) points $\chi \in \mathcal{Y}^\circ$. For any $\varphi^{p\infty} \in \sigma^{p\infty}$ and $\phi^{p\infty} \in \mathcal{S}(\mathbf{V}^{p\infty} \times \mathbf{A}^{p\infty, \times})$, we have

$$\ell_{\varphi^p, \alpha}(\mathcal{I}(\phi^{p\infty}))|_{\mathcal{Y}'^\circ} = L_{p,\alpha}(\sigma_E)|_{\mathcal{Y}'^\circ \times \Psi_p} \prod_{v \nmid p\infty} \mathcal{R}_v^{\natural}(W_v, \phi_v)|_{\mathcal{Y}'^\circ \times \Psi_p} \tag{3.7.1}$$

in $\mathcal{O}_{\mathcal{Y}'^\circ}(\mathcal{Y}'^\circ)^b$, where both \mathcal{I} and \mathcal{R}^{\natural} are constructed using \mathbf{V} . On the right-hand side, the product $\prod_{v \nmid p} \mathcal{R}_v^{\natural}$ makes sense over $\mathcal{Y}'^\circ \times \Psi_p$ by the decomposition $\sigma \cong \mathcal{K}(\sigma^p, \Psi_p) \otimes \mathcal{K}(\sigma_p, \Psi_p)$ induced by the Whittaker functional fixed in the definition of $\ell_{\varphi^p, \alpha}$.²⁴

²⁴ The p -adic L -function $L_{p,\alpha}(\sigma_E)$ does not depend on this choice. Here, letting Ψ'_v denote the space of all non-trivial additive characters of F_v^\times , the space $\mathcal{K}(\sigma^p, \Psi_p)$ is the restriction of $\prod'_{v \nmid p} \mathcal{K}(\sigma_v, \Psi'_v)$ via an embedding $\Psi_p \hookrightarrow \prod_{v \nmid p} \Psi'_v$ obtained as follows: fix any non-trivial character ψ_0 of \mathbf{A}/F in $\mu_{\mathbf{Q}}$; then $\psi_p \mapsto (\psi_0/\psi_p|_{F_v})_v$.

Proof. The definition can be given locally by taking quotients in (3.7.1) for any given $(W^{p\infty}, \phi^{p\infty})$. Note that on the right-hand side of (3.7.1), the product is finite since by Lemma 3.5.2 we have $\mathcal{R}_v^\natural(W_v, \phi_v) = 1$ identically on \mathcal{Y}' if all the data are unramified. The analytic properties of $L_{p,\alpha}(\sigma_E)$ are then a consequence of the following claim. Let S be any finite set of places $v \nmid p\infty$, containing all the ones such that either σ_v is ramified or the subgroup $V_v \subset \mathcal{O}_{E,v}^\times$ fixed in the Introduction is not maximal. Let \mathcal{Y}'° be a connected component and \mathbf{B} be the associated quaternion algebra. Then for each $v \in S$, there exists a finite set of pairs (W_v, ϕ_v) such that the locus of common vanishing of the corresponding functions $\mathcal{R}_v^\natural(W_v, \phi_v)|_{\mathcal{Y}'^\circ}$ is empty.

We prove the claim. Let $\chi_v \in \mathcal{Y}'^\circ$ be any closed point, where $v \in S$ and $\mathcal{Y}'^\circ \subset \mathcal{Y}_v$ is the union of connected components corresponding to \mathcal{Y}'° . By Lemma 5.1.1 below, we have $\mathcal{R}_v^\natural(W_v, \phi_v, \chi_v) = Q_v(\theta_{\psi,v}(W_v, \phi_v), \chi_v)$, where $\theta_{\psi,v}$ is a Shimizu lift sending $\sigma_v \times \mathcal{S}(\mathbf{V}_v \times F^\times)$ onto $\pi_v \otimes \pi_v^\vee$, with π_v the Jacquet–Langlands transfer of σ_v to \mathbf{B}_v^\times . By construction of \mathbf{B}_v and the result mentioned in § 1.1, the functional $Q_v(\cdot, \chi_v)$ is non-vanishing. Therefore, given $\chi_v \in \mathcal{Y}'^\circ$, we can find (W_v, ϕ_v) such that $\mathcal{R}_v^\natural(W_v, \phi_v; \chi_v) \neq 0$.

Consider the set of all functions $\mathcal{R}_v^\natural(W_v, \phi_v)|_{\mathcal{Y}'^\circ}$ for varying (W_v, ϕ_v) . As the locus of their common vanishing is empty, it follows by the Nullstellensatz that finitely many of them generate the unit ideal of $\mathcal{O}(\mathcal{Y}'^\circ)$.²⁵ This completes the proof of the claim.

We now move to the interpolation property. The algebraicity on $\mathcal{Y}'_{M(\alpha)}^{1.c.}$ is clear from the definition just given. By ι , which we will omit from the notation below, we can identify φ with an antiholomorphic automorphic form φ^ι . By the definitions and Proposition 3.5.3, we have

$$\begin{aligned} L_{p,\alpha}(\sigma_E)(\chi', \psi_p) &= \frac{\ell_{\varphi^p, \alpha}(\mathcal{I}(\phi^p, \chi'))}{\prod_{v \nmid p\infty} \mathcal{R}_v^\natural(W_v, \phi_v; \chi'_v, \psi_v)} \\ &= \frac{|D_F|^{1/2} \zeta_F(2)}{2L(1, \sigma, \text{ad})} \cdot \frac{|D_E|^{1/2} |D_F| \alpha^{-r}(\varphi, w_r^{-1} I(\phi, \chi'))}{\prod_{v \nmid p\infty} R_v^\natural(W_v, \phi_v, \chi', \psi_v)} \\ &= \frac{|D_F|^{1/2} \zeta_F(2)}{2L(1, \sigma, \text{ad})} \cdot \frac{L^{(\infty)}(1/2, \sigma_E, \chi')}{\zeta_F^{(\infty)}(2) L^{(\infty)}(1, \eta) \prod_{v|\infty} R_v^\circ(\phi_v, W_v, \chi'_v, \psi_v)} \prod_{v|p} R_{r,v}^\natural(\phi_v, W_v, \chi'_v, \psi_v) \\ &= \frac{\zeta_{F,\infty}(2)}{2^{[F:\mathbf{Q}]} L(1, \sigma_\infty, \text{ad})} \cdot \frac{|D_F|^{1/2} L^{(\infty)}(1/2, \sigma_E, \chi')}{2L^{(\infty)}(1, \eta) L^{(\infty)}(1, \sigma, \text{ad})} \prod_{v|p} R_{r,v}^\natural(W_v, \phi_v, \chi'_v, \psi_v). \end{aligned}$$

Here ψ^p is any additive character such that $\psi = \psi^p \psi_p \psi_\infty$ vanishes on F . For $v|\infty$, we have $\zeta_{F,v}(2)/L(1, \sigma_v, \text{ad}) = \pi^{-1}/(\pi^{-3}/2) = 2\pi^2$, so that the first fraction in the last line equals $\pi^{2[F:\mathbf{Q}]}$. The result follows.

The proof is completed by the identification $R_{r,v}^\natural = Z_v^\circ$ for $v|p$ carried out in Proposition A.2.2. □

4. p -adic heights

We recall the definition and properties of p -adic heights and prove two integrality criteria for them. The material of §§ 4.2–4.3 will not be used until §§ 8–9.

²⁵ Recall that \mathcal{Y}_v is an affine scheme of finite type over M (more precisely, it is a closed subscheme of a split torus).

For background in p -adic Hodge theory, see the summary in [Nek93, §1] and references therein. The notation we use is completely standard; it coincides with that of [Nek93] except that we shall prefer to write \mathbf{D}_{dR} instead of DR for the functor of de Rham periods.

4.1 Local and global height pairings

Let F be a number field and $\mathcal{G}_F := \text{Gal}(\overline{F}/F)$. Let L be a finite extension of \mathbf{Q}_p , and let V be a finite-dimensional L -vector space with a continuous action of \mathcal{G}_F . For each place v of F , we denote by V_v the space V considered as a representation of $\mathcal{G}_{F,v} := \text{Gal}(\overline{F}_v/F_v)$.

Recall that the Bloch–Kato Selmer group of V

$$H_f^1(F, V)$$

is the subset of $H^1(F, V) = \text{Ext}_F^1(L, V)$ (extensions in the category of continuous \mathcal{G}_F -representations over L) consisting of the classes of those extensions $0 \rightarrow V \rightarrow E_1 \rightarrow L \rightarrow 0$ which are unramified at all $v \nmid p$ and crystalline at all $v|p$ (that is, such that E_1 is).

Suppose that:

- V is unramified outside of a finite set of primes of F ;
- V_v is de Rham, and hence potentially semistable, for all $v|p$;
- $H^0(F_v, V) = H^0(F_v, V^*(1)) = 0$ for all $v \nmid p$;
- $\mathbf{D}_{\text{crys}}(V_v)^{\varphi=1} = \mathbf{D}_{\text{crys}}(V_v)^{\varphi=1} = 0$ for all $v|p$ (where φ denotes the crystalline Frobenius).

Under those conditions, Nekovář [Nek93] (to which we refer for more details; see also [Nek06]) constructed a bilinear pairing on the Bloch–Kato Selmer groups

$$\langle \cdot, \cdot \rangle : H_f^1(F, V) \times H_f^1(F, V^*(1)) \rightarrow \Gamma_F \hat{\otimes} L \tag{4.1.1}$$

depending on choices of L -linear splittings of the Hodge filtration

$$\text{Fil}^0 \mathbf{D}_{\text{dR}}(V_v) \subset \mathbf{D}_{\text{dR}}(V_v) \tag{4.1.2}$$

for the primes $v|p$. In fact in [Nek93] it is assumed that V_v is semistable; we will recall the definitions under this assumption, and at the same time see that they can be made compatible with extending the ground field (in particular, to reduce the potentially semistable case to the semistable case). Compare also [Ben14] for a very general treatment.

Post-composing $\langle \cdot, \cdot \rangle$ with a continuous homomorphism $\ell : \Gamma_F \rightarrow L'$, for some L -vector space L' , yields an L' -valued pairing $\langle \cdot, \cdot \rangle_\ell$ (the cases of interest to us are $L' = L$ with any ℓ , or $L' = \Gamma_F \hat{\otimes} L$ with the tautological ℓ). For such an ℓ we write $\ell_v := \ell|_{F_v^\times}$ and we say that ℓ_v is *unramified* if it is trivial on $\mathcal{O}_{F,v}^\times$ (note that this is automatic if $v \nmid p$).

Let $x_1 \in H_f^1(F, V)$, $x_2 \in H_f^1(F, V^*(1))$ and view them as classes $x_1 = e_1 = [E_1]$, $x_2 = e_2 = [E_2]$ of extensions of Galois representations

$$\begin{aligned} e_1 : \quad & 0 \rightarrow V \rightarrow E_1 \rightarrow L \rightarrow 0, \\ e_2 : \quad & 0 \rightarrow V^*(1) \rightarrow E_2 \rightarrow L \rightarrow 0. \end{aligned}$$

For any e_1, e_2 as above, the set of Galois representations E fitting into a commutative diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & L(1) & \longrightarrow & E_2^*(1) & \longrightarrow & V \longrightarrow 0 \\
 & & \parallel & & \downarrow & & \downarrow \\
 0 & \longrightarrow & L(1) & \longrightarrow & E & \longrightarrow & E_1 \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & L & \xlongequal{\quad} & L \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0
 \end{array}$$

is an $H^1(F, L)$ -torsor [Nek93, Proposition 4.4]; any such E is called a *mixed extension* of $e_1, e_2^*(1)$. Depending on the choice of (extensions e_1 and e_2 and) a mixed extension E , there is a decomposition

$$\langle x_1, x_2 \rangle_\ell = \sum_{v \in S_F} \langle x_{1,v}, x_{2,v} \rangle_{\ell_v, E, v} \tag{4.1.3}$$

of the height pairing into a (convergent) sum of local symbols indexed by the *non-archimedean* places of F . We recall the definition of the latter [Nek93, § 7.4]. The representation E can be shown to be automatically semistable at any $v|p$; for each v it then yields a class $[E_v] \in H_*^1(F_v, E_2)$ with $*$ = \emptyset if $v \nmid p$, $*$ = st if $v|p$. This group sits in the following diagram of exact sequences.

$$\begin{array}{ccccccc}
 0 & \longrightarrow & H^1(F_v, L(1)) & \longrightarrow & H_*^1(F_v, E_2) & \longrightarrow & H_f^1(F_v, V) \longrightarrow 0 \\
 & & \uparrow & & \uparrow & & \parallel \\
 0 & \longrightarrow & H_f^1(F_v, L(1)) & \longrightarrow & H_f^1(F_v, E_2) & \longrightarrow & H_f^1(F_v, V) \longrightarrow 0
 \end{array} \tag{4.1.4}$$

If $v|p$, the chosen splitting of (4.1.2) uniquely determines a splitting $s_v : H_*^1(F_v, E_2) \rightarrow H^1(F_v, L(1))$; if $v \nmid p$, there is a canonical splitting independent of choices, also denoted by s_v . In both cases, the local symbol is

$$\langle x_{1,v}, x_{2,v} \rangle_{\ell_v, E, v} := -\ell_v(s_v([E_v])),$$

where we still denote by ℓ_v the composition $H^1(F_v, L(1)) \cong F_v^\times \hat{\otimes} L \rightarrow \Gamma_F \hat{\otimes} L \rightarrow L'$. When $v|p$, we say that $[E_v]$ is *essentially crystalline* if $[E_v] \in H_f^1(F_v, E_2) \subset H_{\text{st}}^1(F_v, E_2)$; equivalently, $s_v([E_v]) \in H_f^1(F_v, L(1))$.

Behaviour under field extensions. If F'_w/F_v is a finite extension of local non-archimedean fields, the pairing

$$\langle \cdot, \cdot \rangle_{\ell_v \circ N_{F'_w/F_v}} : H_f^1(F'_w, V_w) \times H_f^1(F'_w, V_w^*(1)) \rightarrow L' \tag{4.1.5}$$

defined using the induced Hodge splittings and the map $\ell_w := \ell_v \circ N_{F'_w/F_v}$ satisfies

$$\langle \text{cores}_{F'_w}^{F'_w} x_1, x_2 \rangle_{\ell_v} = \langle x_1, \text{res}_{F'_w}^{F'_w} x_2 \rangle_{\ell_v \circ N_{F'_w/F_v}} \tag{4.1.6}$$

for all $x_1 \in H_f^1(F_v, V_v), x_2 \in H_f^1(F'_w, V|_{\mathcal{G}_{F'_w}}^*(1))$.

Back to the global situation, it follows that extending any $\ell : \Gamma_F \rightarrow L'$ to the direct system $(\Gamma_{F'})_{F'/F \text{ finite}}$ by

$$\ell_w = \ell|_{F_w'^\times} := \frac{1}{[F' : F]} \ell_v \circ N_{F_w'/F_v} \tag{4.1.7}$$

we can extend $\langle \cdot, \cdot \rangle_\ell$ to a pairing

$$\langle \cdot, \cdot \rangle_\ell : \varinjlim_{F'} H_f^1(F', V|_{\mathcal{G}_{F'}}) \times H_f^1(F', V|_{\mathcal{G}_{F'}}^*(1)) \rightarrow L', \tag{4.1.8}$$

where the limit is taken with respect to restriction maps. This allows us to define the pairing in the potentially semistable case as well.

Ordinariness. Let $v|p$ be a place of F .

DEFINITION 4.1.1. We say that a de Rham representation V_v of \mathcal{G}_{F_v} satisfies the *Panchishkin condition* or that it is *potentially ordinary* if there is a (necessarily unique) exact sequence of de Rham \mathcal{G}_{F_v} -representations

$$0 \rightarrow V_v^+ \rightarrow V_v \rightarrow V_v^- \rightarrow 0$$

with $\text{Fil}^0 \mathbf{D}_{\text{dR}}(V_v^+) = \mathbf{D}_{\text{dR}}(V_v^-)/\text{Fil}^0 = 0$.

If V_v is potentially ordinary, there is a canonical splitting of (4.1.2) given by

$$\mathbf{D}_{\text{dR}}(V_v) \rightarrow \mathbf{D}_{\text{dR}}(V_v^-) = \text{Fil}^0 \mathbf{D}_{\text{dR}}(V_v). \tag{4.1.9}$$

Abelian varieties. If A/F is an abelian variety with potentially semistable reduction at all $v|p$, then the rational Tate module $V = V_p A$ satisfies the required assumptions, and there is a canonical isomorphism $V^*(1) \cong V_p A^\vee$. Suppose that there is an embedding of a number field $M \hookrightarrow \text{End}^0(A)$; its action on V induces a decomposition $V = \bigoplus_{\mathfrak{p}|p} V_{\mathfrak{p}}$ indexed by the primes of \mathcal{O}_M above p . Given such a prime \mathfrak{p} , a finite extension L of $M_{\mathfrak{p}}$, and splittings of the Hodge filtration on $\mathbf{D}_{\text{dR}}(V_{\mathfrak{p}}|_{\mathcal{G}_{F_v}}) \otimes_{M_{\mathfrak{p}}} L$ for $v|p$, we obtain from the compatible pairings (4.1.8) a height pairing

$$\langle \cdot, \cdot \rangle : A(\overline{F}) \times A^\vee(\overline{F}) \rightarrow \Gamma_F \hat{\otimes} L \tag{4.1.10}$$

via the Kummer maps $\varkappa_{A, F', \mathfrak{p}} : A(F') \rightarrow H_f^1(F', V) \rightarrow H_f^1(F', V_{\mathfrak{p}})$ and $\varkappa_{A^\vee, F', \mathfrak{p}} : A(F') \rightarrow H_f^1(F', V_{\mathfrak{p}}^*(1))$ for any $F \subset F' \subset \overline{F}$.

If \mathfrak{p} is a prime of M above p and $V_{\mathfrak{p}} A \otimes L$ is potentially ordinary for all $v|p$, the height pairing (4.1.10) is then canonical (cf. [Nek06, § 11.3]). Such is the situation of Theorem B. In that case we consider the restriction of (4.1.10) to $A(\chi)$, coming from the Kummer maps

$$\varkappa_{A(\chi)} : A(\chi) \rightarrow H_f^1(E, V_{\mathfrak{p}} A(\chi)), \quad \varkappa_{A^\vee(\chi^{-1})} : A^\vee(\chi^{-1}) \rightarrow H_f^1(E, (V_{\mathfrak{p}} A(\chi))^*(1)),$$

where

$$V_{\mathfrak{p}} A(\chi) := V_{\mathfrak{p}} A|_{\mathcal{G}_E} \otimes L(\chi)_\chi.$$

Note that by the condition $\chi|_{\mathbf{A}^{\infty, \times}} = \omega_A^{-1}$, we have $(V_{\mathfrak{p}} A(\chi))^*(1) \cong V_{\mathfrak{p}} A(\chi)$.

4.2 Heights and intersections on curves

If X/F is a (connected, smooth, proper) curve with semistable reduction at all $v|p$, let $V := H_{\text{ét}}^1(X_{\overline{F}}, \mathbf{Q}_p(1))$. Then V satisfies the relevant assumptions; moreover, it carries a non-degenerate symplectic form by Poincaré duality, inducing an isomorphism $V \cong V^*(1)$. For any finite extension L of \mathbf{Q}_p , any Hodge splittings on $(\mathbf{D}_{\text{dR}}(V_v \otimes L))_{v|p}$, and any continuous homomorphism $\ell : \Gamma_F \rightarrow L'$, we obtain a pairing

$$\langle \cdot, \cdot \rangle_{X,\ell} : \text{Div}^0(X_{\overline{F}}) \times \text{Div}^0(X_{\overline{F}}) \rightarrow L' \tag{4.2.1}$$

via the Kummer maps similarly to the above. The pairing factors through $\text{Div}^0(X_{\overline{F}}) \rightarrow J_X(\overline{F})$, where J_X is the Albanese variety; it corresponds to the height pairing on $J_X(\overline{F}) \times J_X^\vee(\overline{F})$ via the canonical autoduality of J_X .

The restriction of (4.2.1) to the set $(\text{Div}^0(X_{\overline{F}}) \times \text{Div}^0(X_{\overline{F}}))^*$ of pairs of divisors with disjoint supports admits a canonical decomposition

$$\langle \cdot, \cdot \rangle_{X,\ell} = \sum_{w \in S_{F'}} \langle \cdot, \cdot \rangle_{X,\ell_w,w}.$$

Namely, the local symbols are continuous symmetric bi-additive maps given by

$$\langle D_1, D_2 \rangle_{X,\ell_w} := \langle x_1, x_2 \rangle_{\ell_w, E, w}, \tag{4.2.2}$$

where x_i is the class of D_i in $H_f^1(F', V)$ and, if $Z_1, Z_2 \subset X_{\overline{F}}$ are disjoint proper closed subsets of $X_{\overline{F}}$ such that the support of D_i is contained in Z_i , then E, E_1, E_2 are the extensions obtained from the diagram of étale cohomology groups

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H^0(Z_2, L(1)) & \longrightarrow & H^1((X_{\overline{F}}, Z_2), L(1)) & \longrightarrow & H^1(X_{\overline{F}}, L(1)) \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H^0(Z_2, L(1)) & \longrightarrow & H^1((X_{\overline{F}} - Z_1, Z_2), L(1)) & \longrightarrow & H^1(X_{\overline{F}} - Z_1, L(1)) \longrightarrow 0 \\ & & & & \downarrow & & \downarrow \\ & & & & H_{Z_1}^2(X_{\overline{F}}, L(1)) & \xlongequal{\quad} & H_{Z_1}^2(X_{\overline{F}}, L(1)) \\ & & & & \downarrow & & \downarrow \\ & & & & 0 & & 0 \end{array}$$

by pullback along $\text{cl}_{D_1} : L \rightarrow H_{Z_1}^2(X_{\overline{F}}, L(1))$ and pushout along $-\text{Tr}_{D_2} : H^0(Z_2, L(1)) \rightarrow L(1)$.

If X does not have semistable reduction at the primes above p , we can still find a finite extension F'/F such that $X_{F'}$ does, and define the pairing on $X_{F'}$. If $X = \coprod_i X_i$ is a disjoint union of finitely many connected curves, then $\text{Div}^0(X_{\overline{F}})$ will denote the group of divisors having degree zero on each connected component; it affords local and global pairings by direct sum.

A uniqueness principle. Suppose that $D_1 = \text{div}(h)$ is a principal divisor with support disjoint from the support of D_2 , and let $h(D_2) := \prod_P h(P)^{n_P}$. Then the mixed extension $[E_w] = [E_{D_1, D_2, w}]$ is the image of $h(D_2) \otimes 1 \in F_w^{\prime \times} \hat{\otimes} L \cong H^1(F'_w, L(1))$ in $H_*^1(F'_w, E_2)$ under (4.1.4); it follows that

$$\langle D_1, D_2 \rangle_{X,\ell_w} = \ell_w(h(D_2)) \tag{4.2.3}$$

independently of the choice of Hodge splittings. When ℓ_w is unramified, this property in fact suffices to characterise the pairing.

LEMMA 4.2.1. *Let X/F_v be a smooth proper curve over a local field F_v and suppose that $\ell_v : F_v^\times \rightarrow L$ is unramified. Then there exists a unique locally constant symmetric bi-additive pairing*

$$\langle \cdot, \cdot \rangle_{X, \ell_v} : (\text{Div}^0(X_{F_v}) \times \text{Div}^0(X_{F_v}))^* \rightarrow L$$

such that

$$\langle \text{div}(h), D_2 \rangle_{X, \ell_v} = \ell_v(h(D_2))$$

whenever the two arguments have disjoint supports.

Proof. The result is well known, see e.g. [CG89, Proposition 1.2], but for the reader’s convenience we recall the proof. A construction of such a pairing has just been recalled, and a second one will be given below. For the uniqueness, note that the difference of any two such pairings is a locally constant homomorphism $J(F_v) \times J(F_v) \rightarrow L$. As the source is a compact group and the target is torsion free, such a homomorphism must be trivial. \square

Arithmetic intersections. Let $F' \subset \bar{F}$ be a finite extension of F and $\mathcal{X}/\mathcal{O}_{F'}$ be a regular integral model of X . For a divisor $D \in \text{Div}^0(X_{F'})$, we define its *flat extension* to the model \mathcal{X} to be the unique extension of D which has intersection zero with any vertical divisor; it can be uniquely written as $\bar{D} + V$, where \bar{D} is the Zariski closure of D in \mathcal{X} and V is a vertical divisor.

Let $D_1, D_2 \in \text{Div}^0(X_{\bar{F}})$ be divisors with disjoint supports, with each D_i defined over a finite extension F_i ; assume that $F \subset F_2 \subset F_1 \subset \bar{F}$. Let $\mathcal{X}/\mathcal{O}_{F_2}$ be a regular and semistable model. Then, for each finite place $w \in S_{F_1}$, we can define partial local intersection multiplicities i_w, j_w of the flat extensions $\bar{D}_1 + V_1$ of $D_1, \bar{D}_2 + V_2$ of D_2 to $\mathcal{X}_{\mathcal{O}_{F_1, w}}$. If the latter model is still regular, they are defined by

$$\begin{aligned} i_w(D_1, D_2) &= \frac{1}{[F_1 : F]} (\bar{D}_1 \cdot \bar{D}_2)_w, \\ j_w(D_1, D_2) &= \frac{1}{[F_1 : F]} (\bar{D}_1 \cdot V_2)_w, \end{aligned} \tag{4.2.4}$$

where on the right-hand sides $(\cdot)_w$ are the usual \mathbf{Z} -valued intersection multiplicities in $\mathcal{X}_{\mathcal{O}_{F_1, w}}$; see [YZZ12, § 7.1.7] for the generalisation of the definition to the case when $\mathcal{X}_{\mathcal{O}_{F_1, w}}$ is not regular. The total intersection

$$m_w(D_1, D_2) = i_w(D_1, D_2) + j_w(D_1, D_2)$$

is of course independent of the choice of models.

Fix an extension \bar{v} to \bar{F} of the valuation v on F . Then we have pairings $i_{\bar{v}}, j_{\bar{v}}$ on divisors on $X_{\bar{F}_{\bar{v}}}$ with disjoint supports by the above formulas. We can group together the contributions of i and j according to the places of F by

$$\lambda_v(D_1, D_2) = \int_{\text{Gal}(\bar{F}/F)} \lambda_{\bar{v}}(D_1^\sigma, D_2^\sigma) d\sigma$$

for v any finite place of F and $\lambda = i, j$, or $\lambda_v(D_1, D_2) = \langle D_1, D_2 \rangle_v$. Here the integral uses the Haar measure of total volume 1, and reduces to a finite weighted average for any fixed D_1, D_2 .

PROPOSITION 4.2.2. *Suppose that D_1 and D_2 are divisors of degree zero on X , defined over an extension F' of F . Then, for all finite places $w \nmid p$ of F' ,*

$$\langle D_1, D_2 \rangle_{X, \ell_w} = m_w(D_1, D_2) \cdot \ell_w(\varpi_w) = (i_w(D_1, D_2) + j_w(D_1, D_2)) \cdot \ell_w(\varpi_w).$$

Proof. This follows from Lemma 4.2.1 and (4.2.3); the verification that the arithmetic intersection pairing m_w also satisfies the required properties can be found in [Gro84]. \square

4.3 Integrality criteria

The result of Proposition 4.2.2 applies with the same proof if $w|p$ when ℓ_w is an unramified logarithm such as the valuation. In this case we will view it as a first integrality criterion for local heights.

PROPOSITION 4.3.1. *Let $\ell_v : F_v^\times \hat{\otimes} L \rightarrow \Gamma_F \hat{\otimes} L$ be the tautological logarithm and let ℓ_w be as in (4.1.7). Let $v : F_v^\times \hat{\otimes} L \rightarrow L$ be the valuation. Then, for all $D_1, D_2 \in \text{Div}^0(X_{F'})$, we have*

$$v(\langle D_1, D_2 \rangle_{X, \ell_w}) = [F'_w : F_v] \cdot m_w(D_1, D_2).$$

In particular, if $m_w(D_1, D_2) = 0$, then

$$\langle D_1, D_2 \rangle_{X, \ell_w} = \ell_w(s_w([E_{D_1, D_2, w}])) \in \mathcal{O}_{F, v}^\times \hat{\otimes} L = \ell_w(H_f^1(F_w, L(1)));$$

equivalently, the mixed extension $[E_{D_1, D_2, w}]$ is essentially crystalline.

We need a finer integrality property for local heights, slightly generalising [Nek95, Proposition 1.11]. Let F_v and L be finite extensions of \mathbf{Q}_p , let V be a \mathcal{G}_{F_v} -representation on an L -vector space equipped with a splitting of (4.1.2), and let $\ell_v : F_v^\times \rightarrow L$ be a logarithm. Suppose that the following conditions are satisfied:

- (a) $\ell_v : F_v^\times \rightarrow L$ is ramified;
- (b) the space V admits a direct sum decomposition $V = V' \oplus V''$ as \mathcal{G}_{F_v} -representation, such that V' satisfies the Panchishkin condition, with a decomposition $0 \rightarrow V'^+ \rightarrow V' \rightarrow V'^- \rightarrow 0$, and the restriction of the Hodge splitting of $\mathbf{D}_{\text{dR}}(V)$ to $\mathbf{D}_{\text{dR}}(V')$ coincides with the canonical one of (4.1.9);
- (c) $H^0(F_v, V'^-) = H^0(F_v, V'^{+*}(1)) = 0$.

By [Nek93, Proposition 1.28(3)], the last condition is equivalent to $\mathbf{D}_{\text{pst}}(V')^{\varphi=1} = \mathbf{D}_{\text{pst}}(V'^*(1))^{\varphi=1} = 0$, where $\mathbf{D}_{\text{pst}}(V') := \lim_{F \subset F'} \mathbf{D}_{\text{st}}(V'|_{\mathcal{G}_{F'}})^{\text{Gal}(F'/F)}$ (the limit ranging over all sufficiently large finite Galois extensions F'/F).

Let T be a $\mathcal{G}_{F, v}$ -stable \mathcal{O}_L -lattice in V , $T' := T \cap V'$, $T'' = T \cap V''$; let $d_0 \geq 0$ be an integer such that $\mathfrak{p}_L^{d_0} T \subset T' \oplus T'' \subset T$, where $\mathfrak{p}_L \in L$ is the maximal ideal of \mathcal{O}_L .

Let $F_v \subset F_{v, \infty} \subset F_v^{\text{ab}}$ be the intermediate extension determined by $\text{Gal}(F_v^{\text{ab}}/F_{v, \infty}) \cong \ker(\ell_v) \subset F_v^\times$ under the reciprocity isomorphism. Let

$$N_{\infty, \ell_v} H_f^1(F_v, T') := \bigcap_{F'_w} \text{cores}_{F'_w}^{F'_w} (H_f^1(F'_w, T'))$$

be the subgroup of *universal norms*, where the intersection ranges over all finite extensions $F_v \subset F'_w$ contained in $F_{v, \infty}$.

PROPOSITION 4.3.2. *Let $x_1 \in H_f^1(F_v, T)$, $x_2 \in H_f^1(F_v, T^*(1))$, and suppose that the image of x_2 in $H_f^1(F_v, T''^*(1))$ vanishes. Let d_1 be the \mathcal{O}_L -length of $H^1(F_v, T''^*(1))_{\text{tors}}$ and d_2 the length of $H_f^1(F_v, T')/N_{\infty, \ell_v} H_f^1(F_v, T')$. Then*

$$\mathfrak{p}_L^{d_0+d_1+d_2} \langle x_1, x_2 \rangle_{\ell_v, E, v} \subset \ell_v(F_v^\times \hat{\otimes} \mathcal{O}_L)$$

for any mixed extension E . If moreover $[E_v]$ is essentially crystalline, then

$$\mathfrak{p}_L^{d_0+d_1+d_2} \langle x_1, x_2 \rangle_{\ell_v, E, v} \subset \ell_v(\mathcal{O}_{F_v}^\times \hat{\otimes} \mathcal{O}_L).$$

Proof. The first assertion (which implicitly contains the assertion that $H_f^1(F_v, T')/N_{\infty, \ell_v} H_f^1(F_v, T')$ is finite) is identical to [Nek95, Proposition 1.11], whose assumptions however are slightly more stringent. First, L is assumed to be \mathbf{Q}_p ; this requires only cosmetic changes in the proof. Secondly, in [Nek95] the representation V (hence V') is further assumed to be crystalline. This assumption is used via [Nek93, § 6.6] to apply various consequences of the existence of the exact sequence

$$0 \rightarrow H_f^1(F_v, V'^+) \rightarrow H_f^1(F_v, V') \rightarrow H_f^1(F_v, V'^-) \rightarrow 0, \tag{4.3.1}$$

which is established in [Nek93, Proposition 1.25] under the assumption that V' is crystalline. However, (4.3.1) still exists under our assumption that $H^0(F_v, V'^-) = 0$ by [Ben11, Corollary 1.4.6].²⁶

The second assertion follows from the proof of the first one: the height is the image under ℓ_v of an element of $H^1(F_v, L(1)) = F_v^\times \hat{\otimes} L$, which belongs to $H_f^1(F_v, L(1)) = \mathcal{O}_{F_v}^\times \hat{\otimes} L$ if $[E_v]$ is essentially crystalline. \square

5. Generating series and strategy of proof

We introduce the constructions that will serve to prove the main theorem. We have expressed the p -adic L -function as the p -adic Petersson product of a form φ in σ_A and a certain kernel function depending on a Schwartz function ϕ . The connection between the data of (φ, ϕ) and the data of (f_1, f_2) appearing in the main theorem is given by the Shimizu lifting introduced in § 5.1. An arithmetic–geometric analogue of the latter allows us to express also the left-hand side of Theorem B as the image under the p -adic Petersson product of another kernel function, introduced in § 5.3. The main result of this section is thus the reduction of Theorem B to an identity between the two kernel functions (§ 5.4).

5.1 Shimizu’s theta lifting

Let B be a quaternion algebra over a local or global field F , $V = (B, q)$ with the reduced norm q . The action $(h_1, h_2) \cdot x := h_1 x h_2^{-1}$ embeds $(B^\times \times B^\times)/F^\times$ inside $\mathbf{GO}(V)$. If F is a local field, σ is a representation of $\mathbf{GL}_2(F)$, and π is a representation of B^\times , then the space of liftings

$$\text{Hom}_{\mathbf{GL}_2(F) \times B^\times \times B^\times}(\sigma \otimes \mathcal{S}(V \times F^\times), \pi \otimes \pi^\vee)$$

has dimension zero unless either $B = M_2(F)$ and $\pi = \sigma$ or σ is a discrete series and π is its image under the Jacquet–Langlands correspondence; in the latter cases the dimension is one. An explicit generator was constructed by Shimizu in the global coherent case, and we can use it to normalise a generator in the local case and construct a generator in the global incoherent case.

²⁶ The finiteness of $H_f^1(F_v, T')/N_{\infty, \ell_v} H_f^1(F_v, T')$ under our assumption is also in [Nek06, Corollary 8.11.8].

Global lifting. Let B be a quaternion algebra over a number field F , $V = (B, q)$ with the reduced norm. Let σ be a cuspidal automorphic representation of $\mathbf{GL}_2(\mathbf{A}_F)$ which is a discrete series at all places where B is ramified and π the automorphic representation of $B_{\mathbf{A}}^{\times}$ attached to σ by the Jacquet–Langlands correspondence. Fix a non-trivial additive character $\psi : \mathbf{A}/F \rightarrow \mathbf{C}^{\times}$. Consider the theta series

$$\theta(g, h, \Phi) = \sum_{u \in F^{\times}} \sum_{x \in V} r_{\psi}(g, h)\Phi(x, u), \quad g \in \mathbf{GL}_2(\mathbf{A}), h \in (B_{\mathbf{A}}^{\times} \times B_{\mathbf{A}}^{\times})/\mathbf{A}^{\times} \subset \mathbf{GO}(V_{\mathbf{A}})$$

for $\Phi \in \mathcal{S}(V_{\mathbf{A}} \times \mathbf{A}^{\times})$. Then the Shimizu theta lift of any $\varphi \in \sigma$ is defined to be

$$\theta(\varphi, \Phi)(h) := \frac{\zeta_F(2)}{2L(1, \sigma, \text{ad})} \int_{\mathbf{GL}_2(F) \backslash \mathbf{GL}_2(\mathbf{A})} \varphi(g)\theta(g, h, \Phi) dg \in \pi \times \pi^{\vee}$$

and it is independent of the choice of ψ .

If F is totally real, B is totally definite, and $\phi \in \overline{\mathcal{S}}(V_{\mathbf{A}} \times \mathbf{A}^{\times})$, we denote $\theta(\varphi, \phi) := \theta(\varphi, \Phi)$ for any $\mathbf{O}(V_{\infty})$ -invariant preimage Φ of ϕ under (3.1.3). Let $\mathcal{F} : \pi \otimes \pi^{\vee} \rightarrow \mathbf{C}$ be the duality defined by the Petersson bilinear pairing on $B_{\mathbf{A}}^{\times}$ (for the Tamagawa measure). By [Wal85, Proposition 5], we have

$$\begin{aligned} \mathcal{F}\theta(\varphi, \Phi) &= \frac{(\pi^2/2)^{[F:\mathbf{Q}]}}{|D|_F^{3/2} \zeta_F^{\infty}(2)} \prod_{v|\infty} \frac{|d_v|^{-3/2} \zeta_{F,v}(2)^2}{L(1, \sigma_v, \text{ad})} \int_{N(F)_v \backslash \mathbf{GL}_2(F_v)} W_{\varphi,-1,v}(g)r(g)\Phi_v(1, 1) dg \\ &\times \prod_{v|\infty} \frac{2\zeta_{F,v}(2)}{\pi^2 L(1, \sigma_v, \text{ad})} \int_{N(F)_v \backslash \mathbf{GL}_2(F_v)} W_{\varphi,-1,v}(g)r(g)\Phi_v(1, 1) dg. \end{aligned} \tag{5.1.1}$$

The terms in the first line are all rational if $W_{\varphi,-1,v}$ and Φ_v are, and almost all of the factors equal 1.²⁷ For v a real place, if $W_{\varphi,-1,v}$ is the standard antiholomorphic discrete series of weight 2 and Φ_v is a preimage of the standard Schwartz function $\phi_v \in \overline{\mathcal{S}}(V_v \times F_v^{\times})$ under (3.1.2), the terms in the last line are rational too,²⁸ and in fact equal to 1 by the calculation of Lemma 3.5.4.

Local lifting. In the local case, depending on the choice of ψ_v , we can then normalise a generator

$$\theta_v = \theta_{\psi_v} \in \text{Hom}_{\mathbf{GL}_2(F_v) \times B_v^{\times} \times B_v^{\times}}(\mathcal{W}(\sigma_v, \overline{\psi}_v) \otimes \mathcal{S}(V_v \times F_v^{\times}), \pi_v \otimes \pi_v^{\vee})$$

(where $\mathcal{W}(\sigma_v, \overline{\psi}_v)$ is the Whittaker model for σ_v for the conjugate character $\overline{\psi}_v$) by

$$\mathcal{F}_v \theta_v(W, \Phi) = \frac{c_v \zeta_{F,v}(2)}{L(1, \sigma_v, \text{ad})} \int_{N(F)_v \backslash \mathbf{GL}_2(F_v)} W(g)r(g)\Phi(1, 1) dg \tag{5.1.2}$$

with $c_v = |d_v|^{-3/2} \zeta_{F,v}(2)$ if v is finite and $c_v = 2\pi^{-2}$ if v is archimedean. Here the decompositions $\pi = \bigotimes_v \pi_v$, $\pi^{\vee} = \bigotimes_v \pi_v^{\vee}$ are taken to satisfy $\mathcal{F} = \prod_v \mathcal{F}_v$ for the natural dualities $\mathcal{F}_v : \pi_v \otimes \pi_v^{\vee} \rightarrow \mathbf{C}$.

Then by (5.1.1) we have a decomposition

$$\theta = \frac{(\pi^2/2)^{[F:\mathbf{Q}]}}{|D|_F^{3/2} \zeta_F^{\infty}(2)} \otimes_v \theta_v. \tag{5.1.3}$$

As in the global case, we define $\theta_v(W, \phi) := \theta_v(W, \Phi)$ for $\phi = \overline{\Phi} \in \overline{\mathcal{S}}(V_v \times F_v^{\times})$.

²⁷ The analogous assertion in [YZZ12, Proposition 2.3] is incorrect.

²⁸ The analogous statement holds for discrete series of arbitrary weight.

Incoherent lifting. Finally, suppose that F is totally real and \mathbf{B} is a totally definite *incoherent* quaternion algebra over $\mathbf{A} = \mathbf{A}_F$, and let $\mathbf{V} = (\mathbf{B}, q)$. Let σ be a cuspidal automorphic representation of $\mathbf{GL}_2(\mathbf{A})$ which is a discrete series at all places of ramification of \mathbf{B} and let $\pi = \bigotimes_v \pi_v$ be the representation of \mathbf{B}^\times associated to σ by the local Jacquet–Langlands correspondence. Then (5.1.3) defines a lifting

$$\theta \in \text{Hom}_{\mathbf{GL}_2(\mathbf{A}) \times \mathbf{B}^\times \times \mathbf{B}^\times}(\sigma \otimes \overline{\mathcal{S}}(\mathbf{V} \times \mathbf{A}^\times), \pi \otimes \pi^\vee).$$

It coincides with the lifting denoted by the same name in [YZZ12].

Rational liftings. If M is a number field, σ^∞ is an M -rational cuspidal automorphic representation of \mathbf{GL}_2 of weight 2 as in the Introduction, and π is its transfer to an M -rational representation of $\mathbf{B}^{\infty \times}$ under the rational Jacquet–Langlands correspondence of [YZZ12, Theorem 3.4], let σ^ι be the associated complex automorphic representation, and $\pi^\iota = \pi \otimes_{M, \iota} \mathbf{C}$ for any $\iota : M \hookrightarrow \mathbf{C}$. Let $\theta^\iota = \bigotimes_v \theta_v^\iota$ be the liftings just constructed. Then, if \mathbf{B} is coherent, using the algebraic Petersson product of Lemma 2.4.2, there is a lifting $\theta : \sigma^\infty \otimes \overline{\mathcal{S}}(\mathbf{V}^\infty \times \mathbf{A}^{\infty, \times}) \rightarrow \pi \otimes \pi^\vee$, which is defined over M and satisfies

$$\iota\theta(\varphi, \phi^\infty) = \theta^\iota(\varphi^\iota, \phi^\infty \phi_\infty^\iota) \tag{5.1.4}$$

if φ^ι is as described before Lemma 2.4.2 and ϕ_∞^ι is standard. On the left-hand side, we view π and π^\vee indifferently as a representation of $\mathbf{B}^{\infty \times}$ or \mathbf{B}^\times by tensoring on each with generators of the trivial representation at infinite places which pair to 1 under the duality (this ensures compatibility with the decomposition). After base-change to $M \otimes \mathcal{O}_{\Psi_v}(\Psi_v)$, there are local liftings at finite places $\theta_{\psi_{\text{univ}, v}} : \mathcal{W}(\sigma_v, \overline{\psi}_{\text{univ}, v}) \otimes \overline{\mathcal{S}}(V_v \times F_v^\times) \rightarrow \pi_v \otimes \pi_v^\vee \otimes \mathcal{O}_{\Psi_v}(\Psi_v)$ satisfying $\iota\theta_{\psi_{\text{univ}, v}}(W, \phi)(\psi_v) = \theta_{\psi_v}^\iota(W^\iota, \phi^\iota)$. They induce an incoherent global lifting on $\sigma^\infty \otimes \overline{\mathcal{S}}(\mathbf{V}^\infty \times \mathbf{A}^{\infty, \times})$, which is defined over M independently of the choice of an additive character of \mathbf{A} *trivial* on F ,²⁹ and satisfies (5.1.4).

Finally, for an embedding $\iota' : M \hookrightarrow L$ with L a p -adic field, we let

$$\theta_{\iota'} : (\sigma^\infty \otimes \overline{\mathcal{S}}(\mathbf{V}^\infty \times \mathbf{A}^{\infty, \times})) \otimes_M L \rightarrow (\pi \times \pi^\vee) \otimes_{M, \iota'} L$$

be the base-change.

Local toric periods and zeta integrals. Recall from the Introduction that for any dual pair of representations $\pi_v \otimes \pi_v^\vee$ isomorphic (possibly after an extension of scalars) to the local component of $\pi \otimes \pi^\vee$, the normalised toric integrals of matrix coefficients of (1.1.2) are defined, for any $\chi \in \mathcal{Z}_v(\mathbf{C})$, by

$$Q_v(f_{1,v}, f_{2,v}, \chi) = |D|_v^{-1/2} |d|_v^{-1/2} \frac{L(1, \eta_v)L(1, \pi_v, \text{ad})}{\zeta_{F,v}(2)L(1/2, \pi_{E,v}^\iota \otimes \chi_v)} Q_v^\sharp(f_{1,v}, f_{2,v}, \chi_v),$$

$$Q_v^\sharp(f_{1,v}, f_{2,v}, \chi_v) = \int_{E_v^\times / F_v^\times} \chi_v(t)(\pi(t)f_{1,v}, f_{2,v}) dt. \tag{5.1.5}$$

The following lemma follows from the normalisation (5.1.2) and the definitions of the local toric zeta integrals $R_{T,v}^\sharp$ in (3.5.2).

²⁹ In the following sense. Let $\Psi := \prod_v \Psi_v$ and let $\Psi^\circ \subset \Psi$ be defined by $\prod_v \psi_v|_F = 1$. Then the global lifting is first defined on $\sigma^\infty \otimes \overline{\mathcal{S}}(\mathbf{V}^\infty \times \mathbf{A}^{\infty, \times}) \otimes \mathcal{O}_\Psi(\Psi)$. Its restriction to $\sigma^\infty \otimes \overline{\mathcal{S}}(\mathbf{V}^\infty \times \mathbf{A}^{\infty, \times}) \otimes \mathcal{O}_\Psi(\Psi^\circ)$ is invariant under the homogeneous action of \mathcal{O}_F^\times on Ψ° and hence is the base-change of an M -linear map $\sigma^\infty \otimes \overline{\mathcal{S}}(\mathbf{V}^\infty \times \mathbf{A}^{\infty, \times}) \rightarrow \pi \times \pi^\vee$.

LEMMA 5.1.1. Let $\chi \in \mathcal{Y}_{M(\alpha)}^{1.c.}(\mathbf{C})$ and $\psi = \bigotimes_v \psi_v : \mathbf{A}/F \rightarrow \mathbf{C}$ be a non-trivial additive character. Let Q_v and Q_v^\sharp be the pairings defined above for the representation $\theta_{\psi,v}(\mathcal{W}(\sigma_v, \bar{\psi}_v) \otimes \overline{\mathcal{F}}(V_v \times F_v^\times))$. Then, for all $v \nmid \infty$, we have

$$|d|_v^{-3/2} R_{r,v}^\circ(W_v, \phi_v, \chi_v, \psi_v) = Q_v^\sharp(\theta_{\psi,v}(W_v, \iota\alpha_v(\varpi_v)^{-r_v} w_{r,v}^{-1} \phi_v), \chi_v),$$

where as usual if $v \nmid p$, we set $\alpha_v = 1$, $r_v = 0$, $w_{r,v} = 1$.

If $v \nmid p$, we have

$$\mathcal{R}^\sharp(W_v, \phi_v, \chi_v, \psi_v) = Q_v(\theta_{\psi,v}(W_v, \phi_v), \chi_v)$$

and, for the product $Q^p = \prod_{v \nmid p\infty} Q_v$, we have

$$\prod_{v \nmid p\infty} \mathcal{R}_{r,v}^\sharp(W_v, \phi_v, \chi_v, \psi_v) = \frac{D_F^{3/2} \zeta_F^\infty(2)}{(\pi^2/2)^{[F:\mathbf{Q}]}} Q^p(\theta_\psi(\varphi, \alpha(\varpi)^{-r} w_r^{-1} \phi), \chi).$$

5.2 Hecke correspondences and generating series

Referring to [YZZ12, §3.1] for more details, let us recall some basic notions on the Shimura curves X_U . The set of geometrically connected components is $\pi_0(X_{U,\bar{F}}) \cong F_+^\times \backslash \mathbf{A}^{\infty,\times} / q(U)$. The curve X_U admits a canonical divisor class (the *Hodge class*) $\xi_U = (1/\deg L_U)L_U$ of degree 1 on each geometrically connected component; here

$$L_U = \omega_{X_U/F} + \sum_{x \in X_U(\bar{F})} (1 - e_x^{-1})x,$$

a line bundle defined over F ; here $\omega_{X_U/F}$ is the canonical bundle and e_x is the ramification index (see [YZZ12, §3.1.3] for the precise definition) of the point x .

Hecke correspondences. For $x \in \mathbf{B}^{\infty,\times}$, let $T_x : X_{xUx^{-1}} \rightarrow X_U$ be the translation, given in the complex uniformisation by $T_x([z, y]) = [z, yx]$. Let $p : X_{U \cap xUx^{-1}} \rightarrow X_U$ be the projection and let $Z(x)_U$ be the image of

$$(p, p \circ T_x) : X_{U \cap xUx^{-1}} \rightarrow X_U \times X_U.$$

We view $Z(x)_U$ as a correspondence on X_U , and we will sometimes use the same notation for the image of $Z(x)_U$ in $\text{Pic}(X_U \times X_U)$ (such abuses will be made clear in what follows).

We obtain an action of the Hecke algebra $\mathcal{H}_{\mathbf{B}^{\infty,\times},U} := C_c^\infty(\mathbf{B}^{\infty,\times})^{U \times U}$ of U -bi-invariant functions on $\mathbf{B}^{\infty,\times}$ by

$$\mathbf{T}(h)_U = \sum_{x \in U \backslash \mathbf{B}^{\infty,\times} / U} h(x)Z(x)_U.$$

Note the obvious relation $Z(x)_U = \mathbf{T}(\mathbf{1}_{UxU})_U$. The transpose $\mathbf{T}(h)^\dagger$ equals $\mathbf{T}(h^\dagger)$ with $h^\dagger(x) := h(x^{-1})$. It is then easy to verify that if x has trivial components away from the set of places where U is maximal, we have

$$Z(x)_U^\dagger = Z(q(x)^{-1})_U Z(x)_U. \tag{5.2.1}$$

Finally, for any simple quotient A'/F of J with $M' = \text{End}^0(A')$, we have a \mathbf{Q} -linear map

$$\begin{aligned} \mathbf{T}_{\text{alg}} : \pi_{A'} \otimes_{M'} \pi_{A'^\vee} &\rightarrow \text{Hom}^0(J, J^\vee) \\ f_1 \otimes f_2 &\mapsto f_2^\vee \circ f_1. \end{aligned}$$

If $\iota : M' \hookrightarrow L'$ is any embedding into a p -adic field L' , we denote by

$$T_{\text{alg}, \iota} : \pi_{A'} \otimes_{M'} \pi_{A'^\vee} \otimes_{M'} L' \xrightarrow{\iota} \pi_{A'} \otimes_{M'} \pi_{A'^\vee} \otimes_{\mathbf{Q}} L' \xrightarrow{T_{\text{alg}} \otimes 1} \text{Hom}^0(J, J^\vee) \otimes_{\mathbf{Q}} L'$$

the composition in which the first arrow is deduced from the unique L' -linear embedding $L' \hookrightarrow M' \otimes_{\mathbf{Q}} L'$ whose composition with $M' \otimes_{\mathbf{Q}} L' \rightarrow L'$, $x \otimes y \mapsto \iota(x)y$, is $\text{id}_{L'}$.

Generating series. For any $\phi \in \overline{\mathcal{S}}(\mathbf{V} \times \mathbf{A}^\times)$ invariant under $K = U \times U$, define a generating series

$$Z(\phi) := Z_0(\phi)_U + Z_*(\phi)_U,$$

where

$$\begin{aligned} Z_0(\phi)_U &:= - \sum_{\beta \in F_+^\times \setminus \mathbf{A}^{\infty, \times} / q(U)} \sum_{u \in \mu_{U^p}^2 \setminus F^\times} E_0(\beta^{-1}u, \phi) L_{K, \beta}, \\ Z_a(\phi)_U &:= w_U \sum_{x \in K \setminus \mathbf{B}^{\infty, \times}} \phi(x, aq(x)^{-1}) Z(x)_U \quad \text{for } a \in F^\times, \\ Z_*(\phi)_U &:= \sum_{a \in F^\times} Z_a(\phi)_U \end{aligned}$$

with $w_U = |\{\pm 1\} \cap U|$. Here $L_{K, \beta}$ denotes the component of a Hodge class in $\text{Pic}(X_U \times X_U)_{\mathbf{Q}}$ obtained from the classes L_U (see [YZZ12, § 3.4.4]) and

$$E_0(u, \phi) = \phi(0, u) + W_0(u, \phi)$$

is the constant term of the standard Eisenstein series: its intertwining part $W_0(u, \phi)$ is the value at $s = 0$ of

$$W_0(s, u, \phi) = \int_{\mathbf{A}} \delta(\text{wn}(b))^s r(\text{wn}(b)) \phi(0, u) db,$$

where $\delta(g) = |a/d|^{1/2}$ if $g = \begin{pmatrix} a & * \\ & d \end{pmatrix} k$ with $k \in \mathbf{GL}_2(\widehat{\mathcal{O}}_F) \mathbf{SO}(2, F_\infty)$.

For $g \in \mathbf{GL}_2(\mathbf{A})$, define

$$Z(g, \phi) = Z(r(g)\phi),$$

and similarly $Z_0(g, \phi)_U$, $Z_a(g, \phi)_U$, $Z_*(g, \phi)_U$.

Let $U = U^p U_p$ and c_{U^p} be as in (3.4.3). By [YZZ12, § 3.4.6], the normalised versions

$$\tilde{Z}(g, \phi) := c_{U^p} Z(g, \phi)_U, \quad \tilde{Z}_a(g, \phi) := c_{U^p} Z_a(g, \phi)_U, \quad \dots$$

are independent of U^p . A key result, which is essentially a special case of the main theorem of [YZ09], is that the series $\tilde{Z}(g, \phi)$ defines an automorphic form valued in $\text{Pic}(X \times X)_{\mathbf{Q}}$.

THEOREM 5.2.1. *The map*

$$(\phi, g) \mapsto \tilde{Z}(g, \phi)$$

defines an element

$$\tilde{Z} \in \text{Hom}_{\mathbf{B}^\times \times \mathbf{B}^\times}(\overline{\mathcal{S}}(\mathbf{V} \times \mathbf{A}^\times), C^\infty(\mathbf{GL}_2(F) \backslash \mathbf{GL}_2(\mathbf{A})) \otimes \text{Pic}(X \times X)_{\mathbf{Q}}).$$

Here the target denotes the set of $\text{Pic}(X \times X)$ -valued series \tilde{Z} such that for any linear functional $\lambda : \text{Pic}(X \times X)_{\mathbf{Q}} \rightarrow \mathbf{Q}$, the series $\lambda(\tilde{Z})$ is absolutely convergent and defines an automorphic form. Its constant term is non-holomorphic in general (in fact, only when $F = \mathbf{Q}$ and $\Sigma = \infty$). (However, the geometric kernel that we introduce next will always be a holomorphic cusp form of weight 2.) See [YZZ12, Theorem 3.17 and Lemma 3.18] for the proof of the theorem.

Assume from now on that ϕ_{∞} is standard; we accordingly only write $Z(\phi^{\infty})$, $\tilde{Z}(\phi^{\infty})$, \dots . Define, for each $a \in \mathbf{A}^{\times}$,

$$\tilde{Z}_a(\phi^{\infty}) := c_{U^p} w_U |a| \sum_{x \in K \setminus \mathbf{B}^{\infty \times}} \phi^{\infty}(x, aq(x)^{-1}) Z(x)_U \tag{5.2.2}$$

for any sufficiently small U . This extends the previous definition for $a \in F^{\times}$, and it is easy to check that for every $y \in \mathbf{A}^{\infty, \times}$, $a \in F^{\times}$, we have

$$\tilde{Z}_a\left(\begin{pmatrix} y & \\ & 1 \end{pmatrix}, \phi^{\infty}\right) = |ay|_{\infty} \tilde{Z}_{ay}(\phi^{\infty}) \psi(iay_{\infty}).$$

In other words, the images in $\text{Pic}(X_U \times X_U)$ of the $\tilde{Z}_a(\phi^{\infty})$ are the reduced q -expansion coefficients of $\tilde{Z}(\phi^{\infty})$, in the following sense: for any functional λ as above, $(\lambda(\tilde{Z}_0(y, \phi)))_a$, $(\lambda(\tilde{Z}_a(\phi^{\infty})))_a$ are the reduced q -expansion coefficients of the modular form $\lambda(\tilde{Z})(\phi)$.

Hecke operators and Hecke correspondences. For the following lemma, let S be a set of finite places of F such that for all $v \notin S$, \mathbf{B}_v is split, U_v is maximal, and ϕ_v is standard. Fix any isomorphism $\gamma : \mathbf{B}^S \rightarrow M_2(\mathbf{A}^S)$ of \mathbf{A}^S -algebras carrying the reduced norm to the determinant and $\mathcal{O}_{\mathbf{B}^S}$ to $M_2(\widehat{\mathcal{O}}_F^S)$; such an isomorphism is unique up to conjugation by $\mathcal{O}_{\mathbf{B}^S}^{\times}$.

LEMMA 5.2.2. *Let $U'^S = \mathbf{GL}_2(\widehat{\mathcal{O}}_F^S)$, and identify the commutative algebras $\mathcal{H}_{U'^S}^S = \mathcal{H}_{\mathbf{GL}_2(\mathbf{A}^{\infty}), U'^S}^S$ with $\mathcal{H}_{\mathbf{B}^{\infty \times}, U}^S$ via the isomorphism γ^* induced by γ above. Then, for each $h \in \mathcal{H}_{U'^S}^S$, we have*

$$T(h)Z_*(\phi^{\infty})_U = T(\gamma^*h)_U \circ Z_*(\phi^{\infty})_U.$$

In the left-hand side, we view $Z_*(\phi^{\infty})$ as a reduced q -expansion of central character $z \mapsto Z(z)$, and $T(h)$ is the usual Hecke operator acting by (2.2.1); in the right-hand side, the symbol \circ denotes composition of correspondences on X_U .

In particular, the right-hand side is independent of the choice of γ .

Proof. It suffices to check the statements for the set of generators of the algebra $\mathcal{H}_{U'^S}^S$ consisting of elements $h = h_v h^{vS}$, with h^{vS} the unit of $\mathcal{H}_{U'^S}^{vS}$ and $h_v = \mathbf{1}_{U_v x_v U_v}$ for $x_v = \begin{pmatrix} \varpi^v & \\ & 1 \end{pmatrix}$ or $x_v = \begin{pmatrix} \varpi_v^{\pm 1} & \\ & \varpi_v^{\pm 1} \end{pmatrix}$ and $v \notin S$. In the second case the statement is clear.

Suppose then that $x_v = \begin{pmatrix} \varpi^v & \\ & 1 \end{pmatrix}$. Decomposing $Z_a(\phi)_U = Z_{a^v}(\phi^v)_U Z_{a_v}(\phi_v)_U$, the a th coefficient of the left-hand side equals

$$Z_{a\varpi^v}(\phi^{\infty})_U + Z(\varpi_v)_U Z_{a/\varpi^v}(\phi^{\infty})_U = Z_{a^v}(\phi^v)_U \circ (Z_{a_v \varpi^v}(\phi_v)_U + Z(\varpi_v)_U Z_{a_v/\varpi^v}(\phi_v)_U).$$

It is not difficult to identify this with the a th coefficient of the right-hand side using the Cartan decomposition

$$Z_{a_v}(\phi_v)_U = \sum_{0 \leq j \leq i \leq v(a), i+j=v(a)} Z\left(\begin{pmatrix} \varpi_v^i & \\ & \varpi_v^j \end{pmatrix}\right)_U \tag{5.2.3}$$

and the relation

$$Z\left(\begin{pmatrix} \varpi_v & \\ & 1 \end{pmatrix}\right)_U \circ Z\left(\begin{pmatrix} \varpi_v^i & \\ & \varpi_v^j \end{pmatrix}\right)_U = Z\left(\begin{pmatrix} \varpi_v^{i+1} & \\ & \varpi_v^j \end{pmatrix}\right)_U + Z(\varpi_v)_U Z\left(\begin{pmatrix} \varpi_v^{i-1} & \\ & \varpi_v^j \end{pmatrix}\right)_U$$

valid whenever $i > 0$.³⁰ □

5.3 Geometric kernel function

Fix a point $P \in X^{E^\times}(E^{ab})$ as in the Introduction and, for any $h \in \mathbf{B}^{\infty \times}$, denote

$$[h] := T_h P, \quad [h]^\circ = h - \xi_{q(h)},$$

where we identify $\pi_0(X_{U, \overline{F}}) \cong F_+^\times \backslash \mathbf{A}^\times / q(U)$ so that P_U is in the component indexed by 1 (the point $T(h)P$ is then in the component indexed by $q(h)$; see [YZZ12, §3.1.2]).

LEMMA 5.3.1. *Fix L -linear Hodge splittings on all the abelian varieties A'/F parametrised by J and let $\langle \cdot, \cdot \rangle_{A', *}$ be the associated local (for $* = v$) or global ($* = \emptyset$) height pairings. There are unique local and global height pairings*

$$\langle \cdot, \cdot \rangle_{J, *}: J^\vee(\overline{F}) \times J(\overline{F}) \rightarrow \Gamma_F \hat{\otimes} L$$

such that for any A' and $f'_1 \in \pi_{A'}$, $f'_2 \in \pi_{A'}^\vee$, and any $P_1 \in J^\vee(\overline{F})$, $P_2 \in J(\overline{F})$,

$$\langle P_1, P_2 \rangle_* = \langle f'_2 \circ f'_1(P_1), P_2 \rangle_{A', *}$$

Proof. For each fixed level U , there is a decomposition $J_U^\vee \sim \bigoplus_{A'} A'^\vee \otimes \pi_{A'^\vee}^{\vee, U}$ in the isogeny category of abelian varieties, induced by $P_{A'^\vee} \otimes f'^\vee \mapsto f'^\vee(P_{A'})$. Then the Hodge splittings on each A' induce Hodge splittings on J_U^\vee . The associated pairing on $J_U^\vee \times J_U$ is then the unique one satisfying the required property by the projection formula for heights (see [MT83]). The same formula implies the compatibility with respect to changing U . □

We consider the pairing given by the lemma associated with arbitrary Hodge splittings on $V_p A'$ for $A' \neq A$, and any splittings on $V_p A \otimes L = \bigoplus_{p'} V_{p'} A \otimes_{M_{p'}} L$ which induce the canonical one on $V_p A$. The subscript J will be generally omitted when there is no risk of confusion.

Let ϕ be a Schwartz function and $\tilde{Z}(\phi)$ be as above. Each $\tilde{Z}_a(\phi)$ gives a map

$$\tilde{Z}_a(\phi) : J(\overline{F})_{\mathbf{Q}} \rightarrow J^\vee(\overline{F})_{\mathbf{Q}}$$

by the action of Hecke correspondences. When a has trivial components at infinity and ϕ^∞ is standard, we write $\tilde{Z}_a(\phi^\infty) := \tilde{Z}_a(\phi)$. Then, for $g \in \mathbf{GL}_2(\mathbf{A})$, $h_1, h_2 \in \mathbf{B}^{\infty \times}$, we define the height generating series

$$\tilde{Z}(g, h_1, h_2, \phi^\infty) := \langle \tilde{Z}(g, \phi)[h_1]^\circ, [h_2]^\circ \rangle.$$

PROPOSITION 5.3.2. *The series $\tilde{Z}(g, h_1, h_2, \phi)$ is well defined independently of the choice of the point P . It is invariant under the left action of $T(F) \times T(F)$ and it belongs to the space of weight-2 cuspforms $S_2(K', \Gamma_F \hat{\otimes} L)$ for a suitable open compact subgroup $K' \subset \mathbf{GL}_2(\mathbf{A}^\infty)$.*

³⁰Note that a term with $i = 0$ appears in (5.2.3) only in the case $v(a) = 0$, which is easily dealt with separately.

Proof. We explain the modularity with coefficients in the p -adic vector space $\Gamma_F \hat{\otimes} L$. Suppose that U is small enough so that ϕ is invariant under U and U is invariant under the conjugation action of h_1, h_2 . Pick a finite abelian extension E' of E such that $P \in X_U(E')$ and a basis $\{z_i\}$ of $J^\vee(E')_{\mathbf{Q}}$, and let $e_i : J^\vee(E')_{\mathbf{Q}} \rightarrow \mathbf{Q}$ be the projection onto the line spanned by z_i . Then we can write

$$\tilde{Z}(g, (h_1, h_2), \phi^\infty) = \sum_i \langle z_i, [h_2]^0 \rangle \lambda_i(\tilde{Z}(g, \phi^\infty)),$$

where $\lambda_i(T) = e_i(T[h_1]^0)$. Each summand $\lambda_i(\tilde{Z}(g, \phi^\infty))$ is automorphic by Theorem 5.2.1, and in fact a holomorphic cuspform by [YZZ12, Lemma 3.19] (the weight can be easily computed from the shape of ϕ_∞). The other statements are also proved in [YZZ12]. \square

Define

$$\begin{aligned} \tilde{Z}(g, \phi^\infty, \chi) &:= \int_{T(F) \backslash T(\mathbf{A})/Z(A)}^* \chi(t) \tilde{Z}(g, (t, 1), \phi^\infty) d^\circ t \\ &= \int_{T(F) \backslash T(\mathbf{A})/Z(A)}^* \chi(t) \tilde{Z}(g, (1, t^{-1}), \phi^\infty) d^\circ t \\ &= \int_{T(F) \backslash T(\mathbf{A})/Z(A)} \chi(t) \tilde{Z}_{\omega^{-1}}(g, (1, t^{-1}), \phi^\infty) d^\circ t, \end{aligned}$$

where

$$\tilde{Z}_{\omega^{-1}}(g, (1, t^{-1}), \phi^\infty) = \int_{Z(\mathbf{A})} \omega^{-1}(z) \tilde{Z}(g, (1, z^{-1}t^{-1}), \phi^\infty) dz.$$

Note that we have

$$\tilde{Z}(\phi^\infty, \chi) = |D_E|^{1/2} \tilde{Z}^{[YZZ]}(\phi^\infty, \chi) \tag{5.3.1}$$

if $\tilde{Z}^{[YZZ]}(\phi^\infty, \chi)$ is the function denoted by $\tilde{Z}(\chi, \phi)$ in [YZZ12, §§ 3.6.4 and 5.1.2].

5.4 Arithmetic theta lifting and kernel identity

Similarly to [YZZ12], we conclude this section by reducing our main theorem to the form which we will prove, namely as an identity between two kernel functions. The fundamental ingredient is the following theorem of Yuan–Zhang–Zhang.

THEOREM 5.4.1 (Arithmetic theta lifting). *Let σ_A^∞ be the M -rational automorphic representation of $\mathbf{GL}_2(\mathbf{A})$ attached to A . For any $\varphi \in \sigma^\infty$, we have*

$$(\varphi, \tilde{Z}(\phi^\infty))_{\sigma^\infty} = \mathbf{T}_{\text{alg}}(\theta(\varphi, \phi^\infty))$$

in $\text{Hom}(J, J^\vee) \otimes M$.

For any $\varphi^\infty \in \sigma^\infty$, we have

$$(\varphi, \tilde{Z}(\phi^\infty))_{\sigma^\infty} = |D_F| \mathbf{T}_{\text{alg}}(\theta(\varphi, \phi^\infty))$$

in $\text{Hom}(J, J^\vee) \otimes M$.

Let $\iota_p : M \hookrightarrow M_p \subset L$. For each $\varphi^p \in \sigma_A^{p\infty} \otimes L$, completing φ^p to a normalised $(U_v^*)_v|_p$ -eigenform $\varphi \in \sigma \otimes L$ as before Proposition 2.4.4, we have

$$\ell_{\varphi^p, \alpha}(\tilde{Z}(\phi^\infty)) = |D_F| \mathbf{T}_{\text{alg}}(\theta_{\iota_p}(\varphi, \alpha(\varpi)^{-r} w_r \phi^\infty))$$

for any sufficiently large $r \geq 1$.

Proof. In the first identity, both sides in fact belong to $M(\alpha)$, and the result holds if and only if it holds after applying any embedding $\iota : M(\alpha) \hookrightarrow \mathbf{C}$. It is then equivalent to [YZZ12, Theorem 3.22] via Proposition 2.4.4 and [YZZ12, Proposition 3.16]. The second identity follows from the first one and the properties of $\ell_{\varphi^p, \alpha}$. \square

We can now rephrase the main theorem in the form of the following kernel identity.

THEOREM 5.4.2 (Kernel identity). *Let $\varphi^p \in \sigma_A^{p\infty}$ and let $\phi^{p\infty} \in \overline{\mathcal{S}}(\mathbf{V}^{p\infty} \times \mathbf{A}^{p\infty, \times})$. For any compact open subgroup $U_{T,p} = \prod_v U_{T,v} \subset 1 + (\prod_{v|p} \varpi_v) \mathcal{O}_{E,p}$ such that $\chi_p|_{U_{T,p}} = 1$, let $\phi^\infty = \phi^{p\infty} \phi_{p,U_{T,p}}$, where $\phi_{p,U_{T,p}} = \bigotimes_{v|p} \phi_{v,U_{T,v}}$ with*

$$\phi_{v,U_{T,v}}(x, u) = \delta_{1,U_{T,v} \cap \mathbf{V}_1}(x_1) \mathbf{1}_{\mathcal{O}_{\mathbf{V}_2}}(x_2) \mathbf{1}_{d_v^{-1} \mathcal{O}_{\overline{F},v}^\times}$$

for $\delta_{1,U_{T,v}}$ as in (3.4.5).

Suppose that all primes $v|p$ split in E . Then we have

$$\ell_{\varphi^p, \alpha}(d_F \mathcal{I}(\phi^{p\infty}; \chi)) = 2|D_F| L_{(p)}(1, \eta) \cdot \ell_{\varphi^p, \alpha}(\tilde{Z}(\phi^\infty, \chi)).$$

The proof will occupy most of the rest of the paper (cf. the very end of § 8 below).

PROPOSITION 5.4.3. *If Theorem 5.4.2 is true for some $(\varphi^p, \phi^{p\infty})$ such that for all $v \nmid p\infty$, the local integral $R_v(W_v, \phi_v, \chi_v) \neq 0$, then it is true for all $(W^p, \phi^{p\infty})$, and Theorem B is true for all $f_1 \in \pi, f_2 \in \pi^\vee$.*

Proof. Consider the identity

$$\begin{aligned} & \langle \mathbf{T}_{\text{alg}, \iota_p}(f_1 \otimes f_2) P_\chi, P_{\chi^{-1}} \rangle_J \\ &= \frac{\zeta_F^\infty(2)}{2(\pi^2/2)^{[F:\mathbf{Q}]} |D_E|^{1/2} L(1, \eta)} \prod_{v|p} Z_v^\infty(\alpha_v, \chi_v)^{-1} \cdot d_F L_{p,\alpha}(\sigma_{A,E})(\chi) \cdot Q(f_1, f_2, \chi), \end{aligned} \tag{5.4.1}$$

where $\iota_p : M \hookrightarrow L(\chi)$, and we set

$$P_\chi = \int_{[T]} \mathbf{T}_t(P - \xi_P) \chi(t) dt \in J(\overline{F})_{L(\chi)}.$$

The identity (5.4.1) is equivalent to Theorem B by Lemma 5.3.1, but it has the advantage of making sense, by linearity, for any element of $\pi \otimes \pi^\vee$. By the multiplicity-one result, it suffices to prove it for a single element of this space which is not annihilated by the functional $Q(\cdot, \chi)$. Such element will arise as a Shimizu lift. (The similar assertion on the validity of Theorem 5.4.2 for all $(\varphi^p, \phi^{p\infty})$ follows from the uniqueness of the Shimizu lifting.)

By (3.7.1), we can write

$$\ell_{\varphi^p, \alpha}(d_F \mathcal{I}(\phi^{p\infty}; \chi)) = d_F L_{p,\alpha}(\sigma_E)(\chi) \prod_{v \nmid p\infty} \mathcal{R}_v^\sharp(W_v, \phi_v; \chi_v)$$

(note that as the functional $\ell_{\varphi^p, \alpha}$ is bounded, we can interchange it with the differentiation; the fact that the Leibniz rule does not introduce other terms follows from the vanishing of $\mathcal{I}(\phi^{p\infty}; \chi)$, which will be shown in Proposition 7.1.1(3) below). By Lemma 5.1.1, this equals

$$\frac{|D_F|^{3/2} \zeta_F^\infty(2)}{(\pi^2/2)^{[F:\mathbf{Q}]}} \prod_{v|p} Q_v(\theta_v(W_v, \alpha(\varpi_v)^{-r_v} w_{r,v}^{-1} \phi_v), \chi_v)^{-1} \cdot d_F L_{p,\alpha}(\sigma_E)(\chi) \cdot Q(\theta_{\iota_p}(\varphi, \alpha(\varpi)^{-r} w_r^{-1} \phi), \chi).$$

For the geometric kernel, by Theorem 5.4.1 and the calculation of [YZZ12, §3.6.4], we have

$$\ell_{\varphi^p, \alpha}(\tilde{Z}(\phi^\infty, \chi)) = 2|D_F|^{1/2}|D_E|^{1/2}L(1, \eta)\langle T_{\text{alg}, \iota_p}(\theta_{\iota_p}(\varphi, \alpha(\varpi)^{-r}w_r^{-1}\phi))P_\chi, P_\chi^{-1}\rangle.$$

Then (5.4.1) follows from Theorem 5.4.2 provided we show that, for all $v|p$,

$$Q_v(\theta_v(W_v, \alpha_v(\varpi_v)^{-r}w_{r,v}^{-1}\phi_v), \chi_v) = L(1, \eta_v)^{-1} \cdot Z_v^\circ(\chi_v).$$

This is proved by explicit computation in Proposition A.3.1. □

6. Local assumptions

We list here the local assumptions which simplify the computations, while implying the desired identity in general. We recall on the other hand the essential assumption, valid until the end of this paper, that all primes $v|p$ split in E .

Let S_F be the set of *finite* places of F . We partition it as

$$S_F = S_{\text{non-split}} \cup S_{\text{split}}$$

with the obvious meaning according to the behaviour in E , and further as

$$S_F = S_p \cup S_1 \cup S_2 \cup (S_{\text{non-split}} - S_1) \cup (S_{\text{split}} - S_p - S_2),$$

where:

- $S_p \subset S_{\text{split}}$ is the set of places above p ;
- S_1 is a finite subset of $S_{\text{non-split}}$ containing all places where E/F or F/\mathbf{Q} is ramified, or σ is not an unramified principal series, or χ is ramified, or \mathbf{B} is ramified; we assume that $|S_1| \geq 2$;
- S_2 consists of two places in $S_{\text{split}} - S_p$ at which σ and χ are unramified.

We further denote by S_∞ the set of archimedean places of F .

6.1 Assumptions away from p

Consider the following assumptions from [YZZ12, §5.2].

ASSUMPTION 6.1.1 (Cf. [YZZ12, Assumption 5.2]). The Schwartz function $\phi = \bigotimes \phi_v \in \overline{\mathcal{S}}(\mathbf{B} \times \mathbf{A}^\times)$ is a pure tensor, ϕ_v is standard for any $v \in S_\infty$, and ϕ_v has values in \mathbf{Q} for any $v \in S_F$.

ASSUMPTION 6.1.2 [YZZ12, Assumption 5.3]. For all $v \in S_1$, ϕ_v satisfies

$$\phi_v(x, u) = 0 \quad \text{if } v(uq(x)) \geq -v(d_v) \text{ or } v(uq(x_2)) \geq -v(d_v).$$

ASSUMPTION 6.1.3 [YZZ12, Assumption 5.4]. For all $v \in S_2$, ϕ_v satisfies

$$r(g)\phi_v(0, u) = 0 \quad \text{for all } g \in \mathbf{GL}_2(F_v), u \in F_v^\times.$$

See [YZZ12, Lemma 5.10] for an equivalent condition.

ASSUMPTION 6.1.4 [YZZ12, Assumption 5.5]. For all $v \in S_{\text{non-split}} - S_1$, ϕ_v is the standard Schwartz function $\phi_v(x, u) = \mathbf{1}_{\mathcal{O}_{\mathbf{B}_v}}(x)\mathbf{1}_{d_v^{-1}\mathcal{O}_{F,v}^\times}(u)$.

ASSUMPTION 6.1.5 [YZZ12, Assumption 5.6]. The open compact subgroup $U^p = \prod_{v \nmid p} U_v \subset \mathbf{B}(\mathbf{A}^{p\infty})$ satisfies the following:

- (i) U_v is of the form $(1 + \varpi_v^r \mathcal{O}_{\mathbf{B}_v})^\times$ for some $r \geq 0$;
- (ii) χ is invariant under $U_T^p := U^p \cap T(\mathbf{A}^{p\infty})$;
- (iii) ϕ is invariant under $K^p = U^p \times U^p$;
- (iv) U_v is maximal for all $v \in S_{\text{non-split}} - S_1$ and all $v \in S_2$;
- (v) $U^p U_{0,p}$ does not contain -1 ;
- (vi) $U^p U_{0,p}$ is sufficiently small so that each connected component of the complex points of the Shimura curve X_U is an unramified quotient of \mathfrak{H} under the complex uniformisation.

Here we have denoted by $U_{0,p} \subset \mathbf{B}_p^\times$ the maximal compact subgroup.

See [YZZ12, § 5.2.1] for an introductory discussion of the effect of those assumptions.

LEMMA 6.1.6. *For each $v \nmid p\infty$, there exist $W_v \in \sigma_v$ and a Schwartz function ϕ_v satisfying all of Assumptions 6.1.1–6.1.5 such that*

$$R_v^{\natural}(W_v, \phi_v) \neq 0.$$

For all but finitely many places v , we can take W_v to be an unramified vector and ϕ_v to be the standard Schwartz function.

Proof. The existence of the sought-for pairs (W_v, ϕ_v) is proved in [YZZ12, Proposition 5.8]. The second assertion follows from the unramified calculation, Lemma 3.5.2. \square

6.2 Assumptions at p

We make some further assumptions at the places $v \in S_p$. After stating the restrictions on ϕ_v and U_v , we will impose at the end of this section some restrictions on χ_v and on choices of a p -adic logarithm.

Concerning (ϕ_v, U_v) , we need two conditions. On the one hand, that the centre of the open compact subgroup U_v is sufficiently large so that, roughly speaking, for *all but finitely many* characters χ , no non-zero vector in a $T(F_v)$ -representation can be both χ_v -isotypic and invariant under $U_{T,v} := U \cap T(F_v)$; we will apply this in particular for the space generated by the Hodge classes on X_U . On the other hand, we need ϕ_v to be sufficiently close (a condition depending on χ_v) to a Dirac delta, so as to match the Schwartz functions used in the construction of the p -adic analytic kernel. As stated, this is apparently incompatible with the previous condition. However, as $\chi_v|_{F_v^\times} = \omega_v^{-1}$ (fixed), an agreeable compromise can be found. We therefore state two distinct assumptions; while we will eventually work with the second assumption (the ‘compromise’), it will be convenient to reduce some proofs to the situation of the first one.

ASSUMPTION 6.2.1. For each $v \in S_p$, U_v and ϕ_v satisfy the following:

- (i) the subgroup $U_v = 1 + \varpi_v^{r_v} \mathcal{O}_{\mathbf{B}_v}$ for some $r_v \geq 1$;
- (ii) χ_v is invariant under $U_{T,v}$;
- (iii) α_v is invariant under $q(U_v)$;
- (iv) the Schwartz function is

$$\phi_v(x, u) = \delta_{1, U_{T,v}}(x_1) \mathbf{1}_{\mathcal{O}_{\mathbf{V}_2}}(x_2) \delta_{q(U)}(u),$$

where, as in (3.4.5),

$$\delta_{1,U_{T,v}}(x_1) := \frac{\text{vol}(E_v)}{\text{vol}(U_{T,v})} \mathbf{1}_{U_{T,v}}(x_1)$$

and

$$\delta_{q(U)}(u) = \frac{\text{vol}(\mathcal{O}_F^\times)}{\text{vol}(q(U))} \mathbf{1}_{q(U)}(u).$$

ASSUMPTION 6.2.2. For each $v \in S_p$, U_v and ϕ_v satisfy:

- (i) $U_v = U_{F,v}^\circ \tilde{U}_v$ with $U_{F,v}^\circ = (1 + \varpi_v^{n_v} \mathcal{O}_{F,v}^\times) \subset Z(F_v) \subset \mathbf{B}_v^\times$ for some $n_v \geq 1$, and $\tilde{U}_v = 1 + \varpi_v^{r_v} \mathcal{O}_{\mathbf{B},v}$ satisfies (i)–(ii) of Assumption 6.2.1;
- (ii) ω_v is invariant under $U_{F,v}^\circ$;
- (iii) $q(U_v) \subset (U_{F,v}^\circ)^2$;
- (iv) α_v is invariant under $(U_{F,v}^\circ)^2$;
- (v) the Schwartz function ϕ_v is

$$\phi_v := \int_{\mathcal{O}_{F,v}^\times} \omega_v(z) r((z, 1)) \tilde{\phi}_v dz, \tag{6.2.1}$$

where $\tilde{\phi}_v$ is as in Assumption 6.2.1 for \tilde{U}_v .

Remark 6.2.3. By (ii), the function ϕ_v in Assumption 6.2.2 is invariant under $K_v = U_v \times U_v$. The subgroup $U_{F,v}^\circ$ in Assumption 6.2.2 can be chosen *independently* of χ .

In view of the previous remark, we can introduce the following assumption after fixing $U_{F,v}^\circ$.

ASSUMPTION 6.2.4. The character χ is *not* invariant under $V_p^\circ := \prod_{v|p} q^{-1}(U_{F,v}^\circ) \subset \mathcal{O}_{E,p}^\times$.

LEMMA 6.2.5. *The set of finite-order characters $\chi \in \mathcal{Y}$ which do not satisfy Assumption 6.2.4 is finite.*

Proof. Recall that by definition $\mathcal{Y} = \mathcal{Y}_\omega(V^p)$ parametrises some V^p -invariant characters for the open compact subgroup $V^p \subset E_{\mathbf{A}^p}^\times$ fixed (arbitrarily) in the Introduction. Then a character χ as in the lemma factors through

$$E_{\mathbf{A}^\infty}^\times / \overline{E^\times V^p V_p^\circ},$$

a finite group. □

p-adic logarithms. Recall that a p -adic logarithm valued in a finite extension L of \mathbf{Q}_p is a continuous homomorphism

$$\ell : \Gamma_F \rightarrow L;$$

we call it *ramified* if for all $v|p$ the restriction $\ell_v := \ell|_{F_v^\times}$ is ramified, i.e. non-trivial on $\mathcal{O}_{F,v}^\times$.

LEMMA 6.2.6. *For any finite extension L of \mathbf{Q}_p , the vector space of continuous homomorphisms $\text{Hom}(\Gamma_F, L)$ admits a basis consisting of ramified logarithms.*

Proof. If $F = \mathbf{Q}$, then $\text{Hom}(\Gamma_{\mathbf{Q}}, L)$ is one dimensional with generator the cyclotomic logarithm $\ell_{\mathbf{Q}}$, which is ramified. For general F , $\ell_{\mathbf{Q}} \circ N_{F/\mathbf{Q}} : \Gamma_F \rightarrow \Gamma_{\mathbf{Q}} \rightarrow \mathbf{Q}_p$ is ramified (and it generates $\text{Hom}(\Gamma_F, L)$ if the Leopoldt conjecture for F holds). Any other logarithm ℓ can be written as $\ell = a \ell_{\mathbf{Q}} \circ N_{F/\mathbf{Q}} + (\ell - a \ell_{\mathbf{Q}} \circ N_{F/\mathbf{Q}})$ for any $a \in L$; for all but finitely many values of a , both summands are ramified. □

7. Derivative of the analytic kernel

For this section, we retain all the notation of §§ 3.2–3.4, and we keep the assumption that \mathbf{V} is incoherent. We assume that all $v|p$ split in E .

7.1 Whittaker functions for the Eisenstein series

We start by studying the incoherent Eisenstein series \mathcal{E} .

PROPOSITION 7.1.1.

- (1) Let $a \in F_v^\times$.
 - (a) If a is not represented by $(\mathbf{V}_{2,v}, uq)$, then $W_{a,v}^\circ(g, u, \mathbf{1}) = 0$.
 - (b) (Local Siegel–Weil formula.) If there exists $x_a \in \mathbf{V}_{2,v}$ such that $uq(x_a) = a$, then

$$W_{a,v}^\circ\left(\begin{pmatrix} y & \\ & 1 \end{pmatrix}, u, \mathbf{1}\right) = \int_{E_v^1} r\left(\begin{pmatrix} y & \\ & 1 \end{pmatrix}, h\right) \phi_{2,v}(x_a, u) dh.$$

- (2) For any $a, u \in F^\times$, there is a place $v \nmid p$ of F such that a is not represented by (\mathbf{V}_2, uq) .
- (3) For any $\phi_2^{p\infty} \in \mathcal{S}(\mathbf{V}_2^{p\infty} \times \mathbf{A}^{p\infty, \times})$, $u \in F^\times$, we have

$$\mathcal{E}(u, \phi_2^{p\infty}; \mathbf{1}) = 0$$

and consequently

$$\mathcal{I}_F(\phi^{p\infty}; \mathbf{1}) = 0, \quad \mathcal{I}(\phi^{p\infty}; \chi) = 0$$

for any $\phi^{p\infty} \in \mathcal{S}(\mathbf{V}^{p\infty} \times \mathbf{A}^{p\infty, \times})$, $\chi \in \mathcal{Y}_\omega$.

Proof. Part (1) is [YZZ12, Proposition 6.1] rewritten in our normalisation: except for (b) when $v|p$, which is verified by explicit computation of both sides (recall that $\phi_{2,v}$ is standard when $v|p$). Part (2) is a crucial consequence of the incoherence, proved in [YZZ12, Lemma 6.3]. In view of the expansion of Proposition 3.2.1, the vanishing is a consequence of the vanishing of the non-zero Whittaker functions (which is implied by the previous local results) and of

$$W_0(u, \mathbf{1}) = -L^{(p)}(0, \eta) \prod_v W_{0,v}^\circ(u, \mathbf{1}):$$

here we have

$$L^{(p)}(0, \eta) = \frac{L(0, \eta)}{\prod_{v|p} L(0, \eta_v)} = 0$$

as $L(0, \eta)$ is defined and non-zero whereas $L(s, \eta_v)$ has a pole at $s = 0$ when v splits in E . \square

7.2 Decomposition of the derivative

Fix henceforth a tangent vector $\ell \in \text{Hom}(\Gamma_F, L(\chi)) \cong T_1 \mathcal{Y}_F \otimes L(\chi) \cong \mathcal{N}_{\mathcal{Y}'/\mathcal{Y}'|\chi}^*$; we assume that ℓ is ramified when viewed as a p -adic logarithm (cf. Lemma 6.2.6). For any function f on \mathcal{Y}_F , we denote by

$$f'(\mathbf{1}) = D_\ell f(\mathbf{1})$$

the corresponding directional derivative.

Our goal is to compute, for any locally constant χ , the derivative

$$\begin{aligned} \mathcal{I}'(\phi^{p\infty}; \chi) &= \int_{[T]}^* \chi(t) \mathcal{I}'_F((t, 1), \phi^{p\infty}; \mathbf{1})_U d^\circ t \\ &= \int_{[T]}^* \chi(t) \mathcal{I}'_F((1, t^{-1}), \phi^{p\infty}; \mathbf{1})_U d^\circ t, \end{aligned}$$

where the first identity (of q -expansions) follows from the vanishing of the values $\mathcal{I}'_F(\phi^{p\infty}; \mathbf{1})$.

We can decompose the derivative into a sum of q -expansions indexed by the non-split finite places v . For each $u \in F^\times$ and each place v of F , let $F_u(v)$ be the set of those $a \in F^\times$ represented by (\mathbf{V}_2^v, uq) ; by Proposition 7.1.1, we have $W_{a,v}^\circ(u, \mathbf{1}) = 0$ for each $a \in F_u(v)$, and moreover $F_u(v)$ is always empty if v splits in E .

Then

$$\mathcal{E}'(u; \mathbf{1}) = \frac{2^{[F:\mathbf{Q}]} |D_F|^{1/2}}{|D_E|^{1/2} L(p)(1, \eta)} \mathcal{W}'_0(u; \mathbf{1}) - \frac{2^{[F:\mathbf{Q}]} |D|^{1/2}}{|D_E|^{1/2} L(p)(1, \eta)} \sum_{\substack{v \text{ non-split} \\ a \in F_u(v)}} \mathcal{W}'_{a,v}(u; \mathbf{1}) \mathcal{W}_{a,v}^\circ(u; \mathbf{1}) \mathbf{q}^a.$$

For a non-split finite place v , let

$$\begin{aligned} \mathcal{E}'(u, \phi_2^{p\infty}; \mathbf{1})(v) &:= -\frac{2^{[F:\mathbf{Q}]} |D_F|^{1/2}}{|D_{E/F}|^{1/2} L(p)(1, \eta)} \sum_{a \in F_u(v)} \mathcal{W}'_{a,v}(u; \mathbf{1}) \mathcal{W}_{a,v}^\circ(u; \mathbf{1}) \mathbf{q}^a, \\ \mathcal{I}'_F((t_1, t_2), \phi^{p\infty}; \mathbf{1})(v) &:= c_{U^p} \sum_{u \in \mu_{U^p}^2 \setminus F^\times} \theta(u, r(t_1, t_2) \phi_1) \mathcal{E}'(u, \phi_2^{p\infty}; \mathbf{1})(v), \\ \mathcal{I}'(\phi^{p\infty}; \chi)(v) &:= \int_{[T]}^* \chi(t) \mathcal{I}'_F((1, t^{-1}), \phi^{p\infty}; \mathbf{1})(v) d^\circ t \end{aligned}$$

if $\phi^{p\infty} = \phi_1^{p\infty} \otimes \phi_2^{p\infty}$, with ϕ_1 obtained from $\phi_1^{p\infty}$ as in (3.4.5), and extended by linearity in general.

PROPOSITION 7.2.1. Under Assumption 6.1.2, we have

$$\mathcal{I}'_F(\phi^{p\infty}; \mathbf{1}) = \sum_{v \text{ non-split}} \mathcal{I}'_F(\phi^{p\infty}; \mathbf{1})(v).$$

Proof. By the definitions, we only need to show that under our assumptions we have

$$\mathcal{W}'_0(u; \mathbf{1}) = 0.$$

This is proved similarly to [YZZ12, Proposition 6.7]. □

7.3 Main result on the derivative

We give explicit expressions for the local components at good places, and identify the local components at bad places with certain coherent theta series coming from nearby quaternion algebras $\mathbf{B}(v)$; these theta series will be orthogonal to all forms in σ by the Waldspurger formula and the local dichotomy.

PROPOSITION 7.3.1. Let v be a finite place non-split in E . Then, for any $(t_1, t_2) \in T(\mathbf{A})$, we have

$$\mathcal{I}'_F((t_1, t_2), \phi^{p\infty}; \mathbf{1})(v) = 2|D_F| L(p)(1, \eta) \int_{[T]} \mathcal{K}_{\phi^{p\infty}}^{(v)}((tt_1, tt_2)) dt$$

and

$$\mathcal{I}'(\phi^{p\infty}; \chi)(v) = 2|D_F|L_{(p)}(1, \eta) \int_{[T]}^* \int_{[T]} \mathcal{K}_{\phi^{p\infty}}^{(v)}((t, tt_1^{-1})) dt d^{\circ}t_1,$$

where

$$\begin{aligned} \mathcal{K}_{\phi}^{(v)}(y, (t_1, t_2)) &= \mathcal{K}_{r(t_1, t_2)\phi}^{(v)}(y) \\ &= c_{UP} \sum_{u \in \mu_{UP}^2 \setminus F^{\times}} \sum_{x \in V - V_1} k_{r(t_1, t_2)\phi_v}(y, x, u) r\left(\begin{pmatrix} y & \\ & 1 \end{pmatrix}, (t_1, t_2)\right) \phi^{v\infty}(x, u) \mathbf{q}^{uq(x)} \end{aligned}$$

with $k_{\phi_v}(y, x, u)$ the linear function in ϕ_v given when $\phi = \phi_{1,v} \otimes \phi_{2,v}$ by

$$k_{\phi_v}(y, x, u) := -\frac{|d|_v^{1/2}|D|_v^{1/2}}{\text{vol}(E_v^1)} r\left(\begin{pmatrix} y_v & \\ & 1 \end{pmatrix}\right) \phi_{1,v}(x_1, u) \mathcal{W}_{uq(y_2),v}^{\circ'}(y, u, \phi_{2,v}).$$

Proof. This follows from the definitions and the Siegel–Weil formula (Proposition 7.1.1(b)). The computation is as in [YZZ12, Proposition 6.5]. \square

LEMMA 7.3.2. Assume that ϕ^{∞} is \mathbf{Q} -valued. For each non-split finite place v , the values of the function

$$k_{\phi_v}^{\natural}(y, x, u) := \ell(\varpi_v)^{-1} k_{\phi_v}(y, x, u)$$

and the coefficients of the reduced q -expansions

$$\begin{aligned} \mathcal{K}_{\phi^{p\infty}}^{(v)\natural} &:= \ell(\varpi_v)^{-1} \mathcal{K}_{\phi^{p\infty}}^{(v)}, \\ \mathcal{I}'_F{}^{\natural}(\phi^{p\infty})(v) &:= \ell(\varpi_v)^{-1} \mathcal{I}'_F(\phi^{p\infty})(v) \end{aligned}$$

belong to \mathbf{Q} .

Proof. By Lemma 3.3.1, the local Whittaker function $\mathcal{W}_{a,v}^{\circ}(y, u, \phi_{2,v}; \chi_F)$ belongs to $\mathcal{O}(\mathcal{Y}_{F,v}) \cong M[X_v^{\pm 1}]$ and actually to $\mathbf{Q}[X_v^{\pm 1}]$, where $X_v(\chi_{F,v}) := \chi_{F,v}(\varpi_v)$ for any uniformiser ϖ_v . (Recall that the scheme $\mathcal{Y}_{F,v}$ of (3.3.1) parametrises unramified characters of F_v^{\times} .) Therefore, its derivative in the direction ℓ is a rational multiple of $D_{\ell}X_v = \ell(\varpi_v)$. \square

The following is the main result of this section. It is the direct analogue of [YZZ12, Proposition 6.8 and Corollary 6.9] and it is proved in the same way, using Proposition 3.2.3(1). To compare signs with [YZZ12], note that in Proposition 7.3.1 we have preferred to place the minus sign in the definition of k_{ϕ_v} ; and that our $\ell(\varpi_v)$, which is the derivative at $s = 0$ of $\chi_F(\varpi_v)^s$, should be compared with $-\log q_{F,v}$ in [YZZ12] (denoted by $-\log N_v$ there), which is the derivative at $s = 0$ of $|\varpi_v|^s$.

PROPOSITION 7.3.3. Let v be a non-split finite place of F , and let B_v be the quaternion algebra over F_v which is not isomorphic to \mathbf{B}_v .

(1) If $v \in S_{\text{non-split}} - S_1$, then

$$k_{\phi_v}^{\natural}(1, x, u) = \mathbf{1}_{\mathcal{O}_{B_v} \times \mathcal{O}_{F_v}^{\times}}(x, u) \frac{v(q(x_2)) + 1}{2}.$$

- (2) If $v \in S_1$ and ϕ_v satisfies Assumption 6.1.2, then $k_{\phi_v}^{\natural}(y, x, u)$ extends to a rational Schwartz function of $(x, u) \in B_v \times F_v^{\times}$, and we have the identity of q -expansions

$$\mathcal{K}_{\phi}^{(v)\natural}((t_1, t_2)) = \mathfrak{q}\theta((t_1, t_2), k_{\phi_v}^{\natural} \otimes \phi^v),$$

where, for any ϕ' ,

$$\theta(g, (t_1, t_2), \phi') = c_{U^p} \sum_{u \in \mu_{U^p}^2 \setminus F^{\times}} \sum_{x \in V} r(g, (t_1, t_2)) \phi'(x, u)$$

is the usual theta series.

8. Decomposition of the geometric kernel and comparison

We establish a decomposition of the geometric kernel according to the places of F , and compare its local terms away from p with the corresponding local terms in the expansion of the analytic kernel. Together with a result on the local components of the geometric kernel at p proved in §9, this proves the kernel identity of Theorem 5.4.2 (hence Theorem B) when χ satisfies Assumption 6.2.4.

8.1 Vanishing of the contribution of the Hodge classes

Fix a level U as in Assumptions 6.1.5 and 6.2.2.

Recall the height generating series

$$\tilde{Z}((t_1, t_2), \phi^{\infty}) = \langle \tilde{Z}_*(\phi^{\infty})(t_1 - \xi_{q(t_1)}), t_2 - \xi_{q(t_2)} \rangle$$

and the geometric kernel function

$$\tilde{Z}(\phi^{\infty}, \chi) = \int_{[T]}^* \chi(t) \tilde{Z}((1, t^{-1}), \phi^{\infty}) dt.$$

They are modular cuspforms with coefficients in $\Gamma_F \hat{\otimes} L(\chi)$.

PROPOSITION 8.1.1. (1) If Assumption 6.1.3 is satisfied, then

$$\deg \tilde{Z}(\phi^{\infty})_{U, \alpha} = 0$$

for all $\alpha \in F_+^{\times} \setminus \mathbf{A}^{\times} / q(U)$.

- (2) If Assumption 6.1.3 is satisfied, then

$$\tilde{Z}(\phi^{\infty}) \xi_{\alpha} = 0$$

for all $\alpha \in F_+^{\times} \setminus \mathbf{A}^{\times} / q(U)$.

- (3) If Assumption 6.2.4 is satisfied, then

$$\int_{[T]}^* \chi(t) \xi_{U, q(t)} dt = 0.$$

- (4) If Assumptions 6.1.3 and 6.2.4 are both satisfied, then

$$\mathfrak{q} \tilde{Z}(\phi^{\infty}, \chi)_U = \langle \mathfrak{q} \tilde{Z}_*(\phi^{\infty}) \mathbf{1}, t_{\chi} \rangle,$$

where

$$t_{\chi} = \int_{[T]}^* \chi(t) [t^{-1}]_U d^{\circ} t \in \text{Div}^0(X_U)_{L(\chi)}.$$

Proof. Note first that part (4) follows from parts (2) and (3) after expanding and bringing the integration inside. Part (2) follows from part (1) as in [YZZ12, § 7.3.1], and part (1) is proved in [YZZ12, Lemma 7.6].

For part (3), note first that the group V_p° of Assumption 6.2.4 acts trivially on the Hodge classes; in fact, we have $r(1, t^{-1})\xi_{U,\alpha} = \xi_{U,\alpha q(t)}$ and, by definition, $q(V_p^\circ) \subset U$. On the other hand, we are assuming that the character χ is non-trivial on V_p° . It follows that the integration against χ on $V_p^\circ \subset T(\mathbf{A})$ annihilates the Hodge classes. \square

8.2 Decomposition

Let

$$\ell : \Gamma_F \rightarrow L(\chi)$$

be the ramified logarithm fixed in § 7.2. For the rest of this section and in § 9, we will abuse notation by writing $\tilde{Z}(\phi^\infty, \chi)$ for the image of $\tilde{Z}(\phi^\infty, \chi)$ under $\ell : \Gamma_F \hat{\otimes} L(\chi) \rightarrow L(\chi)$.

LEMMA 8.2.1. *If Assumption 6.1.2 is satisfied, then for all $a \in \mathbf{A}^{S_1^\infty, \times}$ and for all $t_1, t_2 \in T(\mathbf{A}^\infty)$, the support of $Z_a(\phi^\infty)t_1$ does not contain $[t_2]$.*

Proof. This is shown in [YZZ12, § 7.2.2]. \square

Let $\bar{\mathbf{S}}' = \bar{\mathbf{S}}'_{S_1}(L(\chi))$ be the quotient space, relative to the set of primes S_1 , introduced after the approximation lemma (Lemma 2.1.2).

PROPOSITION 8.2.2. *Suppose that Assumptions 6.1.2, 6.1.3, and 6.2.4 are satisfied. If H is any sufficiently large finite extension of E and w is a place of H , let $\langle \cdot, \cdot \rangle_{\ell, w}$ be the pairing on $\text{Div}^0(X_{U, H})$ associated with ℓ_w of (4.1.7). Let ${}^{\mathfrak{q}}\tilde{Z}_*(\phi^\infty)$ be the image of $\tilde{Z}_*(\phi^\infty)$ in $\bar{\mathbf{S}}' \otimes \text{Corr}(X \times X)_{\mathbf{Q}}$. Then in $\bar{\mathbf{S}}'$ we have the decomposition*

$${}^{\mathfrak{q}}\tilde{Z}(\phi^\infty, \chi) = \sum_v \tilde{Z}(\phi^\infty, \chi)(v)$$

where

$$\tilde{Z}(\phi^\infty, \chi)(v) = \sum_{w|v} \langle {}^{\mathfrak{q}}\tilde{Z}_*(\phi^\infty)1, t_\chi \rangle_{\ell, w}.$$

Proof. By Proposition 8.1.1(4), we have $\tilde{Z}_a(\phi^\infty, \chi) = \langle \tilde{Z}_a(\phi^\infty)1, t_\chi \rangle$ for all $a \in \mathbf{A}^{\infty, \times}$. If $a \in \mathbf{A}^{S_1^\infty, \times}$, the two divisors have disjoint supports by Lemma 8.2.1, and we can decompose their local height according to (4.2.2). \square

For each place w of E , fix an extension \bar{w} of w to $\bar{F} \supset E$ and, for each finite extension $H \subset \bar{F}$ of E , let $\langle \cdot, \cdot \rangle_{\bar{w}}$ be the pairing associated with $\ell_{\bar{w}} := (1/[H_{\bar{w}} : F_v])\ell_v \circ N_{H_{\bar{w}}/F_v}$. The absence of the field H from the notation is justified by the compatibility deriving from (4.1.6). By the explicit description of the Galois action on CM points, we have

$$\tilde{Z}(\phi^\infty, \chi)(v) = \frac{1}{|S_{E_v}|} \sum_{w \in S_{E_v}} \int_{[T]} \langle {}^{\mathfrak{q}}\tilde{Z}_*(\phi^\infty)t, tt_\chi \rangle_{\bar{w}} dt$$

in $\bar{\mathbf{S}}'$ and, if $v \nmid p$, by Proposition 4.2.2, we can further write

$$\begin{aligned} \tilde{Z}(\phi^\infty, \chi)(v) &= \frac{\ell(\varpi_v)}{|S_{E_v}|} \sum_{w \in S_{E_v}} \int_{[T]}^* \int_{[T]} i_{\bar{w}}({}^{\mathfrak{q}}\tilde{Z}_*(\phi)t, tt_1^{-1})\chi(t_1) dt d^\circ t_1 \\ &\quad + \int_{[T]}^* \int_{[T]} j_{\bar{w}}({}^{\mathfrak{q}}\tilde{Z}_*(\phi)t, tt_1^{-1})\chi(t_1) dt d^\circ t_1. \end{aligned} \tag{8.2.1}$$

8.3 Comparison of kernels

Recall that we want to show the kernel identity

$$\ell_{\varphi^p, \alpha}(\mathcal{I}'(\phi^{p\infty}; \chi)) = 2L_{(p)}(1, \eta)\ell_{\varphi^p, \alpha}(\tilde{Z}(\phi^\infty, \chi)) \tag{8.3.1}$$

of Theorem 5.4.2 (more precisely, we have here projected both sides of that identity to $L(\chi)$ via ℓ).

Similarly to Proposition 8.2.2, we have by Propositions 7.2.1 and 7.3.1 a decomposition of reduced q -expansions

$$\mathcal{I}'(\phi^{p\infty}; \chi) = \sum_{v \text{ non-split}} \mathcal{I}'(\phi^{p\infty}; \chi)(v)$$

with

$$\mathcal{I}'(\phi^{p\infty}; \chi)(v) = 2|D_F|L_{(p)}(1, \eta) \int_{[T]}^* \int_{[T]} \mathcal{K}_{\phi^{p\infty}}^{(v)}(t, tt_1^{-1})\chi(t_1) dt d^\circ t_1, \tag{8.3.2}$$

and the q -expansion $\mathcal{K}_{\phi^{p\infty}}^{(v)\natural} = \mathcal{K}_{\phi^{p\infty}}^{(v)} \cdot \ell(\varpi_v)^{-1}$ has rational coefficients.

We thus state the main theorem on the local components of the kernel function from which the identity (8.3.1) will follow, preceded by a result on the components away from p which facilitates the comparison with [YZZ12].

PROPOSITION 8.3.1. *Suppose that all of the assumptions of § 6.1 are satisfied together with Assumptions 6.2.2. Then for all $t_1, t_2 \in T(\mathbf{A})$ we have the following identities of reduced q -expansions in $\overline{\mathbf{S}}'$.*

(1) *If $v \in S_{\text{split}} - S_p$, then*

$$i_{\bar{v}}(\mathfrak{q}\tilde{Z}_*(\phi^\infty)t_1, t_2) = j_{\bar{v}}(\mathfrak{q}\tilde{Z}_*(\phi^\infty)t_1, t_2) = 0.$$

(2) *If $v \in S_{\text{non-split}} - S_1$, then*

$$i_{\bar{v}}(\mathfrak{q}\tilde{Z}_*(\phi^\infty)t_1, t_2) = \mathcal{K}_{\phi^{p\infty}}^{(v)\natural}(t_1, t_2), \quad j_{\bar{v}}(\mathfrak{q}\tilde{Z}_*(\phi^\infty)t_1, t_2) = 0.$$

(3) *If $v \in S_1$, then there exist Schwartz functions $k_{\phi_v}, m_{\phi_v}, l_{\phi_v} \in \overline{\mathcal{S}}(B(v)_v \times F_v^\times)$ depending on ϕ_v and U_v such that*

$$\begin{aligned} \mathcal{K}_{\phi^{p\infty}}^{(v)\natural}(t_1, t_2) &= \mathfrak{q}\theta((t_1, t_2), k_{\phi_v} \otimes \phi^v), \\ i_{\bar{v}}(\mathfrak{q}\tilde{Z}_*(\phi^\infty)t_1, t_2) &= \mathfrak{q}\theta((t_1, t_2), m_{\phi_v} \otimes \phi^v), \\ j_{\bar{v}}(\mathfrak{q}\tilde{Z}_*(\phi^\infty)t_1, t_2) &= \mathfrak{q}\theta((t_1, t_2), l_{\phi_v} \otimes \phi^v). \end{aligned}$$

Here $B(v)$ is the coherent nearby quaternion algebra to \mathbf{B} obtained by changing invariants at v and, for $\phi' \in \overline{\mathcal{S}}(B(v)_{\mathbf{A}} \times \mathbf{A}^\times)$, we have the automorphic theta series

$$\theta(g, (t_1, t_2), \phi') = c_{U^p} \sum_{u \in \mu_{U^p}^2 \setminus F^\times} \sum_{x \in B(v)} r(g, (t_1, t_2))\phi'(x, u).$$

We denote by

$$I(\phi', \chi)(g) := \int_{[T]}^* \int_{[T]} \chi(t_1)\theta(g, (t, t_1^{-1}t), \phi') dt d^\circ t_1 \tag{8.3.3}$$

the associated coherent theta function.

Proof. Part (1) is [YZZ12, Theorem 7.8(1)]. In part (2), the vanishing of $j_{\bar{v}}$ follows by the definitions; the other identity is obtained by explicit computation of both sides as in [YZZ12, Proposition 8.8], which gives the expression for the geometric side;³¹ on the analytic side we use the result of Proposition 7.3.3(1). Part (3) for $\mathcal{K}_{\phi^{p\infty}}^{(v)\natural}$ is Proposition 7.3.3(2), whereas for $i_{\bar{v}}$ and $j_{\bar{v}}$ it is [YZZ12, Theorem 7.8(4)]. \square

THEOREM 8.3.2. *Suppose that all of the assumptions of § 6.1 are satisfied together with Assumptions 6.2.2 and 6.2.4. Then we have the following identities of reduced q -expansions in $\bar{\mathbf{S}}'$.*

(1) *If $v \in S_{\text{split}} - S_p$, then*

$$\tilde{Z}(\phi^\infty, \chi)(v) = 0.$$

(2) *If $v \in S_{\text{non-split}} - S_1$, then*

$$\mathcal{I}'(\phi^{p\infty}; \chi)(v) = 2|D_F|L_{(p)}(1, \eta)\tilde{Z}(\phi^\infty, \chi)(v).$$

(3) *If $v \in S_1$, then there exist Schwartz functions $k_{\phi_v}, n_{\phi_v} \in \overline{\mathcal{S}}(B(v)_v \times F_v^\times)$ depending on ϕ_v and U_v such that, with the notation (8.3.3),*

$$\begin{aligned} \mathcal{I}'(\phi^{p\infty}; \chi)(v) &= \mathfrak{q}I(k_{\phi_v} \otimes \phi^v, \chi), \\ \tilde{Z}(\phi^\infty, \chi)(v) &= \mathfrak{q}I(n_{\phi_v} \otimes \phi^v, \chi). \end{aligned}$$

(4) *The sum*

$$\tilde{Z}(\phi^\infty, \chi)(p) := \sum_{v \in S_p} \tilde{Z}(\phi^\infty, \chi)(v)$$

belongs to the isomorphic image $\bar{\mathbf{S}} \subset \bar{\mathbf{S}}'$ of the space of p -adic modular forms \mathbf{S} , and we have

$$\ell_{\varphi^p, \alpha}(\tilde{Z}(\phi^\infty, \chi)(p)) = 0.$$

Proof (to be completed in § 9). Parts (1)–(3) follow from Proposition 8.3.1 by integration via (8.2.1) and (8.3.2). They imply the identity in $\bar{\mathbf{S}}'$

$$\begin{aligned} 2|D_F|L_{(p)}(1, \eta)\tilde{Z}(\phi^\infty, \chi)(p) &= 2|D_F|L_{(p)}(1, \eta)\mathfrak{q}\tilde{Z}(\phi^\infty, \chi) - \sum_{v \nmid p} \tilde{Z}(\phi^\infty, \chi)(v) \\ &= 2|D_F|L_{(p)}(1, \eta)\mathfrak{q}\tilde{Z}(\phi^\infty, \chi) - \mathcal{I}'(\phi^\infty; \chi) \\ &\quad - \sum_{v \in S_1} \mathfrak{q}I(\phi^v \otimes d_{\phi_v}), \end{aligned} \tag{8.3.4}$$

where $d_{\phi_v} = 2|D_F|L_{(p)}(1, \eta)n_{\phi_v} - k_{\phi_v}$ for $v \in S_1$. As all terms in the right-hand side belong to $\bar{\mathbf{S}}$, so does $\tilde{Z}(\phi^\infty, \chi)(p)$. The proof of the vanishing statement of part (4) will be given in § 9. \square

Proof of Theorem 5.4.2 under Assumption 6.2.4. We show that Theorem 5.4.2 follows from Theorem 8.3.2, under the same assumptions. By Proposition 5.4.3 and Lemma 6.1.6, only Assumption 6.2.4 on the character χ is restrictive. Moreover by Lemma 6.2.6 it is equivalent

³¹ Recall that, on the geometric side, $i_{\bar{v}}$ is the same \mathbf{Q} -valued intersection multiplicity both in [YZZ12] and in our case.

to show that the desired kernel identity holds after applying to both sides a ramified logarithm $\ell : \Gamma_F \rightarrow L(\chi)$.

By (8.3.4),

$$\ell_{\varphi^p, \alpha}(2|D_F|L(1, \eta) \cdot \mathfrak{q}\tilde{Z}(\phi^\infty, \chi) - \mathcal{S}'(\phi^\infty; \chi)) = \sum_{v \in S_1} \ell_{\varphi^p, \alpha}(\mathfrak{q}I(\phi^v \otimes d_{\phi_v})) + \ell_{\varphi^p, \alpha}(\tilde{Z}(\phi^\infty, \chi)(p)).$$

The vanishing of the terms indexed by S_1 can be shown as in [YZZ12, §7.4.3] to follow from the local result of Tunnell and Saito together with Waldspurger’s formula. The term $\ell_{\varphi^p, \alpha}(\tilde{Z}(\phi^\infty, \chi)(p)) = 0$ by part (4) of Theorem 8.3.2. □

9. Local heights at p

After some preparation in §9.1, in §9.2 we prove the vanishing statement of part 4 of Theorem 8.3.2. We follow a strategy of Nekovář [Nek95] and Shnidman [Shn16].

For each $v|p$, fix isomorphisms

$$E_v := E \otimes_F F_v \cong F_v \oplus F_v \tag{9.0.5}$$

and $B_v \cong M_2(F_v)$ such that the embedding of quadratic spaces $E_v \hookrightarrow B_v$ is identified with $(a, d) \mapsto \begin{pmatrix} a & \\ & d \end{pmatrix}$; then for the decomposition $\mathbf{B}_v = \mathbf{V}_{1,v} \oplus \mathbf{V}_{2,v} = E_v \oplus E_v^\perp$, the first (respectively second) factor consists of the diagonal (respectively antidiagonal) matrices. Let w, w^* be the places of E above v such that E_w (respectively E_{w^*}) corresponds to the projection onto the first (respectively second) factor under (9.0.5). We fix the extension $\bar{v} = \bar{w}$ of v to $\bar{F} \supset E$ to be any one inducing w on E , and we will accordingly view the local reciprocity maps $\text{rec}_w : E_w^\times = F_v^\times \rightarrow \text{Gal}(\bar{F}_{\bar{v}}/F_v)^{\text{ab}} = \text{Gal}(\bar{F}_{\bar{v}}/E_w)^{\text{ab}}$.

9.1 Local Hecke and Galois actions on CM points

Let $U = U_F^\circ \tilde{U} = \prod_v U_v \subset \mathbf{B}^{\infty \times}$ be an open compact subgroup and $\phi \in \overline{\mathcal{S}}(\mathbf{B} \times \mathbf{A}^\times)$ satisfy Assumption 6.2.2 for integers $r = (r_v)_{v|p}$. Fix throughout this subsection a prime $v|p$.

By Lemma 3.1.1, the generating series $\tilde{Z}(\phi^\infty)$ is invariant under $K^1(\varpi^{r'})_v$ for some $r' > 0$. We compute the action of the operator $U_{v,*}$ on it.

LEMMA 9.1.1. *For each $a \in \mathbf{A}^\times, v|p$, the a th reduced q -expansion coefficient of $U_{v,*}\tilde{Z}(\phi^\infty)$ equals*

$$Z(\varpi_v^{-1})\tilde{Z}_{a\varpi_v}(\phi^\infty),$$

where the $(a\varpi_v)$ th q -expansion coefficient of $\tilde{Z}(\phi^\infty)$ is given in (5.2.2).

Proof. We have

$$\begin{aligned} U_{v,*}\phi_v(x, u) &= |\varpi_v| \sum_{j \in \mathcal{O}_{F,v}/\varpi_v} r\left(\begin{pmatrix} \varpi_v & j \\ & 1 \end{pmatrix}\right)\phi_v(x, u) \\ &= |\varpi_v|\phi_v(\varpi_v x, \varpi_v^{-1}u) \end{aligned}$$

under our assumptions on ϕ_v .

Inserting this in the definition of \tilde{Z} and performing the change of variables $x' = \varpi_v x$, we obtain the result. □

We wish to give a more explicit expression for

$$\tilde{Z}_{a\varpi_v^s}(\phi^\infty)[1]_U = c_{U^p} |a\varpi^s| \sum_{x \in U \setminus \mathbf{B}^{\infty \times} / U} \phi^\infty(x, a\varpi_v^s q(x)^{-1})[x]_U \tag{9.1.1}$$

as s varies. Let

$$\Xi(\varpi_v^{r_v}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathcal{O}_{F,v}) \mid a, d \in 1 + \varpi_v^{r_v} \mathcal{O}_{F,v} \right\} U_{F,v}^\circ.$$

Then $\Xi(\varpi_v^{r_v}) \subset \mathbf{B}_v$ is the image of the support of ϕ_v under the natural projection $\mathbf{B}_v \times F_v^\times \rightarrow \mathbf{B}_v$; and, for each $x_v \in U_v \setminus \mathbf{B}_v^\times / U_v$, some $x^v \in \mathbf{B}^{v\infty, \times}$ such that $x^v x_v$ contributes to the sum (9.1.1) exists if and only if x_v belongs to

$$\Xi(\varpi_v^{r_v})_{a\varpi_v^{s_v}} := \{x_v \in \Xi(\varpi_v^{r_v}), q(x_v) \in a\varpi_v^{s_v}(1 + \varpi^r \mathcal{O}_{F,v})\}.$$

LEMMA 9.1.2. *Let U_v, ϕ_v be as in Assumption 6.2.2, and let $a \in F_v^\times$ with $s_v = v(a) \geq r_v$. Then the quotient sets $U_v \setminus \Xi(\varpi_v^{r_v})_a / U_v, \Xi(\varpi_v^{r_v})_a / U_v$, and $U_v \setminus \Xi(\varpi_v^{r_v})_a$ are in bijection and, for each of them, the set of elements*

$$x_v(b_v, a) := \begin{pmatrix} 1 & b_v \\ b_v^{-1}(1-a) & 1 \end{pmatrix} \in M_2(F_v) = \mathbf{B}_v, \quad b_v \in (\mathcal{O}_{F,v} / \varpi_v^{r_v+s_v})^\times$$

is a complete set of representatives.

Proof. We drop the subscripts v . By acting on the right with diagonal elements belonging to U , we can bring any element $x \in \Xi(\varpi^r)_a$ to one of the form $x(b, a')$ with $b \in \mathcal{O}_F^\times, a' \in a(1 + \varpi^r \mathcal{O}_F)$. The right action of an element $\gamma \in U$ sends an element $x(b, a)$ to one of the same form $x(b', a')$ if and only if

$$\gamma = \begin{pmatrix} 1 + \lambda\varpi^r & -b\mu\varpi^r \\ -\lambda\varpi^r & 1 - a \\ \frac{-\lambda\varpi^r}{b} & 1 + \varpi^r\mu \end{pmatrix}$$

for some $\lambda, \mu \in \mathcal{O}_F$; in this case, we have

$$b' = b \frac{1 - a\mu\varpi^r}{1 - a}, \quad a' = a \left(1 - \varpi^r(\lambda + \mu) - \varpi^{2r} \frac{\lambda\mu a}{1 - a\varpi^r} \right).$$

The situation when considering the left action of U is analogous (as can be seen by the symmetry $b \leftrightarrow b^{-1}(1 - u\varpi^s)$). The lemma follows. \square

Henceforth we will just write $x_v(b_v)$ for $x_v(b_v, a)$ unless there is risk of confusion.

LEMMA 9.1.3. *Fix $a \in F_v^\times$ with $v(a) \geq r_v$.*

(1) *Let $x = x^v x_v(b_v), c_v \in \mathcal{O}_{F,v} = \mathcal{O}_{E_w}$. The action of the Galois element $\text{rec}_E(1 + c_v \varpi_v^{r_v})$ is*

$$\text{rec}(1 + c_v \varpi_v^{r_v})[x]_U = [x^v x_v(b_v(1 + c_v \varpi_v^{r_v}))]_U. \tag{9.1.2}$$

(2) *We have*

$$\Xi(\varpi_v^{r_v})_a / U_v = \coprod_{\bar{b} \in (\mathcal{O}_F / \varpi^r)^\times} \text{rec}_E((1 + \varpi^{r_v} \mathcal{O}_{F,v} / 1 + \varpi^{s_v} \mathcal{O}_{F,v})) x_v(b) U_v,$$

where b is any lift of \bar{b} to $\mathcal{O}_{F,v} / \varpi^{r_v+s_v}$.

Proof. Both assertions follow from the explicit description of the Galois action on CM points: we have

$$\text{rec}_E(1 + c_v \varpi_v^{r_v})[x]_U = \left[\begin{pmatrix} 1 + c_v \varpi_v^r & \\ & 1 \end{pmatrix} x \right]_U,$$

and a calculation establishes part (1). In view of Lemma 9.1.2, part (2) is then a restatement of the obvious identity $(\mathcal{O}_{F_v}/\varpi_v^{r+s})^\times = ((1 + \varpi^r \mathcal{O}_{F,v})/(1 + \varpi^s \mathcal{O}_{F,v}))(\mathcal{O}_{F_v}/\varpi_v^r)^\times$. \square

Norm relation for the generating series. Let

$$\tilde{Z}_a^v(\phi^v) := c_{U^v} \sum_{x^v \in \mathbf{B}^{v\infty \times}/U} \phi^{v\infty}(x^v, aq(x^v)^{-1})Z(x^v)_U.$$

Then we have

$$\tilde{Z}_{a\varpi^s}(\phi)[1]_U = |\varpi_v|_v^{-2r_v} \sum_{x_v} \tilde{Z}_a^v(\phi^v)[x_v]_U,$$

where the sum runs over $x_v \in \Xi(\varpi_v^{r_v})_{v(a)+v(d)+s_v}/U_v$.

For $s \geq r_v$, let $H_s \subset E^{\text{ab}}$ be the extension of E with norm group

$$U_F^\circ U_T^v(1 + \varpi_w^s \mathcal{O}_{E,w}),$$

where $U_T = U \cap E_{\mathbf{A}^\infty}^\times$. Let $H_\infty = \bigcup_{s \geq r_v} H_s$. If r_v is sufficiently large, for all $s \geq r_v$ the extension H_s/H_{r_v} is totally ramified at \bar{w} , and we have

$$\text{Gal}(H_s/H_r) \cong \text{Gal}(H_{s_v, \bar{w}}/H_{r_v, \bar{w}}) \cong (1 + \varpi_v^{r_v} \mathcal{O}_{F,v})/(1 + \varpi_v^s \mathcal{O}_{F,v}). \tag{9.1.3}$$

For convenience, we set

$$H'_s := H_{r_v+s}$$

for any $s \in \{0, 1, \dots, \infty\}$ and, when $s < \infty$, we denote by Tr_s , and similarly later N_s , the trace (respectively norm) with respect to the field extension H'_s/H'_0 .

PROPOSITION 9.1.4. *Fix any $a \in \mathbf{A}^{\infty, \times}$ with $v(a) = r_v$. With the notation just defined, we have*

$$\tilde{Z}_{a\varpi^s}(\phi)[1]_U = \sum_{i \in I} \sum_{\bar{b} \in (\mathcal{O}_{F,v}/\varpi^r)^\times} \text{Tr}_s c_i [x_i^v x_v(b, a\varpi^s)]_U,$$

where the finite indexing set I , the constants $c_i \in \mathbf{Q}$, and the cosets $x_i^v U^v$ are independent of s . Moreover, there exists an integer $d \neq 0$ independent of a such that $c_i \in d^{-1}\mathbf{Z}$ for all i .

Proof. By Lemma 9.1.3, we can write

$$\begin{aligned} \tilde{Z}_{a\varpi^s}(\phi)[1]_U &= |\varpi_v|_v^{-2r_v} \sum_{\bar{b} \in (\mathcal{O}_F/\varpi_v^{r_v})^\times} \tilde{Z}_a^v(\phi^v)[\text{Gal}(H'_{s, \bar{w}}/H'_{0, \bar{w}}) \cdot x_v(b, a\varpi^s)]_U \\ &= |\varpi_v|_v^{-2r_v} \sum_{\bar{b} \in (\mathcal{O}_F/\varpi_v^{r_v})^\times} \text{Tr}_s(\tilde{Z}_a^v(\phi^v)[x_v(b, a\varpi^s)]_U) \end{aligned}$$

using (9.1.3), as by construction the correspondence $\tilde{Z}_a^v(\phi^v)_U$ is defined over H'_0 . We obtain the result by writing $\tilde{Z}_a^v(\phi^v)_U = \sum_{i \in I} c_i Z(x_i^v)_U$.

Finally, the existence of d follows from the fact that ϕ^∞ is a Schwartz function. \square

The extension $H_{\infty, \bar{w}}/E_w$. After de Shalit [dSha85], given a non-archimedean local field K , we say that an extension $K' \subset K^{\text{ab}}$ of K is a *relative Lubin–Tate extension* if there is a (necessarily unique) finite unramified extension $K \subset K^\circ \subset K'$ such that $K' \subset K^{\text{ab}}$ is maximal for the property of being totally ramified above K° . By local class field theory, for any relative Lubin–Tate extension K' , there exists a unique element $\varpi \in K^\times$ with $v_K(\varpi) = [K^\circ : K]$ (where v_K is the valuation of K) such that $K' \subset K^{\text{ab}}$ is the subfield cut out by $\langle \varpi \rangle \subset K^\times$ via the reciprocity map of K . We call ϖ the *pseudo-uniformiser* associated with K' .

LEMMA 9.1.5. *The field $H_{\infty, \bar{w}}$ is the relative Lubin–Tate extension of E_w associated with a pseudo-uniformiser $\varpi_{\text{LT}} \in E_w^\times$ which is algebraic over E and satisfies $q_w(\varpi_{\text{LT}}) = 1$.*

Proof. It is easy to verify that $H_{\infty, \bar{w}}$ is a relative Lubin–Tate extension. We only need to identify the corresponding pseudo-uniformiser ϖ_{LT} . It suffices in fact to find an element $\theta \in E^\times$ satisfying $q(\theta) = 1$ and lying in the kernel of $\text{rec}_{E_w} : E_w^\times \rightarrow \text{Gal}(H_{\infty, \bar{w}}/E_w)$, as then ϖ_{LT} must be a root of θ and hence also satisfies the required property.

Let $\varpi_w \in E_w$ be a uniformiser at w , and let $d = [E_{\mathbf{A}^\infty}^\times : E^\times U_T]$, where U_T^v is as before, $U_{T,w} \subset \mathcal{O}_{E,w}^\times$ is arbitrary, and U_{T,w^*} is identified with $U_{F,v}^\circ$ under $F_v \cong E_{w^*}$. Then we can find $t \in E^\times$, $u \in U_T$ such that $\varpi_w^d = tu$. We show that the image of $\theta := t/\bar{t}$ in E_w^\times lies in the kernel of $\text{rec}_{E_w} : E_w^\times \rightarrow \text{Gal}(H_{\infty, \bar{w}}/E_w)$. Letting $\iota_w : E_w^\times \hookrightarrow E_{\mathbf{A}^\infty}^\times$ be the inclusion, we show equivalently that $\iota_w(\theta)$ is in the kernel of $\text{rec}_E : E_{\mathbf{A}^\infty}^\times \rightarrow \text{Gal}(H_{\infty, \bar{w}}/E)$ or concretely that $i_w(t/\bar{t}) \in E^\times U_T^v U_F^\circ$. Now we have

$$\iota_w(t/\bar{t}) = t \cdot u^v \iota_w(u_{w^*}) \iota_{w^*}(u_{w^*}).$$

By construction, $u^v \in U_T^v$, and $\iota_w(u_{w^*}) \iota_{w^*}(u_{w^*})$ belongs to $U_{F,v}^\circ$. This completes the proof. \square

9.2 Annihilation of local heights

Suppose still that the open compact U and the Schwartz function ϕ^∞ satisfy all of the assumptions of § 6.1 together with Assumption 6.2.2. In this subsection, we complete the proof of Theorem 8.3.2 by showing that the element $\tilde{Z}(\phi^\infty, \chi)(p) \in \bar{\mathbf{S}}$ is annihilated by $\ell_{\varphi^p, \alpha}$. Let S be a finite set of non-archimedean places of F such that, for all $v \notin S$, all the data are unramified, U_v is maximal, and ϕ_v is standard. Let $K = K^p K_p$ be the level of the modular form $\tilde{Z}(\phi^\infty)$, and let $T_{\iota_p}(\sigma^\vee) \in \mathcal{H}^S(L) = \mathcal{H}^S(M) \otimes_{M, \iota_p} L$ be any element as in Proposition 2.4.4(4). By that result, it suffices to prove that

$$\ell_{\varphi^p, \alpha}(T_{\iota_p}(\sigma^\vee) \tilde{Z}(\phi^\infty, \chi)(p)) = 0. \tag{9.2.1}$$

We will in fact prove the following.

PROPOSITION 9.2.1. *For all $v|p$, the element $T_{\iota_p}(\sigma^\vee) \tilde{Z}(\phi^\infty, \chi)(v) \in \bar{\mathbf{S}}'$ is v -critical in the sense of Definition 2.4.1.*

Recall from § 2.2 that the commutative ring $\mathcal{H}^S(M)$ acts on the space of reduced q -expansions $\mathbf{S}'(K^p)$ and its quotient $\bar{\mathbf{S}}'_{S_1}(K^p)$, so that the expression $T_{\iota_p}(\sigma^\vee) \tilde{Z}(\phi^\infty, \chi)(v)$ makes sense. Proposition 9.2.1 implies that $T_{\iota_p}(\sigma^\vee) \tilde{Z}(\phi^\infty, \chi)(p)$ is a p -critical element of $\bar{\mathbf{S}}$ and hence it is annihilated by $\ell_{\varphi^p, \alpha}$ by Proposition 2.4.4(3), establishing (9.2.1).

By Lemma 5.2.2, there is a Hecke correspondence $\mathbb{T}(\sigma^\vee)_U$ on X_U (with coefficients in M) such that

$$T_{\iota_p}(\sigma^\vee)^{\mathbf{q}} Z_*(\phi^\infty)_U = \mathbb{T}(\sigma^\vee)_U \circ Z_*(\phi^\infty)_U$$

as correspondences on X_U . Then $T_{\iota_p}(\sigma^\vee)\tilde{Z}(\phi^\infty, \chi)_U(v)$ is an average of

$$\langle T_{\iota_p}(\sigma^\vee)_U \mathfrak{q}\tilde{Z}_*(\phi^\infty)_U[1], t_\chi \rangle_{\bar{w}} = \langle \mathfrak{q}\tilde{Z}_*(\phi^\infty)_U[1], T_{\iota_p}(\sigma^\vee)_U t_\chi \rangle_{\bar{w}} \tag{9.2.2}$$

for $\bar{w}|v$.

We study the class of $T_{\iota_p}(\sigma^\vee)_U t_\chi$ in $H_f^1(E, \text{Ind}_E^H V_p J_U|_{\mathcal{G}_E})$, where $H \subset E^{\text{ab}}$ is any sufficiently large finite extension. Let L' denote $\bar{\mathbf{Q}}_p$ or any sufficiently large finite extension of \mathbf{Q}_p . As \mathcal{G}_F -representations, we have

$$V_p J_U \otimes_{\mathbf{Q}_p} L' = \bigoplus_{A', \iota': M_{A'} \hookrightarrow L'} \pi_{A'}^U \otimes_{M_{A'}} V_p A'^{\iota'}$$

for some pairwise non-isogenous simple abelian varieties A'/F with $\text{End}^0 A' = M_{A'}$; here $V_p A'^{\iota'} := V_p A' \otimes_{M_{A'}, \iota'} L'$. More generally, let

$$V := \text{Ind}_E^H V_p J_U|_{\mathcal{G}_H} \tag{9.2.3}$$

for a finite extension H of E as above; then we have

$$V \otimes_{\mathbf{Q}_p} L' = \bigoplus_{A', \iota': M_{A'} \hookrightarrow L', \chi': \text{Gal}(H/E) \rightarrow L'^{\times}} \pi_{A'}^U \otimes_{M_{A'}} V_p A'^{\iota'}(\chi'), \tag{9.2.4}$$

where $V_p A'^{\iota'}(\chi') := V_p A' \otimes_{M_{A'}, \iota'} L'_{\chi'}$.

Let $V' := \pi_{A^\vee}^U \otimes_M V_p A^{\vee \iota_p}(\chi^{-1}) \subset V$, where $\iota_p : M \hookrightarrow L \subset L(\chi)$ is the usual embedding. Let $V'' \subset V$ be its complement in the direct sum (9.2.4), $V = V' \oplus V''$.

PROPOSITION 9.2.2. *The class of $T_{\iota_p}(\sigma^\vee)_U t_\chi$ in $H_f^1(E, V)$ belongs to the subspace $H_f^1(E, V')$.*

Proof. We may and do replace $L(\chi)$ by a sufficiently large finite extension L' . It is clear that the class of t_χ belongs to $H_f^1(E, V_p J_U(\chi^{-1}))$. Then it is enough to verify that $T_{\iota_p}(\sigma^\vee)_U$ annihilates $\bigoplus_{(A', \iota') \neq (A^\vee, \iota_p)} (V_p A'^{\iota'})^{m_{A'}}$. Equivalently, we show that for any $(A', \iota') \neq (A, \iota_p)$ and any $f_1 \in \pi_{A'}$, $f_2 \in \pi_{A'}$,

$$T_{\text{alg}, \iota'}(f_1, f_2) \circ T_{\iota_p}(\sigma^\vee)_U = 0$$

in $\text{Hom}(J_U, J_U^\vee)$.

We have

$$T_{\text{alg}, \iota'}(f_1, f_2) \circ T_{\iota_p}(\sigma^\vee)_U = T_{\iota_p}(\sigma^\vee)_U \circ T_{\text{alg}, \iota'}(f_1, f_2) = T_{\iota_p}(\sigma^\vee)_U \circ T_{\text{alg}, \iota'}(\theta_{\iota'}(\varphi', \phi^{\infty'}))$$

for some $\varphi' \in \sigma_{A'}^\infty$ and some rational Schwartz function $\phi^{\infty'}$. By Theorem 5.4.1, this can be rewritten as

$$T_{\iota_p}(\sigma^\vee)_U \circ (\iota_p \varphi', \tilde{Z}_*(\phi^{\infty'}))_{\sigma_{A'}^\infty} = (\iota_p \varphi', T_{\iota_p}(\sigma^\vee)_U \circ \tilde{Z}_*(\phi^{\infty'}))_{\sigma_{A'}^\infty}$$

using an obvious commutativity. Applying Lemma 5.2.2 again, we have

$$T_{\iota_p}(\sigma^\vee)_U \circ \tilde{Z}_*(\phi^{\infty'}) = \iota_p T(\sigma) \tilde{Z}_*(\phi^{\infty'}),$$

where in view of (5.2.1), it is easy to see that we are justified in calling $T(\sigma)$ the Hecke operator corresponding to $T(\sigma^\vee)_U$; i.e. this Hecke operator acts as the idempotent projection onto $\sigma^K \subset M_2(K, M)$ for the appropriate level K . It is then clear that modular forms in the image of $T(\sigma)$ are in the right kernel of $(,)_{\sigma_{A'}^\infty}$ if $\sigma_{A'} \neq \sigma_{A^\vee}$ or equivalently (as $A' \mapsto \sigma_{A'}$ is injective) if $A' \neq A^\vee$.

If $A' = A^\vee$, then the expression of interest is the image of $\varphi' \otimes T(\sigma) \tilde{Z}_*(\phi^{\infty'}) \in \sigma_{A^\vee} \otimes (\text{Hom}(J_U, J_U^\vee) \otimes S_2(M))$ under the M -linear algebraic Petersson product and two projections applied to the two factors, induced respectively from $\iota : M \otimes L' \rightarrow L'$ and $\iota_p : M \otimes L' \rightarrow L'$. If $\iota \neq \iota_p$, their combination is zero. \square

PROPOSITION 9.2.3. For each $\bar{w}|p$ and each $a \in \mathbf{A}^{S_1\infty, \times}$, the mixed extension E associated with the divisors $\tilde{Z}_a(\phi^\infty)_U[1]$ and $\mathrm{T}_{\ell_p}(\sigma^\vee)t_\chi$ is essentially crystalline at \bar{w} .

Proof. By Proposition 4.3.1, it is equivalent to show that

$$m_{\bar{w}}(\tilde{Z}_a(\phi^\infty)_U[1], \mathrm{T}_{\ell_p}(\sigma^\vee)t_\chi) = 0.$$

Under Assumptions 6.1.2 and 6.1.3 (which are local at $S_1 \cup S_2$ and hence unaffected by the action of $\mathrm{T}_{\ell_p}(\sigma^\vee)$), this is proved in [YZZ12, Proposition 8.15, § 8.5.1]. \square

Proof of Proposition 9.2.1. For each $\bar{w}|v$, by the discussion preceding (9.2.2) it suffices to show that

$$\langle \mathfrak{q}\tilde{Z}_*(\phi^\infty)_U[1], \mathrm{T}_{\ell_p}(\sigma^\vee)t_\chi \rangle_{\bar{w}}$$

is a v -critical element of $\bar{\mathbf{S}}'$, that is (by the definition and Lemma 9.1.1), that

$$\langle \tilde{Z}_{a\varpi^s}(\phi^\infty)_U[1], \mathrm{T}_{\ell_p}(\sigma^\vee)t_\chi \rangle_{\bar{w}} \in q_{F,v}^{s-c} \mathcal{O}_{L(\chi)}$$

for all a with $v(a) = r_v$ and some constant c independent of a and s .

We may assume that \bar{w} extends the place w of E fixed above.

By Proposition 9.1.4, when $v(a) = r_v$, the divisor $\tilde{Z}_{a\varpi^s}(\phi^\infty)_U[1]$ is a finite sum, with p -adically bounded coefficients, of elements $\mathrm{Tr}_s[x_{j,s}]_U$, where Tr_s denotes the trace map on divisors for the field extension H'_s/H'_0 , and $[x_{j,s}]_U \in \mathrm{Div}^0(X_{U,H'_s})$.

Recall that $\langle \cdot, \cdot \rangle_{\bar{w}} = \langle \cdot, \cdot \rangle_{\ell_{\bar{w}}}$ for a fixed $\ell_{\bar{w}} : \bar{F}_{\bar{w}}^\times \rightarrow L(\chi)$.³² By (4.1.6), we have

$$\langle \mathrm{Tr}_s[x_{j,s}]_U, \mathrm{T}_{\ell_p}(\sigma^\vee)t_\chi \rangle_{\ell_{\bar{w}}} = \langle [x_{j,s}]_U, \mathrm{T}_{\ell_p}(\sigma^\vee)t_\chi \rangle_{\ell_{\bar{w}} \circ N_{s,\bar{w}}}, \tag{9.2.5}$$

where we recall that $N_s = N_{s,\bar{w}}$ is the norm for $H'_{s,\bar{w}}/H'_{0,\bar{w}}$.

Now take the field H of (9.2.3) to be $H = H'_s$, and consider the \mathcal{G}_{E_w} -representations $V' \subset V = V_s$ over $L(\chi)$ defined above Proposition 9.2.2. By our assumptions, V' satisfies the condition $\mathbf{D}_{\mathrm{pst}}(V')^{\varphi=1} = 0$ and the Panchishkin condition, with an exact sequence of \mathcal{G}_{E_w} -representations

$$0 \rightarrow V'^+ \rightarrow V' \rightarrow V'^- \rightarrow 0.$$

The representation V_s has a natural \mathcal{G}_E -stable lattice, namely $T_s := \mathrm{Ind}_E^{H'_s} T_p J_U|_{\mathcal{G}_{H'_s}}$. Let $T' := T \cap V'$, $T'' := T_s \cap V''$, $T'^+ := T' \cap V'^+$, $T'^- = T'/T'^+$; note that V', T', T'^\pm are independent of s (hence the notation).

Let $\tilde{E}_{\bar{w}}/E_w$ be a finite extension over which $V_p A$ (hence V') becomes semistable. We may assume that the extension $\tilde{E}_{\bar{w}} \subset E^{\mathrm{ab}}$ is abelian and totally ramified (see [Nek06, Propositions 12.11.5(iv) and 12.5.10 and their proofs]). For $s \in \{0, 1, \dots, \infty\}$, let $\tilde{H}'_{s,\bar{w}} := \tilde{E}_{\bar{w}} H'_{s,\bar{w}}$ be the compositum.

We can then apply Proposition 4.3.2, with the fields \tilde{H}'_s playing the role of the field denoted there by ' F_v '; together with Proposition 9.2.3, it implies that (9.2.5) belongs to

$$\mathfrak{p}_{L(\chi)}^{-(d_0+d_{1,s}+d_2)} \ell_{\bar{w}} \circ \tilde{N}_s(\mathcal{O}_{H'_{s,\bar{w}}}^\times \hat{\otimes} \mathcal{O}_{L(\chi)}) \subset \mathfrak{p}_{L(\chi)}^{-(d_0+d_{1,s}+d_2+d')} q_{F,v}^{s-c_0} \mathcal{O}_{L(\chi)},$$

where $d_0, d_{1,s}, d_2$ are the integers of Proposition 4.3.2,³³ \tilde{N}_s is the norm of $\tilde{H}'_s/\tilde{H}'_0$, and d' accounts for the denominators coming from (4.1.7). The containment follows from the fact that

³² We only need to consider $\ell_{\bar{w}}$ on the field H'_0 recalled just below.

³³ Note that by construction there is an obvious direct sum decomposition $T_s = T_0 \oplus T^{s-1}$ for a complementary subspace T^{s-1} ; so that the integer d_0 is independent of s .

the extension H'_s/H'_0 is totally ramified at \bar{w} of degree $q_{F,v}^s$, so that $\tilde{H}'_s/\tilde{H}'_0$ has ramification degree at least $q_{F,v}^{s-c_0}$ for some constant c_0 .

To complete the proof, we need to establish the boundedness of the integer sequence

$$d_{1,s} = \text{length}_{\mathcal{O}_{L(\chi)}} H^1(\tilde{H}'_{0,\bar{w}}, T_s''^*(1) \otimes L(\chi)/\mathcal{O}_{L(\chi)})_{\text{tors}}.$$

As

$$\begin{aligned} & H^1(\tilde{H}'_{0,\bar{w}}, T_s''^*(1) \otimes L(\chi)/\mathcal{O}_{L(\chi)})_{\text{tors}} \\ & \cong H^0(\tilde{H}'_{0,\bar{w}}, T_s''^*(1)) \subset H^0(E_w, T_p J_U^*(1) \otimes \mathcal{O}_{L(\chi)}[\text{Gal}(\tilde{H}'_{s,\bar{w}}/E_w)]), \end{aligned}$$

the boundedness follows from the next lemma. □

LEMMA 9.2.4. *Let $\Gamma'_{\text{LT}} := \text{Gal}(\tilde{H}'_{\infty,\bar{w}}/E_w)$. Then*

$$H^0(E_w, V_p J_U^*(1) \otimes \mathcal{O}_{L(\chi)}[\Gamma'_{\text{LT}}]) = 0.$$

Proof. We use the results and notation of Lemma 9.1.5. Note first that we may safely replace $L(\chi)$ by a finite extension L' splitting E_w/\mathbf{Q}_p . As $V_p J_U = V_p J_U^*(1)$ is Hodge–Tate, we have

$$H^0(E_w, V_p J_U \otimes_{\mathcal{O}_{L(\chi)}} \mathcal{O}_{L(\chi)}[\Gamma'_{\text{LT}}]) \subset \bigoplus_{\psi} H^0(E_w, V_p J_U(\psi))(\psi^{-1}),$$

where ψ runs through the Hodge–Tate characters of \mathcal{G}_E factoring through Γ'_{LT} . Since the latter is a quotient of E_w^\times dominating $\Gamma_{\text{LT}} = \text{Gal}(H_{\infty,\bar{w}}/E_w) \cong E_w^\times / \langle \varpi_{\text{LT}} \rangle$, we have $\Gamma'_{\text{LT}} \cong E_w^\times / \langle \varpi_{\text{LT}}^e \rangle$ for some $e \geq 1$. Then the condition that ψ factors through Γ'_{LT} is equivalent to $\psi \circ \text{rec}_{E,w}(\varpi_{\text{LT}}^e) = 1$ for the pseudo-uniformiser $\omega_{\text{LT}} \in E_w^\times$.

By [Ser68, Appendix III.A], a character ψ of $\mathcal{G}_{E,w}$ is Hodge–Tate of some weight $\underline{n} \in \mathbf{Z}[\text{Hom}(E_w, L')]$ if and only if the maps $E_w^\times \rightarrow L'^\times$ given by $\psi \circ \text{rec}_{E,w}$ and $x \mapsto x^{-\underline{n}} := \prod_{\tau \in \text{Hom}(E_w, L')} \tau(x)^{-\underline{n}(\tau)}$ coincide near $1 \in E_w^\times$. By [Con, Proposition B.4(i)], ψ is crystalline if and only if those maps coincide on $\mathcal{O}_{E,w}^\times$; therefore, we may write any Hodge–Tate character ψ as $\psi = \psi_0 \psi_1$ with ψ_0 of finite order and ψ_1 crystalline.

Then, letting $E_{w,0}$ be the maximal unramified extension of \mathbf{Q}_p contained in E_w and $d = [E_w : E_{w,0}]$, we can first write

$$H^0(E_w, V_p J_U(\psi)) = \mathbf{D}_{\text{crys}}(V_p J_U(\psi))^{\varphi=1} \subset \mathbf{D}_{\text{crys}}(V_p J_U(\psi))^{\varphi^d=1}$$

for the $E_{w,0}$ -linear endomorphism φ^d (where φ is the crystalline Frobenius) and then

$$\mathbf{D}_{\text{crys}}(V_p J_U(\psi))^{\varphi^d=1} = \mathbf{D}_{\text{crys}}(V_p J_U(\psi_0))^{\varphi^d=\lambda_1^{-1}},$$

where $\lambda_1 \in L'$ is the scalar giving the action of φ^d on $\mathbf{D}_{\text{crys}}(\psi_1)$.

By [Mok93, Theorem 5.3], all eigenvalues of φ^d on $\mathbf{D}_{\text{crys}}(V_p J_U(\psi_0))$ are Weil $q_{E,w}$ -numbers of strictly negative weight. To conclude that $\mathbf{D}_{\text{crys}}(V_p J_U(\psi))^{\varphi^d=1} = 0$, it thus suffices to show that λ_1 is an algebraic number of weight 0.

By [Con, Proposition B.4(ii)], we have

$$\lambda_1 = \psi_1 \circ \text{rec}_{E,w}(\varpi_w)^{-1} \cdot \varpi_w^{-\underline{n}},$$

where $\varpi_w \in E_w^\times$ is any uniformiser and \underline{n} are the Hodge–Tate weights. Writing $\varpi^m = u\varpi_{\text{LT}}$ for $m = w(\varpi_{\text{LT}})$ and some $u \in \mathcal{O}_{E,w}^\times$, we have

$$\lambda_1^m = \psi_1 \circ \text{rec}_{E,w}(u\varpi_{\text{LT}})^{-1} \cdot u^{-\underline{n}} \varpi_{\text{LT}}^{-\underline{n}}.$$

Now $\psi_1 \circ \text{rec}_{E,w}(u) = u^{-\underline{n}}$ by the crystalline condition, and $\psi_1^e \circ \text{rec}_{E,w}(\varpi_{\text{LT}}) = 1$ as ψ_1 factors through Γ'_{LT} . Hence, $\lambda_1^{em} = \varpi_{\text{LT}}^{-e\underline{n}}$. By Lemma 9.1.5, ϖ_{LT} is an algebraic number of weight 0 and hence so is λ_1 . □

10. Formulas in anticyclotomic families

In §§ 10.1–10.2, we prove Theorem C after filling in some details in its setup. It is largely a corollary of Theorem B (which we have proved for all but finitely many characters), once a construction of the Heegner–theta elements interpolating automorphic toric periods and Heegner points is carried out. Finally, Theorem B for the missing characters will be recovered as a corollary of Theorem C.

In § 10.3, we prove Theorem D.

We invite the reader to go back to § 1.4 for the setup and notation that we are going to use in this section (except for the preliminary lemma (Lemma 10.1.1)).

10.1 A local construction

Let L and F be finite extensions of \mathbf{Q}_p , and denote by v the valuation of F and by $\varpi \in F$ a fixed uniformiser. Let π be a smooth representation of $\mathbf{GL}_2(F)$ on an L -vector space with central character ω and a stable \mathcal{O}_L -lattice $\pi_{\mathcal{O}_L}$. Let $E^\times \subset \mathbf{GL}_2(F)$ be the diagonal torus. Assume that π is nearly ordinary in the sense of Definition 1.2.2 with unit character $\alpha : F^\times \rightarrow L^\times$. Let $f_\alpha^\circ \in \pi - \{0\}$ be any non-zero element satisfying $U_v^* f_\alpha^\circ = \alpha(\varpi) f_\alpha^\circ$, which is unique up to multiplication by L^\times . For $r \geq 1$, let $s_r = \begin{pmatrix} \varpi^r & \\ & 1 \end{pmatrix}$ and

$$f_{\alpha,r} := |\varpi|^{-r} \alpha(\varpi)^{-r} s_r f_\alpha^\circ.$$

It is easy to check that $f_{\alpha,r}$ is independent of the choice of ϖ , and it is invariant under $V_r = \begin{pmatrix} 1 + \varpi^r \mathcal{O}_{F,v} & \\ & 1 \end{pmatrix} V_F$, where $V_F := \text{Ker}(\omega) \subset Z(F)$.

LEMMA 10.1.1.

(1) The collection $f_{\alpha,V_r} := f_{\alpha,r}$, for $r \geq 0$, defines an element

$$f_\alpha = (f_{\alpha,V}) \in \varprojlim_V \pi^V,$$

where the inverse system runs over compact open subgroups $V_F \subset V \subset E^\times$, and the transition maps $\pi^{V'} \rightarrow \pi^V$ are given by

$$f \mapsto \int_{V/V'} \pi(t) f dt.$$

(2) Let $\pi_{\mathcal{O}_L} \subset \pi$ be a $\mathbf{GL}_2(F)$ -stable \mathcal{O}_L -lattice containing f_α° . The collection of elements $\tilde{f}_{\alpha,V_r} := |\varpi|^r f_{\alpha,r} = \alpha(\varpi)^{-r} s_r f_\alpha^\circ$ defines an element

$$\tilde{f}_\alpha \in \varprojlim_V \pi_{\mathcal{O}_L}^V,$$

where the transition maps $\pi^{V'} \rightarrow \pi^V$ are given by

$$f \mapsto \sum_{t \in V/V'} \pi(t) f.$$

Proof. We need to prove that

$$\int_{V_r/V_{r+1}} \pi(t) f_{\alpha,r+1} dt = f_{\alpha,r}. \tag{10.1.1}$$

A set of representatives for V_r/V_{r+1} is $\{(1+j\varpi^r \ 1)\}_{j \in \mathcal{O}_F/\varpi}$; on the other hand, recall that $U_v^* = \sum_{j \in \mathcal{O}_F/\varpi_v} \binom{\varpi \ j}{1}$. From the identity

$$\begin{pmatrix} 1+j\varpi^r & \\ & 1 \end{pmatrix} \begin{pmatrix} \varpi^{r+1} & 1 \\ & 1 \end{pmatrix} = \begin{pmatrix} \varpi^r & 1 \\ & 1 \end{pmatrix} \begin{pmatrix} \varpi & j \\ & 1 \end{pmatrix} \begin{pmatrix} 1+j\varpi^r & \\ & 1 \end{pmatrix},$$

we obtain

$$\int_{V_r/V_{r+1}} \pi(t) s_{r+1} f_\alpha^\circ = q_{F,v}^{-1} s_r U_v^* f_\alpha^\circ = |\varpi|^{-1} \alpha(\varpi) s_r f_\alpha^\circ,$$

as desired. The integrality statement of part 2 is clear as $\alpha(\varpi) \in \mathcal{O}_L^\times$. □

Let us restore the notation $\pi^+ = \pi$, $\pi^- = \pi^\vee$, employing it in the current local setting. Then if π^+ is as in the previous lemma, it is easy to check that π^- is also nearly \mathfrak{p} ordinary. Explicitly, the element

$$f_\alpha^{+,\circ}(y) = \mathbf{1}_{\mathcal{O}_F - \{0\}}(y) |y|_v \alpha_v(y) \tag{10.1.2}$$

in the L -rational subspace³⁴ of any Kirillov model of π_v^+ satisfies $U_v^* f_\alpha^{+,\circ} = \alpha_v(\varpi_v) f_\alpha^{+,\circ}$, and the element

$$f_\alpha^{-,\circ}(y) = \mathbf{1}_{\mathcal{O}_F - \{0\}}(y) |y|_v \omega^{-1}(y) \alpha_v(y) \tag{10.1.3}$$

in the L -rational subspace of any Kirillov model of π_v^- satisfies $U_{v,*} f_\alpha^{-,\circ} = \alpha_v(\varpi_v) f_\alpha^{-,\circ}$.

We can then construct an element $f_\alpha^- = (f_{\alpha,V}^-)_V = (f_{\alpha,r}^-)_r$ with the property of the previous lemma as $f_{\alpha,r}^- := |\varpi|^{-r} \alpha(\varpi)^{-r} s_r^* f_\alpha^{-,\circ}$ with $s_r^* = \begin{pmatrix} 1 & \varpi^{-r} \\ & 1 \end{pmatrix}$.

Local toric periods. Let us restore the subscripts v . Recall the universal Kirillov models $\mathcal{K}(\pi_v^\pm, \psi_{\text{univ},v})$ of § 2.3. Then the elements $f_{\alpha,v}^{\pm,\circ}$ of (10.1.2) and (10.1.3) yield, by the proof of the lemma, explicit elements

$$f_{\alpha,v}^\pm \in \varprojlim_V \mathcal{K}(\pi_v^\pm, \psi_{\text{univ},v})^V, \tag{10.1.4}$$

where the transition maps are given by averages.

Recall the local toric period $Q_v(f_v^+, f_v^-, \chi_v)$ of (1.1.2), for a character $\chi_v \in \mathcal{Y}_v^{1.c.}$, which we define on $\mathcal{K}(\pi_v^+, \psi_{\text{univ},v}) \otimes \mathcal{K}(\pi_v^-, \psi_{\text{univ},v})$ using the canonical pairing of Lemma 2.3.2 on the universal Kirillov models.

By the previous discussion and the defining property of f_α^\pm , for any character $\chi_v \in \mathcal{Y}_v^{1.c.}$, the element

$$Q_v(f_{\alpha,v}^+, f_{\alpha,v}^-, \chi_v) := \lim_V \frac{L(1, \eta_v) L(1, \pi_v, \text{ad})}{\zeta_{F,v}(2) L(1/2, \pi_{E,v} \otimes \chi_v)_v} \int_{E_v^\times/F_v^\times} \chi_v(t) (\pi(t) f_{\alpha,v,V}^+, f_{\alpha,v,V}^-) d^\circ t$$

is well defined and it belongs to $M(\chi_v) \otimes \mathcal{O}_{\Psi_v}(\omega_v)$. In fact, if $M(\alpha_v, \chi_v) \subset L$ is a subfield containing the values of α_v , ω_v , and χ_v , then $Q_v(f_{\alpha,v}^+, f_{\alpha,v}^-, \chi_v)$ belongs to $\mathcal{O}_{\Psi_v, M(\alpha_v, \chi_v)}(\omega_v)$.

LEMMA 10.1.2. *With notation as above, we have*

$$Q_v(f_{\alpha,v}^+, f_{\alpha,v}^-) = \zeta_{F,v}(2)^{-1} \cdot Z_v^\circ$$

as sections of $\mathcal{O}_{\mathcal{Y}_v \times \Psi_v}(\omega_v)$; here Z_v° is as in Theorem A.

³⁴ Cf. § 2.3.

Proof. It suffices to show that the result holds at any complex geometric point $(\chi_v, \psi_v) \in \mathcal{Z}_v^{1.c.} \times \Psi_v(\mathbf{C})$. Drop all subscripts v , and fix a sufficiently large integer r (depending on χ). Recalling the pairing of Lemma 2.3.2, we have by definition

$$Q(f_\alpha^+, f_\alpha^-, \chi) = \frac{L(1, \eta)L(1, \pi, \text{ad})}{\zeta_F(2)L(1/2, \pi_E, \chi)} \int_{E^\times/F^\times} \chi(t)(\pi(t)f_\alpha^+, f_\alpha^-) \frac{dt}{|d|^{1/2}}.$$

We denote by E_w (respectively E_{w^*}) the image of F under the map $F \rightarrow M_2(F)$ sending $t \mapsto \begin{pmatrix} t & \\ & 1 \end{pmatrix}$ (respectively $t \mapsto \begin{pmatrix} 1 & \\ & t \end{pmatrix}$), and by χ_w (respectively χ_{w^*}) the restriction of χ to E_w^\times (respectively $E_{w^*}^\times$).

We can then compute that

$$\frac{\zeta_F(1)L(1/2, \pi_E, \chi)}{L(1, \eta)} Q(f_\alpha^+, f_\alpha^-, \chi)$$

equals

$$\begin{aligned} & |d|^{-1} \int_{F^\times} \int_{F^\times} |\varpi|^{-r} \alpha(\varpi)^{-r} s_r f_\alpha^{+, \circ}(ty) \cdot |\varpi|^{-r} \alpha(\varpi)^{-r} s_r^* f_\alpha^{-, \circ}(y) \chi_w(t) d^\times y d^\times t \\ &= |d|^{-1} \int_{F^\times} \int_{F^\times} |\varpi|^{-r} \alpha(\varpi)^{-r} \psi(-ty) |ty\varpi^r| \alpha(ty\varpi^r) \mathbf{1}_{\mathcal{O}_F - \{0\}}(ty\varpi^r) \\ &\quad \times |\varpi|^{-r} \alpha(\varpi)^{-r} \psi(y) \omega(\varpi)^r |y\varpi^r| \alpha(y\varpi^r) \omega^{-1}(y\varpi^r) \mathbf{1}_{\mathcal{O}_F - \{0\}}(y\varpi^r) \chi_w(t) d^\times y d^\times t. \end{aligned}$$

We now perform the change of variables $t' = ty$ and observe that $\chi_w(t) = \chi_w(t')\chi_w^{-1}(y) = \chi_w(t')\omega(y)\chi_{w^*}(y)$; we conclude after simplification that the above expression equals

$$|d|^{-1} \int_{v(t') \geq -r} \psi(-t') |t'| \alpha(t') \chi_w(t') d^\times t' \int_{v(y) \geq -r} \psi(-y) |y| \alpha(y) \chi_{w^*}(y) d^\times y.$$

If r is sufficiently large, the domains of integration can be replaced by F^\times . The computation of the integrals is carried out in Lemma A.1.1. We obtain

$$Q_v(f_{\alpha,v}^+, f_{\alpha,v}^-, \chi_v) = \frac{L(1, \eta_v)}{\zeta_{F,v}(1)L(1/2, \pi_{E,v}, \chi_v)} \zeta_{E,v}(1) \prod_{w|v} Z_w(\chi_w) = \zeta_{F,v}(2)^{-1} \cdot Z_v^\circ(\chi_v). \quad \square$$

10.2 Gross–Zagier and Waldspurger formulas

Here we prove Theorem C. We continue with the notation of the previous subsection, and we suppose that π_v^\pm is isomorphic to the local component at $v|p$ of the representation π^\pm of the Introduction. Let w, w^* be the two places of E above v , and fix an isomorphism $\mathbf{B}_v \cong M_2(F_v)$ such that the map $E_v \cong E_w \oplus E_{w^*} \rightarrow \mathbf{B}_v$ is identified with the map $F_v \oplus F_v \rightarrow M_2(F_v)$ given by $(t_1, t_2) \mapsto \begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix}$.

We go back to the global situation with the notation and assumption of § 1.4. Choose a universal Whittaker (or Kirillov) functional for π^+ at p , that is, a \mathbf{B}_p^\times -equivariant map $\mathcal{K}_p^+ : \pi^+ \otimes \mathcal{O}_{\Psi_p}(\Psi_p) \rightarrow \otimes_{v|p} \mathcal{K}(\pi^+, \psi_{\text{univ},v})$. By the natural dualities of π^\pm and the Kirillov models, it induces a \mathbf{B}_p^\times -equivariant map $\mathcal{K}_p^{+\vee} : \otimes_{v|p} \mathcal{K}(\pi^-, \psi_{\text{univ},v}) \rightarrow \pi^- \otimes \mathcal{O}_{\Psi_p}(\Psi_p)$, whose inverse \mathcal{K}_p^- is a universal Kirillov functional for π^- . Letting $\pi_{p, \mathcal{O}_{\Psi_p}(\Psi_p)}^\pm := \otimes_{v|p} \mathcal{K}(\pi^\pm, \psi_{\text{univ},v})$, we obtain a unique decomposition $\pi^\pm \otimes \mathcal{O}_{\Psi_p}(\Psi_p) \cong \pi_{p, \mathcal{O}_{\Psi_p}(\Psi_p)}^{\pm, p} \otimes \pi_{p, \mathcal{O}_{\Psi_p}(\Psi_p)}^\pm$. The decomposition arises from a decomposition of the natural M -rational subspaces $\pi^\pm \cong \pi^{\pm, p} \otimes \pi_p^\pm$.³⁵

³⁵ However, the M -rational subspaces are not stable under the $\mathbf{B}^{\infty \times}$ -action.

Heegner–theta elements. For each $f^{\pm,p} \in \pi^{\pm p} \otimes M(\alpha)$, let

$$f_{\alpha}^{\pm,p} := f^{\pm,p} \otimes f_{\alpha,v}^{\pm} \in \pi_{M(\alpha)}^{\pm,p} \otimes \varprojlim_{V \subset \mathcal{O}_{E,p}^{\times}} \pi_{p, \mathcal{O}_{M(\alpha)}}^{\pm,V},$$

where $f_{\alpha,v}^{\pm} = \bigotimes_{v|p} f_{\alpha,v}^{\pm}$ with $f_{\alpha,v}^{\pm}$ the elements (10.1.4).

Fix a component $\mathcal{Y}_{\pm}^{\circ} \subset \mathcal{Y}_{\pm}$ of type $\varepsilon \in \{+1, -1\}$ as in § 1.4. Then the elements

$$\Theta_{\alpha}^{\pm}(f^{\pm,p}) := \int_{E^{\times} \backslash E_{\mathbf{A}^{\infty}}^{\times}} f_{\alpha}^{p,\pm}(i(t)) \chi_{\text{univ}}(t) dt \in \mathcal{O}_{\mathcal{Y}_{\pm}}(\mathcal{Y}_{\pm})^b, \tag{10.2.1}$$

$$\begin{aligned} \mathcal{P}_{\alpha}^{\pm}(f^{\pm,p}) &:= \lim_{U_T, p \rightarrow \{1\}} \int_{E^{\times} \backslash E_{\mathbf{A}^{\infty}}^{\times} / U_T} \kappa(f_{\alpha}^{p,\pm}(\text{rec}_E(t) \iota_{\xi}(P))) \otimes \chi_{\text{univ}, U_T}^{\pm}(t) dt \\ &\in \mathbf{S}_p(A_E, \chi_{\text{univ}}^{\pm}, \mathcal{Y}_{\pm})^b, \end{aligned} \tag{10.2.2}$$

or rather their restriction to $\mathcal{Y}_{\pm}^{\circ} \subset \mathcal{Y}_{\pm}$, satisfy the property of Theorem C(1). Here $\chi_{\text{univ}, U_T}^{\pm} : \Gamma \rightarrow \mathcal{O}^{\times}(\mathcal{Y}_{\pm})^{U_T}$ is the convolution of χ_{univ}^{\pm} with the finest U_T -invariant approximation to a delta function at $\mathbf{1} \in \Gamma$.

We explain the boundedness in the case of $\mathcal{P}_{\alpha}^{\pm}(f^{\pm,p})$. The rigid space \mathcal{Y}_{\pm} is the generic fibre of an \mathcal{O}_L -formal scheme $\mathfrak{Y}_{\pm} := \text{Spf } \mathcal{O}_L[[\Gamma]] / ((\omega^{\pm 1}(\gamma)[\gamma] - [1])_{\gamma \in \mathbf{A}^{\infty, \times}})$.³⁶ (Similarly, each geometric connected component $\mathcal{Y}_{\pm}^{\circ}$ has a formal model $\mathfrak{Y}_{\pm}^{\circ} \subset \mathfrak{Y}_{\pm}$.) This identifies $\mathcal{O}_{\mathcal{Y}_{\pm}}(\mathcal{Y}_{\pm})^b = \mathcal{O}_{\mathfrak{Y}_{\pm}}(\mathfrak{Y}_{\pm}) \otimes_{\mathcal{O}_L} L$. By Lemma 10.1.1(2) applied to the natural lattice $\pi_{\mathcal{O}_L}^{\pm} \subset \pi^{\pm} \otimes L$ given by $\text{Hom}(J, A^{\pm}) \otimes_{\text{End}(A)} \mathcal{O}_L \subset \text{Hom}^0(J, A^{\pm}) \otimes L$, after possibly replacing $f^{\pm,p}$ by a fixed multiple, the elements $\tilde{f}_{\alpha}^{p,\pm}(\text{rec}_E(t) \iota_{\xi}(P))$ belong to $A^{\pm}(E^{\text{ab}})$. Then each term in the sequence at the right-hand side of (10.2.2) is a fixed multiple of

$$\sum_{t \in E^{\times} \backslash E_{\mathbf{A}^{\infty}}^{\times} / U_T} \kappa(\tilde{f}_{\alpha}^{p,\pm}(\text{rec}_E(t) \iota_{\xi}(P))) \otimes \chi_{\text{univ}, U_T}^{\pm}(t), \tag{10.2.3}$$

which belongs to the \mathcal{O}_L -module $H_f^1(E, T_p A^{\pm} \otimes \mathcal{O}_{\mathfrak{Y}_{\pm}}(\mathfrak{Y}_{\pm})^{U_T}(\chi_{\text{univ}, U_T}^{\pm}))$. Hence, some non-zero multiple of $\mathcal{P}_{\alpha}^{\pm}(f^{\pm,p})$ belongs to the limit $\varprojlim_{U_T} H_f^1(E, T_p A^{\pm} \otimes \mathcal{O}_{\mathfrak{Y}_{\pm}}(\mathfrak{Y}_{\pm})^{U_T}(\chi_{\text{univ}, U_T}^{\pm}))$, whose tensor product with L is indeed $\mathbf{S}_p(A_E, \chi_{\text{univ}}^{\pm}, \mathcal{Y}_{\pm})^b$.

Local toric periods away from p . Given the chosen decomposition $\gamma^{\pm} : \pi^{\pm} \otimes \mathcal{O}_{\Psi_p}(\Psi_p) \cong \pi_{\mathcal{O}_{\Psi_p}(\Psi_p)}^{\pm,p} \otimes \pi_{\mathcal{O}_{\Psi_p}(\Psi_p)}^{\mp,p}$, let $(,)^p$ be the unique pairing on $\pi_{\mathcal{O}_{\Psi_p}(\Psi_p)}^{\pm,p} \otimes \pi_{\mathcal{O}_{\Psi_p}(\Psi_p)}^{\mp,p}$ which makes $\gamma^+ \otimes \gamma^-$ into an isometry for the natural pairings on π^{\pm} and $\pi_{p, \mathcal{O}_{\Psi_p}(\Psi_p)}^{\pm}$.³⁷ Then, for each $\chi = \chi^p \chi_p \in \mathcal{Y}^{\circ \text{ l.c.}}$, the toric period Q^p of (1.4.7) is defined. By Lemma 5.1.1, Theorem C(2) then follows from Proposition 3.6.1. It is also proved in slightly different language in [LZZ15, Lemma 4.6(ii)].

Formulas. We prove the anticyclotomic formulas of Theorem C, and at the same time complete the proof of Theorem B for the characters who do not satisfy Assumption 6.2.4.

LEMMA 10.2.1. *Let L be a non-archimedean local field with ring of integers \mathcal{O}_L , $n \geq 1$, and let \mathcal{D}_n be the rigid analytic polydisc over L in n variables, that is, the generic fibre of $\text{Spf } \mathcal{O}_L[[X_1, \dots, X_n]]$. Let $\Sigma_n \subset \mathcal{D}_n(\bar{L})$ be the set of points of the form $x = (\zeta_1 - 1, \dots, \zeta_n - 1)$*

³⁶ The quotienting ideal finitely generated as the image of $\mathbf{A}^{\infty, \times}$ in Γ is a finitely generated \mathbf{Z}_p -submodule.

³⁷ Recall that the natural pairing on the factor at p comes from its description as a Kirillov model.

with each ζ_i a root of unity of p -power order. Let

$$f \in \mathcal{O}(\mathcal{D}_n)^b = \mathcal{O}_L[[X_1, \dots, X_n]] \otimes L$$

be such that $f(x) = 0$ for all but finitely many $x \in \Sigma_n$. Then $f = 0$.

Proof. This is by induction on n , the case $n = 1$ being well known [AV75]; we will abbreviate $X = (X_1, \dots, X_n)$ and $X' = (X_1, \dots, X_{n-1})$. Up to multiplying f by a suitable non-zero polynomial, we may assume that $f \in \mathcal{O}_L[[X]]$ and that it vanishes on all of Σ_n . We may write, with multi-index notation, $f = \sum_{J \subset \mathbf{N}^n} a_J X^J$ (where $\mathbf{N} = \{0, 1, 2, \dots\}$) with each $a_J \in \mathcal{O}_L$. Let $f_j(X') := \sum_{J' \subset \mathbf{N}^{n-1}} a_{J'j}(X')^{J'} \in \mathcal{O}_L[[X']]$; then

$$f(X) = \sum_{j=0}^{\infty} f_j(X') X_n^j.$$

By assumption, $f_0(X') = f(X', 0)$ vanishes on all of Σ_{n-1} and hence by the induction hypothesis $f_0 = 0$ and $X_n | f$. By induction on j , repeatedly replacing f by $X_n^{-1} f$, we find that each $f_j = 0$ and hence $f = 0$. \square

LEMMA 10.2.2. Let $\mathcal{Y}^\circ \subset \mathcal{Y}$ be a connected component of type $\varepsilon = -1$, and let $\mathcal{Y}^{\circ'} \subset \mathcal{Y}'$ be the connected component containing \mathcal{Y}° . The p -adic L -function $L_{p,\alpha}(\sigma_E)|_{\mathcal{Y}^{\circ'}}$ is a section of $\mathcal{I}_{\mathcal{Y}} \subset \mathcal{O}_{\mathcal{Y}'}$.

Proof. By the interpolation property and the functional equation, $L_{p,\alpha}(\sigma_E)$ vanishes on $\mathcal{Y}^\circ \cap \mathcal{Y}^{\text{l.c.,an}}(\bar{L})$. We conclude by applying Lemma 10.2.1, noting that after base-change to a finite extension of L , there is an isomorphism $\mathcal{Y} \rightarrow \coprod_{i \in I} \mathcal{D}_{[F:\mathbf{Q}]}^{(i)}$ to a finite disjoint union of rigid polydiscs, taking $\mathcal{Y}^{\text{l.c.,an}}(\bar{L})$ to $\coprod_{i \in I} \Sigma_{[F:\mathbf{Q}]}^{(i)}$. \square

PROPOSITION 10.2.3. The following are equivalent:

- (1) Theorem B is true for all f_1, f_2 , and all locally constant characters $\chi \in \mathcal{Y}_L^{\text{l.c.}}$;
- (2) Theorem B is true for all f_1, f_2 , and all but finitely many locally constant characters $\chi \in \mathcal{Y}_L^{\text{l.c.}}$;
- (3) Theorem C(4) is true for all $f^{+,p}$ and $f^{-,p}$.

Proof. It is clear that (1) implies (2). That (2) implies (3) follows from Lemma 10.2.1 applied to the difference of the two sides of the desired equality, together with the interpolation properties and the evaluation of the local toric integrals in Lemma 10.1.2. Finally, the multiplicity-one result together with Lemma 10.1.2 shows that (3) implies (1). \square

Since we have already shown at the end of § 8.3 that Theorem B is true for all but finitely many finite-order characters, this completes the proof of Theorem B in general and proves Theorem C(4). Finally, the anticyclotomic Waldspurger formula of Theorem C(3) follows from the Waldspurger formula at finite-order characters (1.4.2) by the argument in the proof of Proposition 10.2.3.

10.3 Birch and Swinnerton-Dyer formula

Theorem D follows immediately from combining the first and second parts of the following proposition. We abbreviate $\mathbf{S}_p^\pm := \mathbf{S}_p(A_E^\pm, \chi_{\text{univ}}^\pm, \mathcal{Y}^\circ)^b$ and remark that, under the assumption $\omega = \mathbf{1}$ of Theorem D, we have $A = A^+ = A^-$ and $\pi = \pi^+ = \pi^-$.

PROPOSITION 10.3.1. *Under the assumptions and notation of Theorem D, the following hold.*

(1) Let

$$\underline{\mathcal{H}} \subset \mathbf{S}_p^+ \otimes \mathbf{S}_p^{-,t}$$

be the saturated Λ -submodule generated by the Heegner points $\mathcal{P}_\alpha^+(f^p) \otimes \mathcal{P}_\alpha^-(f^p)$ for $f^p \in \pi^p$. The Λ -modules \mathbf{S}_p^\pm are generically of rank 1, and moreover $\underline{\mathcal{H}}$ is free of rank 1 over Λ , generated by an explicit element $\mathcal{P}_\alpha^+ \otimes \mathcal{P}_\alpha^{-,t}$.

(2) We have the divisibility of Λ -ideals

$$\text{char}_\Lambda \tilde{H}_f^2(E, V_p A \otimes \Lambda(\chi_{\text{univ}}))_{\text{tors}} \quad | \quad \text{char}_\Lambda(\mathbf{S}_p^+ \otimes_\Lambda \mathbf{S}_p^{-,t} / \underline{\mathcal{H}}).$$

(3) Letting $\langle \cdot \rangle$ denote the height pairing (1.4.6), we have

$$\langle \mathcal{P}_\alpha^+ \otimes \mathcal{P}_\alpha^{-,t} \rangle = \frac{c_E}{2} \cdot d_{FL_{p,\alpha}}(\sigma_E)|_{\mathcal{Y}^\circ}.$$

Proof. We define $\tilde{Q}_v(f_v, \chi_v) := Q_v(f_v, f_v, \chi_v)$ for $f_v \in \pi_v$, and similarly $\tilde{Q}(f, \chi) := \prod_v \tilde{Q}(f_v, \chi_v)$ if $f = \otimes_v f_v \in \pi$. By [Wal85, Lemme 13] (possibly applied to twists $(\pi_v \otimes \mu_v^{-t}, \chi_v \cdot (\mu_v \circ q_w))$) for some character μ_v of F_v^\times , the spaces $H(\pi_v, \chi_v) = \text{Hom}_{E_v^\times}(\pi_v \otimes \chi_v, L(\chi_v))$ are non-zero if and only if $\tilde{Q}_v(\cdot, \chi_v)$ is non-zero on π_v . We also define $\tilde{\mathcal{Q}}_v(f_v) := \mathcal{Q}_v(f_v, f_v)$ and $\tilde{\mathcal{Q}}(f^p) := \zeta_{F,p}(2)^{-1} \prod_{v|p} \tilde{\mathcal{Q}}(f_v) \in \Lambda$ for $f^p = \otimes f_v \in \pi^p$. If the local conditions (1.1.1) are satisfied, as we assume, then the spaces $H(\pi_v, \chi_v)$ are non-zero for all $\chi \in \mathcal{Y}^\circ$, and $\tilde{\mathcal{Q}}$ is not identically zero on π^p .

We will invoke the results of Fouquet in [Fou13] after comparing our setup with his. Let $U_r = U^p \prod_{v|p} U_{r,v} \subset \mathbf{B}^{\infty,\times}$ be such that $U_{r,v} = K_0(\varpi_v^r)$ for $v|p$, $r_v \geq 1$. Let $eH_{\text{ét}}^1(X_{U_r, \bar{F}}, \mathcal{O}_L(1))$ be the image of $H_{\text{ét}}^1(X_{U_r, \bar{F}}, \mathcal{O}_L(1))$ under the product e of the projectors $e_v := \lim_n U_v^n$. Let $\mathcal{I}_\pi \subset \mathcal{H}_{\mathbf{B}^{\infty,\times}}^{\text{sph}}$ be the annihilator of π viewed as a module over the spherical Hecke algebra $\mathcal{H}_{\mathbf{B}^{\infty,\times}}^{\text{sph}}$ for $\mathbf{B}^{\infty,\times}$, and let

$$eH_{\text{ét}}^1(X_{U_p, \bar{F}}, \mathcal{O}_L(1))[\pi] := eH_{\text{ét}}^1(X_{U_r, \bar{F}}, \mathcal{O}_L(1)) / \mathcal{I}_\pi,$$

a Galois module which is independent of $r \geq 1$. The operators U_v act invertibly on $eH^1(X_{U_r, \bar{F}}, \mathcal{O}_L(1))$, and in fact by $\alpha_v(\varpi_v)$ on $eH_{\text{ét}}^1(X_{U_p, \bar{F}}, \mathcal{O}_L(1))[\pi]$. Let $f^p \in \pi^p$ be such that $\tilde{\mathcal{Q}}(f^p) \neq 0$. Denote by κ the Abel–Jacobi functor and by $f_\alpha^\circ := f^p \otimes f_{\alpha,p}^\circ$, with $f_{\alpha,p}^\circ$ the product of the elements $f_{\alpha,v}^\circ$ of (10.1.2) for $v|p$. Then, up to a fixed non-zero multiple, the class $\mathcal{P}_\alpha^+(f^p)$ is the image under $\kappa(f_\alpha^\circ)$ of the limit of the compatible sequence

$$\mathcal{P}_{\alpha,r} := U_p^{-r} \sum_{t \in E^\times \setminus E_{\mathbf{A}^\infty}^\times / U_T} \kappa(\text{rec}_E(t) T_{s_r, t\xi}(P)) \otimes \chi_{\text{univ}, U_T}^\pm(t)$$

of integral elements of $H_f^1(E, eH_{\text{ét}}^1(X_{U_p, \bar{F}}, \mathcal{O}_L(1)) \otimes \Lambda^{U_T}(\chi_{\text{univ}, U_T}))$. Fouquet takes as input a certain compatible sequence $(z(\mathfrak{c}_p, S))_S$ [Fou13, Definitions 4.11 and 4.14] of classes in the latter space to construct, via suitable local modifications according to the method of Kolyvagin, an Euler system [Fou13, § 5]. Noting that the local modifications occur at *well-chosen, good* primes ℓ of E , his construction can equally well be applied to the sequence $(\mathcal{P}_{\alpha,r})_r$ in place of $(z(\mathfrak{c}_p, S))_S$. This Euler system can then be projected via $\kappa(f_\alpha)$ to yield an Euler system for $V_p A \otimes \Lambda$. Under the condition that the first element $\mathcal{P}_\alpha^+(f^p) \in \mathbf{S}_p^+$ (corresponding to $z_{f,\infty}$ in [Fou13]) of the projected Euler system is non-torsion, it is proved in [Fou13, Theorem B(ii)] that \mathbf{S}_p^\pm have generic rank 1 over Λ . By the main result of [AN10], generalising [CV05], the family of points

$\mathcal{P}_\alpha^+(f^p) \in \mathbf{S}_p^+$ is indeed not Λ -torsion provided $\tilde{\mathcal{Q}}(f^p) \neq 0$ (i.e. f_v is a ‘local test vector’ for all $v \nmid p$). We conclude as desired that \mathbf{S}_p^\pm have generic rank 1 over Λ and that the same is true of the submodule \mathcal{H} .

We now proceed to complete the proof of part 1 by showing that the ‘Heegner submodule’ \mathcal{H} is in fact *free* of rank 1 and constructing a ‘canonical’ generator. First we note that by [Fou13, Theorem 6.1], each special fibre $\mathcal{H}|_\chi$ (for arbitrary $\chi \in \mathcal{Y}^\circ$) has dimension either 0 (we will soon exclude this case) or 1 over $L(\chi)$. Let $\{\mathcal{P}_\alpha^+(f_i^p) \otimes \mathcal{P}_\alpha^-(f_i^p)^\iota : i \in I\}$ be finitely many sections of \mathcal{H} . By [YZZ12], for each $\chi \in \mathcal{Y}^{\circ, \text{l.c.}} := \mathcal{Y}^\circ \cap \mathcal{Y}^{\text{l.c., an}}$, the specialisation $\mathcal{P}_\alpha^+(f_i^p) \otimes \mathcal{P}_\alpha^-(f_i^p)^\iota(\chi)$ is non-zero if and only if $\tilde{\mathcal{Q}}(f_i^p)(\chi) \neq 0$, and moreover the images of the specialisations at χ of the global sections of \mathcal{H}

$$\prod_{j \in I, j \neq i} \tilde{\mathcal{Q}}(f_j^p) \cdot \mathcal{P}_\alpha^+(f_i^p) \otimes \mathcal{P}_\alpha^-(f_i^p)^\iota \tag{10.3.1}$$

under the Néron–Tate height pairing (after choosing any embedding $L(\chi) \hookrightarrow \mathbf{C}$) coincide. As $\mathcal{H}|_\chi$ has dimension at most 1, the Néron–Tate height pairing on $\mathcal{H}|_\chi \otimes \mathbf{C}$ is an isomorphism onto its image in \mathbf{C} , and we deduce that the elements (10.3.1) coincide over $\mathcal{Y}^{\circ, \text{l.c.}}$. Since the latter set is dense in $\text{Spec } \Lambda$ by Lemma 10.2.1, they coincide everywhere and glue to a global section $(\mathcal{P}_\alpha^+ \otimes \mathcal{P}_\alpha^{-, \iota})'$ of \mathcal{H} over \mathcal{Y}° .³⁸

Similarly to what is claimed in the proof of Theorem 3.7.1, there exists a *finite* set $\{f_i^p : i \in I\} \subset \pi^p$ such that the open sets $\mathcal{U}_{f_i^p} := \{\tilde{\mathcal{Q}}(f_i^p) \neq 0\} \subset \mathcal{Y}^\circ$ cover \mathcal{Y}° . Then the section

$$\mathcal{P}_\alpha^+ \otimes \mathcal{P}_\alpha^{-, \iota} := \prod_{i \in I} \tilde{\mathcal{Q}}(f_i^p)^{-1} \cdot (\mathcal{P}_\alpha^+ \otimes \mathcal{P}_\alpha^{-, \iota})'$$

is nowhere vanishing and a generator of \mathcal{H} . It is independent of choices since for any $f^p \in \pi^p$ it coincides with $\tilde{\mathcal{Q}}(f^p)^{-1} \cdot \mathcal{P}_\alpha^+(f^p) \otimes \mathcal{P}_\alpha^-(f^p)^\iota$ over \mathcal{U}_{f^p} . This completes the proof of Part 1.

Part 2 is [Fou13, Theorem B(ii)] with \mathcal{H} replaced by its submodule generated by a non-torsion element $z_{f, \infty} \otimes z_{f, \infty} \in \mathcal{H}$; as noted above, we can replace this element by any of the elements $\mathcal{P}_\alpha^+(f^p) \otimes \mathcal{P}_\alpha^-(f^p)^\iota$, and by a glueing argument with $\mathcal{P}_\alpha^+ \otimes \mathcal{P}_\alpha^{-, \iota}$.

Part 3 is an immediate consequence of Theorem C(4). □

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³⁸ We remark that a similar argument, in conjunction with the previous observation that $\tilde{\mathcal{Q}} \neq 0$ on π^p if and only if $\mathcal{Q} \neq 0$ on $\pi^p \otimes \pi^p$, shows that \mathcal{H} equals the *a priori* larger saturated submodule $\mathcal{H}' \subset \mathbf{S}_p^+ \otimes_\Lambda \mathbf{S}_p^{-, \iota}$ generated by the $\mathcal{P}_\alpha^+(f^{+, p}) \otimes \mathcal{P}_\alpha^-(f^{-, p})^\iota$ for possibly different $f^{\pm, p} \in \pi$.

Appendix. Local integrals

A.1 Basic integral

All the integrals computed in the appendix will ultimately reduce to the following.

LEMMA A.1.1. *Let F_v be a non-archimedean local field, and let E_w/F_v be an extension of degree $fe \leq 2$, with f the inertia degree and e the ramification degree. Let q_w be the relative norm and $D \in \mathcal{O}_{F,v}$ be a generator of the relative discriminant.*

Let L be a field of characteristic zero, let $\alpha_v : F_v^\times \rightarrow L^\times$ and $\chi' : E_w^\times \rightarrow L^\times$ be multiplicative characters, $\psi_v : F_v \rightarrow L^\times$ be an additive character of level 0, and $\psi_{E,w} = \psi_v \circ \text{Tr}_{E_w/F_v}$. Define

$$Z_w(\chi', \psi_v) := \int_{E_w^\times} \alpha \circ q(t) \chi'(t) \psi_{E_w}(t) \frac{dt}{|D|^{1/2} |d|_v^{f/2}},$$

where dt is the restriction of the standard measure on E_w .

Then we have

$$Z_w(\chi', \psi_v) = \begin{cases} \alpha_v(\varpi_v)^{-v(D)} \chi'_w(\varpi_w)^{-v(D)} \frac{1 - \alpha_v(\varpi_v)^{-f} \chi'_w(\varpi_w)^{-1}}{1 - \alpha_v(\varpi_v)^f \chi'_w(\varpi_w) q_{F,v}^{-f}} & \text{if } \chi'_w \cdot \alpha_v \circ q \text{ is unramified,} \\ \tau(\chi'_w \cdot \alpha_v \circ q, \psi_{E_w}) & \text{if } \chi'_w \cdot \alpha_v \circ q \text{ is ramified.} \end{cases}$$

Here for any character $\tilde{\chi}'_w$ of conductor \mathfrak{f} ,

$$\tau(\tilde{\chi}', \psi_{E_w}) = \int_{w(t)=-w(\mathfrak{f})} \tilde{\chi}'_w(t) \psi_{E,w}(t) \frac{dt}{|D|^{1/2} |d|_v^{f/2}},$$

with $n = -w(\mathfrak{f}(\chi'_w)) - w(d_{E,w})$.

Proof. If χ' is unramified, only the subset $\{w(t) \geq -1 - w(d) - v(D)\} \subset E_w^\times$ contributes to the integral, and we have

$$\begin{aligned} Z_w(\chi', \psi) &= \alpha^{-v(D)} \chi'_w(\varpi_w)^{-ev(d)-v(D)} \left(\frac{\zeta_E(1)^{-1}}{1 - \alpha^f \chi'(\varpi)} - \frac{1}{q_{F,v}^f} \alpha^{-f} \chi'(\varpi)^{-1} \right) \\ &= \alpha^{-v(D)} \chi'_w(\varpi_w)^{-ev(d)-v(D)} \frac{1 - \alpha^{-f} \chi'(\varpi)^{-1} q_{F,v}^{-f}}{1 - \alpha^f \chi'(\varpi)}. \end{aligned}$$

If χ' is ramified of conductor $\mathfrak{f} = \mathfrak{f}(\chi')$, then only the annulus $w(t) = -w(\mathfrak{f}) - w(d) - v(D)$ contributes, and we get

$$Z_w(\chi', \psi) = \alpha_v^{-fw(\mathfrak{f})-fv(D)} \tau(\chi', \psi_E). \quad \square$$

A.2 Interpolation factor

We compute the integral giving the interpolation factor for the p -adic L -function.

The following Iwahori decomposition can be proved similarly to [Hu16, Lemma A.1].

LEMMA A.2.1. *For a local field F with uniformiser ϖ and for any $r \geq 1$, the double quotient*

$$N(F)A(F)Z(F) \backslash \mathbf{GL}_2(F) / K_1^1(\varpi^r)$$

admits the set of representatives

$$\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & \\ c^{(i)} \varpi^{r-i} & 1 \end{pmatrix}, \quad 1 \leq i \leq r, \quad c^{(i)} \in (\mathcal{O}_F/\varpi^i)^\times.$$

Note that we may also replace the first representative by $(\frac{1}{\varpi^r} \ 1) \in K_1^1(\varpi^r)$.

PROPOSITION A.2.2. *Let $\chi' \in \mathcal{Y}_{M(\alpha)}^{1,c}(\mathbf{C})$ and let $\iota : M(\alpha) \hookrightarrow \mathbf{C}$ be the induced embedding. Let $v|p$, and let ϕ_v be either as in Assumption 6.2.1 for some sufficiently small $U_{T,v} \subset \mathcal{O}_{E,v}^\times$, or as in Assumption 6.2.2. Then, for any sufficiently large integer r , the normalised integral $R_{r,v}^\natural(W_v, \phi_v, \chi'_v)$ of (3.5.2) equals*

$$R_{r,v}^\natural(W_v, \phi_v, \chi'_v) = Z_v^\circ(\chi'_v) := \frac{\zeta_{F,v}(2)L(1, \eta_v)^2}{L(1/2, \sigma_{E,v}^\iota, \chi'_v)} \prod_{w|v} Z_w(\chi'_w),$$

with $Z_w(\chi'_w)$ as in Lemma A.1.1.

Proof. We omit the subscripts v and the embedding ι in the calculations which follow. By definition, we need to show that the integral $R_{r,v}^\circ$ of Proposition 3.5.1 equals

$$R_{r,v}^\circ = R_r^\circ(W, \phi, \chi') = |D|^{1/2}|d|^2 L(1, \eta) \prod_{w|v} Z_w(\chi'_w).$$

Note that the assertion in the case of Assumption 6.2.2 is implied by the assertion in the case of Assumption 6.2.1 by (6.2.1), so we will place ourselves in the latter situation.

By the decomposition of Lemma A.2.1, and observing that $\delta_{\chi_F,r}$ vanishes on $K - K_0(\varpi^r)$, we have

$$R_{r,v}^\circ = \alpha(\varpi)^{-r} \int_{F^\times} W_{-1}\left(\begin{pmatrix} y & \\ & 1 \end{pmatrix}\right) \delta_{\chi_F,r}\left(\begin{pmatrix} y & \\ & 1 \end{pmatrix}\right) \\ \times \int_{T(F)} \chi'(t) \int_{P(\varpi^r) \setminus K_1^1(\varpi^r)} |y|r(kw_r^{-1})\phi(yt^{-1}, y^{-1}q(t)) dk d^\times t \frac{d^\times y}{|y|},$$

where $P(\varpi^r) = P \cap K_1^1(\varpi^r)$ (recall that $P = NZA$). Here we have preferred to denote by $d^\times t$ the standard Haar measure on $T(F_v) = E_v^\times$; later dt will denote the additive measure on E_v .

Changing variables $k' = w_rkw_r^{-1}$, we observe that by Lemma 3.1.1 the group $K_1^1(\varpi^r)$ acts trivially for sufficiently large r . Then we can insert

$$W_{-1}\left(\begin{pmatrix} y & \\ & 1 \end{pmatrix}\right) = \mathbf{1}_{\mathcal{O}_F - \{0\}}(y)|y|\alpha(y)$$

and

$$r(w_r^{-1})\phi(x, u) = |\varpi^{-r}|\psi_{E,U}(ux_1)\mathbf{1}_{\mathcal{O}_{\mathbf{V}_2}}(x_2)\delta_{q(U)}(\varpi^r u),$$

where $\psi_{E,U} = r(\text{vol}(U)^{-1}\mathbf{1}_U)\psi_E$ for the extension of r to functions on K (so that $\psi_{E,U}$ is the finest U -invariant approximation to ψ_E). We obtain

$$R_{r,v}^\circ = \alpha(\varpi)^{-r}|d|^{1/2}\zeta_{F,v}(1)^{-1} \int_{\mathcal{O}_F - \{0\}} |y|\alpha(y) \int_{T(F)} \chi'(t)\psi_{E,U}(t)\delta_{q(U)}(\varpi^r y^{-1}q(t)) dt d^\times y,$$

where $|d|^{1/2}\zeta_{F,v}(1)^{-1}$ appears as $\text{vol}(P(\varpi^r) \setminus K_1^1(\varpi^r))|\varpi|^{-r}$. We get

$$R_{r,v}^\circ = |d|\zeta_{F,v}(1)^{-1} \int_{v(q(t)) \geq -r} |q(t)|\alpha(q(t))\chi'(t)\psi_E(t) d^\times t.$$

If r is sufficiently large, the domain of integration can be replaced with all of $T(F)$. Switching to the additive measure, and using the isomorphism $E_v^\times = E_w^\times \times E_{w^*}^\times$ in the split case, the integral equals

$$R_{r,v}^\circ = |d|L(1, \eta) \int_{E_v^\times} \alpha(q(t))\chi'(t)\psi_E(t) dt = |D|^{1/2}|d|^2 L(1, \eta) \prod_{w|v} Z_w(\chi'_w),$$

as desired. □

A.3 Toric period

We compare the normalised toric periods with the interpolation factor.

PROPOSITION A.3.1. *Suppose that $v|p$ splits in E . Then, for any finite-order character $\chi \in \mathcal{Y}^{1.c}$, we have*

$$Q_v(\theta_v(W_v, \alpha_v^{-r_v} w_{r,v}^{-1} \phi_v), \chi_v) = L(1, \eta_v)^{-1} \cdot Z_v^\circ(\alpha_v, \chi_v)$$

for any ϕ_v as in Proposition A.2.2 and any sufficiently large $r_v \geq 1$.

For consistency with the proof of Proposition A.2.2, in the proof we will denote by $d \times t$ the Haar measure on $T(F)$ denoted by dt in the rest of the paper.

Proof. By the definitions and Proposition A.2.2, it suffices to show that for any $\chi \in \mathcal{Y}^{1.c}(\mathbf{C})$, we have

$$|d|_v^{3/2} Q_v^\sharp(\theta_v(W_v, \alpha_v^{-r_v} w_{r,v}^{-1} \phi_v), \chi_v) = L(1, \eta_v)^{-1} \cdot R_{r,v}^\circ(W_v, \phi_v, \chi_v),$$

where Q_v^\sharp is the toric integral of (5.1.5).

By the Shimizu lifting (Lemma 5.1.1) and Lemma A.2.1, we can write

$$Q_v^\sharp := |d|_v^{3/2} Q_v(\theta_v(W_v, \alpha_v^{-r_v} w_{r,v}^{-1} \phi_v), \chi_v) = Q_v^{\sharp(0,1)} + \sum_{i=1}^r \sum_{c \in (\mathcal{O}_F/\varpi^i)^\times} Q_v^{\sharp(i,c)},$$

where, for each (i, c) , omitting the subscripts v ,

$$Q_v^{\sharp(i,c)} := \alpha(\varpi)^{-r} \int_{F^\times} W_{-1} \left(\begin{pmatrix} y & \\ & 1 \end{pmatrix} n^-(c\varpi^{r-i}) \right) \times \int_{T(F)} \chi'(t) \int_{P(\varpi^r) \backslash K_1^1(\varpi^r)} |y|r(n^-(c\varpi^{r-i})kw_r^{-1})\phi(yt^{-1}, y^{-1}q(t)) dk d^\times t \frac{d^\times y}{|y|}.$$

Note that $Q_v^{\sharp(0,1)} = R_v^\circ$, where R_v° is as in the previous proposition, since $n^-(\varpi^r) \in K_1^1(\varpi^r)$. We will compute the other terms.

We have $\begin{pmatrix} 1 & \\ c\varpi^{r-i} & 1 \end{pmatrix} w_r^{-1} = w_r^{-1} \begin{pmatrix} 1 & -c\varpi^{-i} \\ & 1 \end{pmatrix}$ and, when $x = (x_1, x_2)$ with $x_2 = 0$,

$$r(n^-(c\varpi^{r-i})w_r^{-1})\phi(x, u) = |\varpi^{-r}| \int_E \psi_E(ux_1\xi_1)\psi(-uc\varpi^{r-i}q(\xi_1))\delta_U(\xi_1)\delta_{q(U)}(\varpi^r u) d\xi_1 \times \int_{\mathcal{O}_{\mathbf{V}_2}} \psi(-uc\varpi^{r-i}q(\xi_2))\delta_{q(U)}(\varpi^r u) d\xi_2 = |\varpi|^{i-r} \psi_{E,U}(ux_1)\psi_{q(U)}(-c\varpi^{-i})\delta_{q(U)}(\varpi^r u).$$

Inserting this, we obtain

$$Q_v^{\sharp(i,c)} = |\varpi|^i \alpha(\varpi)^{-r} |d|^{1/2} \zeta_{F,v}(1)^{-1} \int_{F^\times} W \left(\begin{pmatrix} y & \\ & 1 \end{pmatrix} n^-(c\varpi^{r-i}) \right) \int_{T(F)} \chi(t) \times \psi_{E,U}(t)\psi_{q(U)}(-c\varpi^{-i})\delta_{q(U)}(\varpi^r y^{-1}q(t)) d^\times t d^\times y.$$

We have already noted that $Q_v^{\sharp(0,1)} = R_v^\circ$. For $i = 1$, if r is sufficiently large, then W is still invariant under $K_1^1(\varpi^{r-1})$; then $\sum_c Q_v^{\sharp(1,c)}$ equals $C \cdot R_v^\circ$ with $C = |\varpi| \sum_{c \in (\mathcal{O}/\varpi)^\times} \psi_{q(U)}(-c\varpi^{-1}) = -|\varpi|$.

Finally, we claim that for each $i \geq 2$, $\sum_c Q_v^{\sharp(i,c)} = 0$. Indeed, let $q(U) = 1 + \varpi^n \mathcal{O}_{F,v}$. If $i \geq n + 1$, then $\psi_{q(U)}(-c\varpi^{-i}) = 0$; if $i \leq n$, then if r is sufficiently large W is still invariant

under $K_1^1(\varpi^{r-i}) \subset K_1^1(\varpi^{r-n})$, and summing the only terms depending on c produces a factor $\sum_{c \in (\mathcal{O}/\varpi^i)^\times} \psi(c\varpi^{-i}) = 0$.

Summing up, we have

$$Q_v^\sharp = Q_v^{\sharp(0,1)} + \sum_{c \in (\mathcal{O}_F/\varpi_v)^\times} Q_v^{\sharp(1,c)} = (1 - |\varpi|)R_{r,v}^\circ = L(1, \eta_v)^{-1} \cdot R_{r,v}^\circ,$$

as desired. □

Question A.3.2. A comparison between Propositions 10.1.2 and A.3.1 suggests that the identity

$$\lim_{r \rightarrow \infty} L(1, \eta_v) \cdot \theta_v(W_v, \alpha_v^{-r} w_r^{-1} \phi_v) = \zeta_{F,v}(2) \cdot f_{\alpha,v}^+ \otimes f_{\alpha,v}^-$$

might hold in $\varprojlim_V (\pi_v^+)^V \otimes \varprojlim_V (\pi_v^-)^V$ (with notation as in Lemma 10.1.1). Is this the case?

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