Standard Functors and Isomorphisms

Tensor functors. Fix a pair of rings Λ, Γ. A bimodule ${}_\Lambda M_\Gamma$ yields an adjoint pair of functors

$$- \otimes_\Lambda M \colon \operatorname{Mod}\Lambda \longrightarrow \operatorname{Mod}\Gamma \qquad \text{and} \qquad \operatorname{Hom}_\Gamma(M, -) \colon \operatorname{Mod}\Gamma \longrightarrow \operatorname{Mod}\Lambda.$$

An additive functor $F \colon \operatorname{Mod}\Lambda \to \operatorname{Mod}\Gamma$ is of the form $F = - \otimes_\Lambda M$ for some bimodule ${}_\Lambda M_\Gamma$ if and only if F preserves all coproducts and cokernels. In that case $M = F(\Lambda)$ with Λ acting via $\Lambda = \operatorname{End}_\Lambda(\Lambda) \to \operatorname{End}_\Gamma(F(\Lambda))$.

Tensor-hom adjunction. Fix a pair of rings Λ, Γ and modules $(X_\Lambda, Y_\Gamma, {}_\Lambda M_\Gamma)$. Then there is a natural isomorphism

$$\operatorname{Hom}_\Lambda(X, \operatorname{Hom}_\Gamma(M, Y)) \xrightarrow{\sim} \operatorname{Hom}_\Gamma(X \otimes_\Lambda M, Y)$$

given by

$$\phi \longmapsto (x \otimes m \mapsto \phi(x)(m)).$$

Modules over algebras. Fix an algebra Λ over a commutative ring k and modules $(X_\Lambda, {}_\Lambda Y, Z_k)$. Then there are natural isomorphisms

$$\operatorname{Hom}_\Lambda(X, \operatorname{Hom}_k(Y, Z)) \cong \operatorname{Hom}_k(X \otimes_\Lambda Y, Z) \cong \operatorname{Hom}_\Lambda(Y, \operatorname{Hom}_k(X, Z)).$$

Finitely generated projective modules. Fix a pair of rings Λ, Γ and modules $(X_\Lambda, Y_\Gamma, {}_\Lambda M_\Gamma)$. Then there is a natural homomorphism

$$X \otimes_\Lambda \operatorname{Hom}_\Gamma(Y, M) \longrightarrow \operatorname{Hom}_\Gamma(Y, X \otimes_\Lambda M)$$

given by

$$x \otimes \phi \longmapsto (y \mapsto x \otimes \phi(y)),$$

which is invertible if X or Y is finitely generated projective.

https://doi.org/10.1017/9781108979108.004 Published online by Cambridge University Press

Duality. Fix a pair of rings Λ, Γ and modules $(X_\Lambda, {}_\Gamma Y, {}_\Gamma M_\Lambda)$. Then there is a natural homomorphism

$$X \otimes_\Lambda \operatorname{Hom}_\Gamma(M, Y) \longrightarrow \operatorname{Hom}_\Gamma(\operatorname{Hom}_\Lambda(X, M), Y)$$

given by

$$x \otimes \phi \longmapsto (\psi \mapsto \phi(\psi(x))),$$

which is invertible if X is finitely generated projective or if X is finitely presented and Y is injective.

Change of rings. Let $\phi\colon \Lambda \to \Gamma$ be a ring homomorphism, which yields canonical bimodules ${}_\Lambda\Gamma_\Gamma$ and ${}_\Gamma\Gamma_\Lambda$. Then the functor $\operatorname{Mod}\Gamma \to \operatorname{Mod}\Lambda$ given by *restriction of scalars*

$$\phi^* := \operatorname{Hom}_\Gamma(\Gamma, -) \cong - \otimes_\Gamma \Gamma =: \phi^!$$

admits a left adjoint $\phi_!$ (*extension of scalars*) and a right adjoint ϕ_*

$$\operatorname{Mod}\Gamma \;\underset{\underset{\phi_*}{\longleftarrow}}{\overset{\overset{\phi_!}{\longleftarrow}}{\xrightarrow{\phi^*=\phi^!}}}\; \operatorname{Mod}\Lambda$$

which are given by

$$\phi_! := - \otimes_\Lambda \Gamma \qquad \text{and} \qquad \phi_* := \operatorname{Hom}_\Lambda(\Gamma, -).$$

Change of categories. Let $f\colon \mathcal{C} \to \mathcal{D}$ be an additive functor between additive categories. Then the functor $f^*\colon \operatorname{Mod}\mathcal{D} \to \operatorname{Mod}\mathcal{C}$ given by $Y \mapsto Y \circ f$ admits a left adjoint $f_!$ and a right adjoint f_*

$$\operatorname{Mod}\mathcal{D} \;\underset{\underset{f_*}{\longleftarrow}}{\overset{\overset{f_!}{\longleftarrow}}{\xrightarrow{f^*=f^!}}}\; \operatorname{Mod}\mathcal{C}$$

which for $X \in \operatorname{Mod}\mathcal{C}$ with presentation

$$\coprod_j \operatorname{Hom}_\mathcal{C}(-, C_j) \longrightarrow \coprod_i \operatorname{Hom}_\mathcal{C}(-, C_i) \longrightarrow X \longrightarrow 0$$

are given by the presentation

$$\coprod_j \operatorname{Hom}_\mathcal{D}(-, f(C_j)) \longrightarrow \coprod_i \operatorname{Hom}_\mathcal{D}(-, f(C_i)) \longrightarrow f_!(X) \longrightarrow 0$$

and

$$f_*(X)(D) = \operatorname{Hom}(\operatorname{Hom}_\mathcal{D}(f-, D), X) \qquad (D \in \mathcal{D}).$$

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