$1+A_{40}=$	18·14242			
	52502			
	3628484			
	90712			
	3628			
	907			
	372·3731	372.3731		
Numerator =		505·3987	log.	2·7036341
$v^{20}\,.\,p_{40\,.\,20}$	as above.		log.	$\bar{1}$·5992787
$1+a_{60}=$	11·49139 . . . (Jones, p. 312)		„	1·0603725
		4·56721	„	0·6596512
$1+a_{40}=$	(Jones, p. 311)	18·14242		
Denominator =		13·57521	log.	1·1327465
			colog.	$\bar{2}$·8672535
Numerator, as above	. . . .		log.	2·7036341
P′ =		37·22953	„	1·5708876

Both methods, of course, give for the required premium the same value, £37 4s. 7d.; but the second is attended with at least four times the amount of labour* required by the first.

Another most important advantage possessed by the new method, is the facility it affords for the extension of the data. Columns of *present* values, unlimited in number, can be found by mere addition, while, by the old method, the formation of every additional column is attended with as much labour as the formation of the first.

P. GRAY.

Baker Street, Lloyd Square,
Nov. 23, 1850.

ON THE VALUE OF ANNUITIES CERTAIN, OF WHICH THE SUCCESSIVE PAYMENTS ARE THE FIGURATE NUMBERS.

To the Editors of the Assurance Magazine.

Gentlemen,—The following remarks will probably be found useful to many of your readers, and I therefore place them at your disposal.

I call 1, 1, 1; 1, 2, 3; 1, 3, 6, &c., figurate numbers, of 1th, 2th, 3th orders; and denote the respective annuities by Σv^n $\Sigma^2 v^n$ $\Sigma^3 v^n$ . . $\Sigma^p v^n$.

The first few cases are deduced from each other by finite integration, from which cases the general form is found by induction.

$$\sum v^n = \frac{1}{i} - \frac{v^n}{i} \quad \sum^2 v^n = \frac{n}{i} - \frac{1}{i}\left(\frac{1}{i} - \frac{v^n}{i}\right) = \frac{n}{i} - \frac{1}{i^2} + \frac{v^2}{i^2}$$

$$\sum^3 v^n = \frac{n(n+1)}{1\,.\,2\,.\,i} - \frac{n}{i^2} + \frac{1}{i^2}\left(\frac{1}{i} - \frac{v^2}{i}\right) = \frac{n(n+1)}{1\,.\,2\,.\,i} - \frac{n}{i^2} + \frac{1}{i^3} - \frac{v^2}{i^3}$$

and generally,

$$\sum^p v^n = \frac{n\,(n+1)\,.\,.\,.\,.\,(n+p-2)}{1\,.\,2\,.\,.\,(p-1)\;i} - \frac{n\,(n+1)\;(n+p-3)}{1\,.\,2\,.\,(p-2)\;i^2} + \,.\,.\,.\,.\, \pm \frac{n}{i^{p-1}} \mp \frac{1}{i^p} \pm \frac{v^n}{i^n}$$

in which the first term is invariably plus, then $- + - +$ &c.

* This would be considerably increased if the value of A_{60} were not assumed.

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Multiplying both sides by i^p and altering the arrangement of the terms,

$$i^p \times \sum^p v^n = \pm v^n \pm 1 \pm ni \pm \frac{n(n+1)}{1 . 2} i^2 \mp \ldots + \frac{n(n+1)(n+p-2)}{1 . 2 . 3 . (p-1)} i^{p-1}$$

the second side commencing with its second term is evidently the first p terms in the development of $\mp (1+i)^{-n}$ if this be denoted by $\mp \left[(1+i)^{-n} p\right]$

$$\sum^p v^n = \frac{1}{i^p} \left(\pm v^n \mp \left[(1+i)^{-n} p\right]\right)$$

The number of terms and consequently the labour of computing $\Sigma^p v^n$ increases with p, but another form may be obtained for it, which has the singularity of being more readily computed when p is a large number, than when it is a small one.

$$\mp \left[(1+i)^{-n} p\right] = \mp (1+i)^{-n} - \left(-\frac{n(n+1)\ \ (n+p-1)}{1 . 2 . \ \ p} i^p + - + \text{\&c.}\right)$$

$$\therefore \sum^p v^n = \frac{1}{i^p} \left(\pm v^n \mp v^n + \frac{n(n+1)\ \ (n+p-1)}{1 . 2 . p} i^p - + - \text{\&c.}\right)$$

$$\text{or} \sum^p v^n = \frac{n(n+1) \ldots (n+p-1)}{1 . 2 . 3 . p} \left(1 - \left(\frac{n+p}{p+1}\right) i + \frac{(n+p)(n+p+1)}{(p+1)(p+2)} i^2 - \text{\&c.}\right)$$

an infinite series, but in all ordinary cases sufficiently convergent to give the value of $\Sigma^p v^n$ when p is not one of the first few numbers, much more readily than the previous formula.

It is often necessary to compute at the same time Σv^n, $\Sigma^2 v^n$, $\Sigma^3 v^n$, $\Sigma^p v^n$. When n is a given number, this may be done continuously by an adaptation of the primary formula.

Comparison of the values of Σv^n, $\Sigma^2 v^n$, $\Sigma^3 v^n$, given before, will show that

$$\sum^2 v^n = \frac{1}{i} \left(n - \Sigma v^n\right)$$

$$\sum^3 v^n = \frac{1}{i} \left(\frac{n(n+1)}{1 . 2} - \Sigma^2 v^n\right)$$

and that this will hold generally, which indicates the method to be followed.

If Σv^n, $\Sigma^2 v^n$, $\Sigma^p v^n$, are computed in this continuous manner, they may be all verified by computation of $\Sigma^p v^n$ directly from either of the formulæ given for that purpose.

Yours, &c.

November, 1850. W. ORCHARD, F.I.A.

To the Editors of the Assurance Magazine.

Gentlemen,—At the last meeting of the Institute, an interesting paper was read by Mr. Hardy, one of the vice-presidents. It had reference to a case that frequently arises in practice, and which is this:—I purchase an annuity certain, for, say n years, for which I pay its tabular value, at say five per cent. Now, reserving this five per cent. on the purchase money, out of the yearly rents as they are received, it is obvious that the balance must be immediately invested, *also at five per cent.*, in order to reproduce my capital at the end of the term. Unless it is so invested—if it is invested at a lower rate, say three per cent.—my capital will not be reproduced, and I shall not have realized five per cent. on my original investment. The question then arises, how ought the purchase money to be regulated, with reference to the annuity to be bought, and the rate at which the yearly balances can be invested, so that I may realize a specified interest on it during the entire term? Or, if we call the purchase money m, the annual rent p, the rate that can be realized on the yearly balances r', and the rate we desire to make on the whole transaction and during the entire term r, it is required to determine the relation between m and p.

The yearly rent being p, and a year's interest on the purchase-money mr, reserving this we have $p - mr$, for the balance to be yearly invested at the rate r'. Hence, if we denote the amount of the annuity of £1 for n years, at this rate, by A, we have for the amount of the annuity $p - mr$, at the end of the term $(p - mr)$ A. But, by the condition, this is equal to m, the purchase-money. That is,

$$(p - mr)\,\mathrm{A} = m;$$
or, $$\mathrm{A}p - \mathrm{A}mr = m.$$

Hence, if m, the sum to be invested, is known, and the yearly rent to be demanded is required, we have,

$$p = m\left(\frac{1}{\mathrm{A}} + r\right);$$

or, if p, the annuity to be purchased, is given, and m, the purchase-money, is required, we have,

$$m = p\,\frac{1}{\frac{1}{\mathrm{A}} + r}.$$

These are formulas of easy application, which is a fortunate thing, as there are so many variable elements (r, r' and n), that it would obviously be impracticable to meet every case by the use of previously constructed tables.

But, as it appears to me, a more practicable question remains behind. Although I realize only the rate r, on my original investment, my customer pays more than r on the advance to him. For while that is the rate he pays on the yearly decreasing portion of it in his hands, he also pays the difference between r and r', to the end of the term, on his yearly repayments, making on the whole an average rate* higher than r, since, by hypothesis, r' is less than r. It, therefore, clearly would not be safe to profess to be charging only the rate r for an advance, made in the circumstances supposed. A contract, specifying this as the rate, certainly could not be enforced. In practice, then, the arrangement will be made without reference to a rate of interest as between the parties, but each will make the best bargain for himself he can. Preliminaries being arranged, therefore, the sum to be advanced, and the annuity to be paid being fixed, it is a question of importance to the lender to be able readily to ascertain the rate he will have realized by the transaction. That is, he will want to know how much he can reserve from the yearly payments of the annuity as interest on his advance, so as to leave just sufficient to replace his capital at the end of the term. For this purpose, in the foregoing expressions, A being known (since n, the term, and r', the rate at which he can invest his yearly repayments, are known) we have to consider m and p, as given, and thence to determine r. Thus,

$$\mathrm{A}p - \mathrm{A}mr = m;$$
$$\therefore \quad \mathrm{A}mr = \mathrm{A}p - m,$$
and $$r = \frac{\mathrm{A}p - m}{\mathrm{A}m} = \frac{p}{m} - \frac{1}{\mathrm{A}}$$

This formula, like the others, will be of sufficiently easy application.

I am, Sir, your obedient servant,

London, Nov. 30, 1850. A SUBSCRIBER.

[NOTE.—Our correspondent does not seem to have anticipated that Mr. Hardy's paper would appear in this *Magazine*. We insert his letter, notwithstanding, since it gives a clear and simple account of the matter. The problem has been investigated also by the late Mr. Morgan, as will be seen on reference

* This rate may be found from the tables, approximately at least. It is the rate which gives m as the present value of an annuity of p, or $\frac{m}{p}$ as the present value of an annuity of £1, for n years.

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to his work on "Assurances," page 321, in the Appendix. It does not, however, seem to have occurred to any of these writers, that by far the most simple way to treat the question would be, to construct tables, showing the annual payments required at practicable rates to produce £1, at the end of n years. Calling these results r', we should have the relation between p and m (to use our correspondent's notation) by inspection, and $m \quad \frac{p}{r+r'}$, $p=m\left(r+r'\right)$, and $r=\frac{p}{m}-r'$. Ed. *A. M.*]

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