Bull. Aust. Math. Soc. **95** (2017), 467–475
doi:10.1017/S0004972716001313

ON THE EVENTUAL PERIODICITY OF PIECEWISE LINEAR CHAOTIC MAPS

M. ALI KHAN✉ and ASHVIN V. RAJAN

(Received 23 August 2016; accepted 12 November 2016; first published online 16 February 2017)

Abstract

We present a family of continuous piecewise linear maps of the unit interval into itself that are all chaotic in the sense of Li and Yorke [‘Period three implies chaos’, *Amer. Math. Monthly* **82** (1975), 985–992] and for which almost every point (in the sense of Lebesgue) in the unit interval is an eventually periodic point of period p, $p \geq 3$, for a member of the family.

2010 *Mathematics subject classification*: primary 37B20; secondary 65Q10, 91B55.

Keywords and phrases: Li–Yorke chaos, piecewise linear functions, eventual convergence, almost everywhere convergence, dynamical systems, rationalisability.

1. Introduction

In [17], for any integer $p > 2$, Nathanson presented a piecewise linear bimodal map from the unit interval to itself, with parallel left and right arms both of unit slope, that is chaotic in the sense of Li–Yorke [14], and yet almost all points in its domain converge in a finite number of periods to an interval in which every point has period p (see, for example, [4, 14] for background detail). In [6, 7], Du complemented Nathanson’s work by constructing chaotic piecewise linear continuous interval mappings for which almost all points in the interval converge to a fixed point, and similar mappings for which almost all points converge to a point of order 3 (see also [8]). For the latter, Du worked with a trapezoidal mapping with two flat bottoms (see [15] for details on trapezoidal maps). All three examples are illustrations of the fact that Li–Yorke chaos is *trivial* (in the words of Nathanson) or, from a physical point of view, *essentially unobservable* (in the words of Colett–Eckmann) (see [1, 3]).

The robustness and the relationship of these results have remained an open question. In particular, one can ask whether there exists a family of piecewise linear maps that yields under particular parametric assumptions all of the aforementioned results and consolidates them in the sense that (i) the result of [7] holds for a point with a given arbitrary number of periods k and (ii) that of [17] is invariant with respect to perturbations of a particular map within a specified family. This question, of interest in itself, is also motivated by issues in economic dynamics where a *rationalisation* of a particular map as a solution to an optimisation problem is of substantive consequence (see, for example, [2, 5, 9, 17]). We will give a complete answer to this question.

From a technical point of view, our results attain significance because the maps that we construct are almost everywhere eventually periodic, as opposed to being asymptotically periodic, and are not smooth in the sense of having a positive Schwarzian derivative at their turning points and falling within the class isolated in [10]. Furthermore, the proof of our theorem is a testimony to the recursive reasoning originally laid out in [17], and supplemented here through the use of generating functions that allow a key step in the proof of the main theorem to be seen as a consequence of the powers of a relevant matrix converging to zero (see [20] for a standard introduction). In addition to a desirable unification of the arguments that lead to the two results, this clarification may encourage the use of this methodology in other related contexts. In particular, the specific forms of the mappings in terms of two parallel arms, as in [17], or flat bottoms, as in [6, 7], have prevented the applications of the Nathanson–Du results to questions in economic dynamics where dynamical systems have to be deduced as a consequence of an infinite-horizon optimal program (see [2, 5, 16]). This is no longer true for the theorem presented here, and we pursue elsewhere its implications for the Leontief–Shinkai and Robinson–Solow–Srinivasan (RSS) models studied in [5, 9, 11–13, 16, 18] and subsequent work. For the importance of recursion-theoretic, as opposed to set-theoretic, methods in economics, see [19] and the references therein.

2. The principal result

We begin by defining the notation that we shall need to state the main theorem of this paper. Let $p \geq 3$ be any fixed integer and $m > 1$ any fixed real number such that

$$0 < \frac{m-1}{m^p - 1} < b < \frac{m-1}{m^p-1} < 1 \quad \text{and} \quad d = -1\Big/\Big(b\frac{m^{p-1}-1}{m-1} - \frac{1-b}{m}\Big).$$

The family of functions that form the backbone for this paper is

$$f(x) = \begin{cases} mx + b & \text{if } 0 \leq x \leq \dfrac{1-b}{m}, \\ d\Big(x - b\Big(\dfrac{m^{p-1}-1}{m-1}\Big)\Big) & \text{if } \dfrac{1-b}{m} \leq x \leq b\Big(\dfrac{m^{p-1}-1}{m-1}\Big), \\ \dfrac{1}{m^{p-1}}\Big(x - b\Big(\dfrac{m^{p-1}-1}{m-1}\Big)\Big) & \text{if } b\Big(\dfrac{m^{p-1}-1}{m-1}\Big) \leq x \leq 1. \end{cases}$$

Note that $f : [0, 1] \to [0, 1]$ is a bimodal, piecewise linear continuous map.

Next, we define p mutually disjoint closed subintervals $C_0, C_1, \ldots, C_{p-1}$ of $[0, 1]$ and $p-1$ mutually disjoint open subintervals $U_{-(p-2)}, U_{-(p-3)}, \ldots, U_0$ as follows:

$$C_j = \Big[b\Big(\frac{m^j-1}{m-1}\Big), \frac{1}{m^{p-(j+1)}}\Big(1 - b\Big(\frac{m^{p-(j+1)}-1}{m-1}\Big)\Big)\Big] \quad \text{for } 0 \leq j \leq (p-1),$$

$$U_{-[(p-1)-j]} = \Big(\frac{1}{m^{p-j}}\Big(1 - b\Big(\frac{m^{p-j}-1}{m-1}\Big)\Big), b\Big(\frac{m^j-1}{m-1}\Big)\Big) \quad \text{for } 1 \leq j \leq (p-1).$$

https://doi.org/10.1017/S0004972716001313 Published online by Cambridge University Press

Then the following maps are linear and onto:

$$C_0 \xrightarrow{f} C_1 \xrightarrow{f} C_2 \xrightarrow{f} \cdots \xrightarrow{f} C_{p-1} \xrightarrow{f} C_0,$$
$$U_{-(p-2)} \xrightarrow{f} U_{-(p-3)} \xrightarrow{f} \cdots \xrightarrow{f} U_{-1} \xrightarrow{f} U_0 \xrightarrow{f} (0,1).$$

Let $C = \bigcup_{j=0}^{p-1} C_j$. Every element of C is a point of period p for f.

Lemma 2.1. *For the values of m and b specified above, f is chaotic in the sense of Li and Yorke.*

Proof. Let $\xi = (1 - b(m+1))/m^2$. Then

$$f(\xi) = \frac{1-b}{m}, \quad f^2(\xi) = 1 \quad \text{and} \quad f^3(\xi) = \frac{1}{m^{p-1}}\Big(1 - b\Big(\frac{m^{p-1}-1}{m-1}\Big)\Big).$$

The Li–Yorke conditions [14] are satisfied by virtue of the fact that

$$0 < f^3(\xi) < \xi < f(\xi) < f^2(\xi).$$

Hence, f must be a chaotic map. □

We now turn to the principal result of this paper, and to its corollaries.

Theorem 2.2. *Assume that $p > 2$ is an integer. To almost every real number $x \in [0,1]$ there corresponds a positive integer n_x such that $f^{n_x}(x) \in C$.*

Corollary 2.3 (Nathanson [17]). *There exists a continuous piecewise linear mapping $f_1 : [0,1] \to [0,1]$ such that to almost every real number $x \in [0,1]$ there corresponds an integer $n_x > 0$ such that $f_1^{n_x}(x)$ is a point of period p.*

Corollary 2.4. *Assume that $p > 2$ is an integer. There exists a continuous piecewise linear mapping $f_0 : [0,1] \to [0,1]$ such that, to almost every real number $x \in [0,1]$, there corresponds a positive integer n_x such that $f_0^{n_x}(x) = 0$, where 0 is a point of period p, $f_0(x) = 0$ for $x \in C_{p-1}$ and $f_0(x) = f(x)$ elsewhere.*

Proof. Consider a mapping that modifies f in its third arm:

$$g(x) = \begin{cases} mx + b & \text{if } 0 \le x \le \dfrac{1-b}{m}, \\ d\Big(x - b\Big(\dfrac{m^{p-1}-1}{m-1}\Big)\Big) & \text{if } \dfrac{1-b}{m} \le x \le b\Big(\dfrac{m^{p-1}-1}{m-1}\Big), \\ 0 & \text{if } b\Big(\dfrac{m^{p-1}-1}{m-1}\Big) \le x \le 1. \end{cases}$$

Then, for $\xi = (1 - (b(1+m))/m^2$,

$$g(\xi) = \frac{1-b}{m}, \quad g^2(\xi) = 1 \quad \text{and} \quad g^3(\xi) = 0,$$

and the Li–Yorke conditions [14] are satisfied. Furthermore,

$$C_0 \xrightarrow{g} C_1 \xrightarrow{g} C_2 \xrightarrow{g} \cdots \xrightarrow{g} C_{p-1} \xrightarrow{g} \{0\} \subseteq C_0.$$

The proof of the theorem goes through unchanged for g. □

https://doi.org/10.1017/S0004972716001313 Published online by Cambridge University Press

Setting $p = 3$ in Corollary 2, we obtain the following result.

COROLLARY 2.5 (Du [7]). *There exists a family of piecewise linear continuous mappings* $f : [0, 1] \to [0, 1]$ *that are all chaotic in the sense of Li and Yorke such that to almost every real number* $x \in [0, 1]$ *there corresponds a positive integer* n_x *such that* $f^{n_x}(x) = 0$, *where* 0 *is a point of period* 3.

We conclude this section with the assertion that the dynamics identified by the family of piecewise linear maps is robust (structurally stable) in the sense that 'small' perturbations of the particular member of the family will leave the dynamics invariant (see also the Remark 3.2 at the end of Section 3).

3. Proofs of results

We begin with the proof of the theorem by defining the following subsets of $[0, 1]$:

$$C^\star = \{x \in [0, 1] : f^n(x) \in C \text{ for some integer } n \geq 0\}$$

and $U^\star = [0, 1]\backslash C^\star$. With μ denoting Lebesgue measure on $[0, 1]$, we wish to prove that $\mu(C^\star) = 1$ or, equivalently, $\mu(U^\star) = 0$. To this end, we define

$$U_n = \{x \in U_0 : f^n(x) \notin C\} \quad \text{for } n \geq 0.$$

Since $f(C) \subset C$, it follows that

$$U_0 \supset U_1 \supset U_2 \supset \cdots .$$

We maintain that as $n \to \infty$, $\mu(U_n) \to 0$. We need the following proposition.

PROPOSITION 3.1. *Fix a nonnegative integer* n *and let* $\mu(U_0) = \delta$. *Then:*

(1) U_n *is composed of a finite collection of mutually disjoint open intervals of length* δ^{j+1}/m^k, *where* k *is a natural number and* $j \in \{0, \ldots, n\}$;

(2) f^n *maps every interval of* U_n *linearly onto exactly one of the intervals* U_{-j} *for* $0 \leq j \leq p - 2$.

PROOF. We argue by mathematical induction on n. In the first place, the statement is clear for $n = 0$, since f^0 is the identity mapping and U_0 is an interval of length δ which is mapped by f^0 linearly onto itself. Assume next, as our induction hypothesis, that U_{n-1} is composed of a finite collection of mutually disjoint open intervals of length δ^{j+1}/m^k, where $j \in \{0, 1, \ldots, n-1\}, k \geq 0$ is an integer and f^{n-1} maps each such open interval in U_{n-1} linearly onto exactly one of the open intervals $U_{-(p-2)}, \ldots, U_{-1}$, or U_0.

We must now prove that the two assertions of the proposition hold for U_n and f^n. In order to understand the behaviour of f^n on U_n, it suffices to understand the behaviour of f^n on U_{n-1} as $U_{n-1} \supset U_n$. Assume therefore that I is any one of the disjoint open intervals that comprise U_{n-1}, and that I has length δ^{j+1}/m^k. If $f^{n-1}(I) = U_{-j}$ for $j > 0$, then $f^n(I) = U_{-(j-1)}$. In this case, the two assertions of the proposition hold for f^n.

https://doi.org/10.1017/S0004972716001313 Published online by Cambridge University Press

If on the other hand $f^{n-1}(I) = U_0$, then $f^n(I) = (0, 1)$ (since $f(U_0) = (0, 1)$). Since I was assumed to have length δ^{j+1}/m^k by hypothesis, the *slope* of f^n is $1/(\delta^{j+1}/m^k) = m^k/\delta^{j+1}$. Since $f^n : I \to (0, 1)$ is a linear onto mapping, it follows that I must contain $p-1$ mutually disjoint subintervals $\{I_0, I_{-1}, \ldots, I_{-(p-3)}, I_{-(p-2)}\}$ such that $f^n(I_{-s}) = U_{-s}$ for $0 \le s \le p-2$. Fix an arbitrary $s \in \{0, \ldots, p-2\}$. Since f^n has slope m^k/δ^{j+1} and since $\mu(U_{-s}) = \delta/m^s$, we conclude that

$$\mu(I_{-s}) = \frac{(\delta/m^s)}{(\delta^{j+1}/m^k)} = \frac{\delta^{j+2}}{m^{s+k}}, \quad \text{where } j \in \{0, 1, \ldots, n-1\} \text{ and } k \ge 0 \text{ is an integer.}$$

Finally, since we have assumed that $j \in \{0, \ldots, n-1\}$ as our induction hypothesis, it must be that $j + 2 \le n + 1$. Hence, U_n is the union of a finite collection of mutually disjoint open intervals of length δ^{j+1}/m^k with $j \in \{0, \ldots, n\}$ such that f^n maps each such interval linearly onto exactly one of the open segments $\{U_{-(p-2)}, \ldots, U_{-1}, U_0\}$. The statement of the proposition therefore follows by mathematical induction. □

Proof of Theorem 2.2. In order to proceed with the proof of the main theorem, we shall need to estimate the number of disjoint subintervals that comprise the sets U_n. We therefore introduce a key piece of notation. For any nonnegative integers n and k, fix $j \in \{0, 1, \ldots, n\}$ and $s \in \{0, 1, \ldots, p-2\}$; let $A^{(-s)}_{n,j,k}$ denote the number of disjoint open intervals I in U_n of length δ^{j+1}/m^k such that f^n maps I linearly onto U_{-s}. We begin our proof by deriving recurrence relations for $A^{(-s)}_{n,j,k}$.

Assume that $1 \le s < p-2$ and let f^{n-1} map an interval I of length δ^{j+1}/m^k linearly onto $U_{-(s+1)}$. Then f^n would have to map I linearly onto U_{-s}. There are exactly $A^{-(s+1)}_{(n-1),j,k}$ such intervals that f^{n-1} maps linearly onto $U_{-(s+1)}$. On the other hand, if f^{n-1} maps an interval I of length δ^j/m^{k-1} linearly onto U_0, then $f^n(I) = (0, 1)$ and, as $f^n : I \to (0, 1)$ is a linear onto mapping, I must contain a *unique subinterval* I_{-s} of length δ^{j+1}/m^k that is mapped linearly onto U_{-s}. Since there are exactly $A^{(0)}_{(n-1),(j-1),(k-1)}$ intervals mapped by f^{n-1} linearly onto U_0, it follows that

$$A^{(-s)}_{n,j,k} = A^{-(s+1)}_{(n-1),j,k} + A^{(0)}_{(n-1),(j-1),(k-1)}.$$

We now treat the case $s = p-2$. Assume that f^{n-1} maps an interval I of length δ^{j+1}/m^j linearly onto U_{-j} for any $0 < j \le p-2$; then f^n must map I linearly onto $U_{-(j-1)}$. Hence, $f^n(I)$ can never be $U_{-(p-2)}$. On the other hand, if f^{n-1} mapped an interval I of length $\delta^j/m^{k-(p-2)}$ linearly onto U_0, then $f^n(I) = (0, 1)$ and, as $f^n : I \to (0, 1)$ is a linear onto mapping, I would contain a unique subinterval of length δ^{j+1}/m^k that f^n maps linearly onto $U_{-(p-2)}$. The number of intervals of length $\delta^j/m^{k-(p-2)}$ mapped by f^{n-1} linearly onto U_0 is $A^{(0)}_{(n-1),(j-1),(k-(p-2))}$, and so

$$A^{-(p-2)}_{n,j,k} = A^{(0)}_{(n-1),(j-1),(k-(p-2))}.$$

https://doi.org/10.1017/S0004972716001313 Published online by Cambridge University Press

Finally, we consider the case $s = 0$. If f^{n-1} maps an interval I of length δ^{j+1}/m^k linearly onto U_{-1}, then f^n maps I linearly onto U_0. The number of intervals of length δ^{j+1}/m^k mapped by f^{n-1} linearly onto U_{-1} is $A^{(-1)}_{(n-1),j,k}$. If, on the other hand, f^{n-1} maps an interval I of length δ^j/m^k linearly onto U_0, then $f^n(I) = (0,1)$ and I contains a unique subinterval I_0 of length δ^{j+1}/m^k such that f^n maps I_0 linearly onto U_0. Since there are $A^{(0)}_{(n-1),(j-k),k}$ intervals of length δ^j/m^k mapped linearly onto U_0 by f^{n-1},

$$A^{(0)}_{n,j,k} = A^{(0)}_{(n-1),(j-1),k} + A^{(-1)}_{(n-1),j,k}.$$

We have obtained the system of linear recurrence relations (for $1 \le \ell \le p-3$):

$$\begin{aligned}
A^{(0)}_{n,j,k} &= A^{(0)}_{(n-1),(j-1),k} + A^{(-1)}_{(n-1),j,k},\\
&\vdots\\
A^{(-\ell)}_{n,j,k} &= A^{(0)}_{(n-1),(j-\ell),(k-1)} + A^{(-(\ell+1))}_{n-1,j,k},\\
&\vdots\\
A^{(-(p-2))}_{n,j,k} &= A^{(0)}_{n-1,(j-(p-2)),k-1}.
\end{aligned}$$

We also observe that $A^{(0)}_{0,0,0} = 1$ and

$$\begin{aligned}
A^{(-s)}_{0,j,0} &= 0 \quad \text{for } j > 0 \text{ and } s \in \{1, \ldots, (p-2)\},\\
A^{(0)}_{0,j,k} &= 0 \quad \text{for } (j,k) \in \mathbf{Z}_+ \times \mathbf{Z}_+, \text{ where } \mathbf{Z}_+ = \{1, 2, \ldots\} \text{ and}\\
A^{(-s)}_{n,j,k} &= 0 \quad \text{for all } s \text{ if any one of } n, j \text{ or } k \text{ is a negative integer.}
\end{aligned}$$

We therefore obtain the system (for $1 \le \ell \le p-3$)

$$\begin{aligned}
\alpha = \begin{bmatrix} A^{(0)}(X,Y,Z) \\ \vdots \\ A^{(-\ell)}(X,Y,Z) \\ \vdots \\ A^{(-(p-2))}(X,Y,Z) \end{bmatrix} &= \begin{bmatrix} \sum_{(n,j,k)} A^{(0)}_{n,j,k} X^n Y^j Z^k \\ \vdots \\ \sum_{(n,j,k)} A^{(-\ell)}_{n,j,k} X^n Y^j Z^k \\ \vdots \\ \sum_{(n,j,k)} A^{(-(p-2))}_{n,j,k} X^n Y^j Z^k \end{bmatrix}\\
&= \begin{bmatrix} \sum_{(n,j,k)} (A^{(0)}_{(n-1),(j-1),k} + A^{(-1)}_{(n-1),j,k}) X^n Y^j Z^k \\ \vdots \\ \sum_{(n,j,k)} (A^{(0)}_{(n-1),(j-\ell),(k-1)} + A^{(-(\ell+1))}_{n-1,j,k}) X^n Y^j Z^k \\ \vdots \\ \sum_{(n,j,k)} (A^{(0)}_{n-1,(j-(p-2)),k-1}) X^n Y^j Z^k \end{bmatrix}.
\end{aligned}$$

https://doi.org/10.1017/S0004972716001313 Published online by Cambridge University Press

Let us write

$$\alpha = \begin{bmatrix} A^{(0)}(X,Y,Z) \\ \vdots \\ A^{(-\ell)}(X,Y,Z) \\ \vdots \\ A^{(-(p-2))}(X,Y,Z) \end{bmatrix}, \quad \mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \quad M = \begin{bmatrix} XZ & X & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ XY^{\ell}Z & 0 & 0 & 0 & X & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ XY^{(p-3)}Z & 0 & 0 & 0 & 0 & \dots & 0 & X \\ XY^{(p-2)}Z & 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}.$$

Our system of equations can then be written

$$\alpha = M\alpha + \mathbf{e}_1 \quad \text{or} \quad \alpha(I_{(p-1)} - M) = \mathbf{e}_1,$$

where $I_{(p-1)}$ denotes the $(p-1) \times (p-1)$ identity matrix. Formally,

$$\alpha = \sum_{n=0}^{\infty} M^n \mathbf{e}_1.$$

What is crucial is the recognition that if we set $\delta = Y$ and $1/m = Z$, then

$$M = X \begin{bmatrix} \delta & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ \delta/m^{\ell} & 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ \delta/m^{(p-3)} & 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ \delta/m^{(p-2)} & 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}.$$

The sum of the terms that comprise the first column of M^n coincides with $\mu(U_n)$ for $n \geq 0$. To prove that $\mu(U_n) \to 0$ as $n \to \infty$, it therefore suffices to prove that all the eigenvalues of the matrix M lie in the *open unit disc*, for then M^n converges uniformly to the zero matrix as $n \to \infty$. The characteristic polynomial of M is

$$\begin{aligned} g(\lambda) &= \det \begin{vmatrix} \lambda - Z & -1 & 0 & 0 & 0 & \dots & 0 & 0 \\ -YZ & \lambda & -1 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ -Y^{\ell}Z & 0 & 0 & \cdots & \lambda & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \dots & \ddots & \vdots \\ -Y^{(p-3)}Z & 0 & 0 & 0 & 0 & \dots & \lambda & -1 \\ -Y^{(p-2)}Z & 0 & 0 & 0 & 0 & \dots & 0 & \lambda \end{vmatrix} \\ &= \lambda^{p-1} + a_{p-2}\lambda^{p-2} + a_{p-3}\lambda^{p-3} + \cdots + a_1\lambda + a_0, \end{aligned}$$

https://doi.org/10.1017/S0004972716001313 Published online by Cambridge University Press

say, where $a_j = \pm Y^{(p-2)-j}Z$. Since $|a_j| = \delta/m^{(p-2)-j}$ for $Y = 1/m$ and $Z = \delta$, we conclude that $|a_j| = \delta/m^{(p-2)-j}$ for $0 \le j \le p-2$. Thus,

$$\sum_{j=0}^{(p-2)} |a_j| = \sum_{j=0}^{(p-2)} \mu(U_{-j}) < 1.$$

To see that all the roots of $g(\lambda)$ lie in the open unit disc, we argue as follows. If $|z| \ge 1$, then

$$|z^{p-1}| > \Big(\sum_{j=0}^{p-2} |-a_j|\Big)|z|^{p-1} \ge \sum_{j=0}^{p-2} |-a_j||z|^j \ge \Big|\sum_{j=0}^{p-2} -a_j z^j\Big|.$$

Thus,

$$|g(z)| = \Big|z^{p-1} - \sum_{j=0}^{p-2}(-a_j)z^j\Big| \ge |z^{p-1}| - \Big|\sum_{j=0}^{p-2} -a_j z^j\Big| > 0.$$

Thus, $g(z) = 0$ implies that $|z| < 1$. We conclude that M^n converges to the $(p-1)\times(p-1)$ zero matrix as $n \to 0$. Hence, $\mu(U_n) \to 0$ as $n \to \infty$. Since $U_0 \supset U_1 \supset \cdots$, it is clear from the definition of U_n that $U^\star \cap U_0 = \bigcap_{n=1}^{\infty} U_n$. Thus, we have proved $\mu(U^\star \cap U_0) = 0$.

Next, we observe that $U^\star = \bigcup_{s=0}^{p-2}(U^\star \cap U_{-s})$ and define

$$U_{n,-s} = \{y \in U_{-s} : f^n(y) \notin C\} \quad \text{and} \quad x_{-s} = \frac{x - b(1 + m + \cdots + m^{s-1})}{m^s} \quad \text{for } x \in [0, 1].$$

Then $x \in U_0$ if and only if $x_{-s} \in U_{-s}$ for $1 \le s \le p-2$. Furthermore, $f^n(x) = f^{n+1}(x_{-s})$. So, $f^n(x) \notin C$ if and only if $f^{n+1}(x_{-s}) \notin C$. Therefore, $x \in U_n$ if and only if $x_{-s} \in U_{(n+s),-s}$, and so $\mu(U_{(n+s),-s}) = (1/m^s)\mu(U_n)$. Thus, as $n \to \infty$, $\mu(U_n) \to 0$ if and only if $\mu(U_{n,-s}) \to 0$ for every $s \in \{1, \ldots, (p-2)\}$.

Now let

$$U_n^\star = \{x \in [0, 1] \mid f^n(x) \notin C\}.$$

Since $f(C) \subset C$, we have $U_0^\star \supset U_1^\star \supset U_2^\star \cdots$ and $U^\star = \bigcap_{n=1}^{\infty} U_n^\star$. Furthermore,

$$U_n^\star = U_n \cup \bigcup_{s=1}^{p-2} U_{n,-s} \Longrightarrow \mu(U_n^\star) = \mu(U_n) + \sum_{s=1}^{p-2} \mu(U_{n,-s}).$$

Since $\mu(U_n) \to 0$ and $\mu(U_{n,-s}) \to 0$, this gives $\mu(U_n^\star) \to 0$ as $n \to \infty$. Finally, since $U^\star = \bigcap_{n=1}^{\infty}$, we have $\mu(U^\star) = 0$, and the proof is finished. □

REMARK 3.2. We observe that $C_{p-1} \subset C^\star$, and so, provided that $f(C_{p-1}) \subset C^\star$, the precise value of the function $f(x)$ on C_{p-1} has no bearing on $\mu(U^\star)$ provided that we can prove that f is chaotic in the sense of Li and Yorke. Hence, the method first initiated by Nathanson and essentially subscribed to in this paper is flexible enough to accommodate the different forms of almost everywhere eventually periodic behaviour that we have specified in the theorems and corollaries in Section 2 above.

Acknowledgement

The authors thank Nitya Kitchloo for several helpful discussions and for an observation that led to the replacement of an elementary but notationally complicated proof, and Laura Gardini for her careful reading.

References

[1] F. Balibrea and V. Jiménez-López, 'The measure of scrambled sets: a survey', *Acta Univ. M. Belii Ser. Math.* **7** (1999), 3–11.
[2] J. Benhabib, 'Chaotic dynamics in economics', in: *The New Palgrave Dictionary of Economics* (eds. S. N. Durlauf and L. E. Blume) (Macmillan, London, 2008).
[3] A. Blaya and V. Jiménez-López, 'Is trivial dynamics that trivial?', *Amer. Math. Monthly* **113** (2006), 109–133.
[4] K. M. Brucks and H. Bruin, *Topics from One-Dimensional Dynamics* (Cambridge University Press, Cambridge, 2004).
[5] R. A. Dana, C. Le Van, T. Mitra and K. Nishimura (eds.), *Handbook of Optimal Growth,* Vol. 1 (Springer, Berlin, 2006).
[6] B. Du, 'Are chaotic functions really chaotic?', *Bull. Aust. Math. Soc.* **28** (1983), 53–66.
[7] B. Du, 'Almost all points are eventually periodic with minimal period 3', *Bull. Inst. Math. Acad. Sin. (N.S.)* **12** (1984), 405–411.
[8] B. Du, 'Topological entropy and chaos of interval maps', *Nonlinear Anal.* **11** (1987), 105–114.
[9] M. Fujio, 'Optimal transition dynamics in the Leontief two-sector growth model with durable capital: the case of capital-intensive consumption goods', *Jpn. Econ. Rev.* **60** (2009), 490–511.
[10] V. Jimenez-López, 'Order and chaos for a class of piecewise linear maps', *Internat. J. Bifur. Chaos* **5** (1995), 1379–1394.
[11] M. Ali Khan and T. Mitra, 'On choice of technique in the Robinson–Solow–Srinivasan model', *Int. J. Econ. Theory* **1** (2005), 83–109.
[12] M. Ali Khan and T. Mitra, 'On topological chaos in the Robinson–Solow–Srinivasan model', *Econom. Lett.* **88** (2005), 127–133.
[13] M. Ali Khan and A. Piazza, 'Turnpike theory: a current perspective', in: *The New Palgrave Dictionary of Economics Online* (eds. S. N. Durlauf and L. E. Blume) (Macmillan, London, 2012).
[14] T. Li and J. Yorke, 'Period three implies chaos', *Amer. Math. Monthly* **82** (1975), 985–992.
[15] J. D. Louck and N. Metropolis, *Symbolic Dynamics of Trapezoidal Maps* (D. Reidel, Dordrecht, 1986).
[16] M. Majumdar, T. Mitra and K. Nishimura, *Optimization and Chaos* (Springer, Berlin, 2000).
[17] M. B. Nathanson, 'Piecewise linear functions with almost all points eventually periodic', *Proc. Amer. Math. Soc.* **60** (1976), 75–81.
[18] K. Nishimura and M. Yano, 'Nonlinear dynamics and chaos in optimal growth: an example', *Econometrica* **63** (1995), 981–1001.
[19] V. Velupillai, *Computable Economics* (Oxford University Press, Oxford, 2000).
[20] H. S. Wilf, *Generatingfunctionology* (Academic Press, New York, 1994).

M. ALI KHAN, Department of Economics,
Johns Hopkins University,
Baltimore, MD 21218, USA
e-mail: akhan@jhu.edu

ASHVIN V. RAJAN, 3935 Cloverhill Road,
Baltimore, MD 21218, USA
e-mail: ashvinrj@aol.com