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ON ALGEBRAIC INVARIANTS FOR FREE ACTIONS ON HOMOTOPY SPHERES

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Abstract

We investigate conjectures and questions regarding topological phenomena related to free actions on homotopy spheres and present some affirmative answers.

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1. Introduction

The purpose of this paper is to investigate the following conjectures and questions regarding topological phenomena related to free actions on homotopy spheres.

CONJECTURE I [31]. A group G has periodic cohomology after some steps if and only if G admits a finite-dimensional free G-CW-complex which is homotopy equivalent to a sphere.

QUESTION A. Suppose there exists a nonnegative integer k such that for each proper subgroup $H < G$ of finite projective complete cohomological dimension, $\operatorname{pccd} H \leq k$. Is it true that $\operatorname{pccd} G < \infty$?

QUESTION B. For which groups is the following statement true?

A group G has periodic cohomology of period q after some steps with periodicity isomorphisms induced by the cup product with an element in $H^q(G, \mathbb{Z})$ if and only if G has periodic homology of period q after some steps with periodicity isomorphisms induced by the cap product with an element in $H^q(G, \mathbb{Z})$.

CONJECTURE II [33]. If G is an elementary amenable group, then $\operatorname{Gcd}_{\mathbb{Q}} G = \operatorname{Gcd} G$.

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Suppose that a group G admits a finite dimensional free G-CW-complex which is homotopy equivalent to a sphere. Then it can be seen that there exists an exact sequence of $\mathbb{Z}G$-modules

$$0 \to \mathbb{Z} \to A \to P_{n-1} \to \cdots \to P_0 \to \mathbb{Z} \to 0,$$

where each P_i is projective and $\text{proj.dim}_{\mathbb{Z}G} A < \infty$ (see [24, Proposition 5.10]), and then it follows from [27, Theorem 2.6] that G has periodic cohomology of period q after k-steps, that is, $H^i(G, -)$ and $H^{i+q}(G, -)$ are naturally equivalent for all $i > k$. Thus the essential part of Conjecture I is whether the periodicity of the group cohomology is the algebraic characterisation of those groups G which admit a finite dimensional free G-CW-complex which is homotopy equivalent to a sphere. Mislin and Talelli [24] describe the history of this conjecture and prove Conjecture I when G belongs to the class $\mathbf{H}\mathfrak{F}_b$ of groups.

The class $\mathbf{H}\mathfrak{F}$ is the smallest class of groups that contains finite groups and contains a group G whenever G admits a finite dimensional contractible G-C-complex whose stabilisers are already in $\mathbf{H}\mathfrak{F}$ [20]. The class $\mathbf{H}\mathfrak{F}_b$ is the subclass consisting of those groups in $\mathbf{H}\mathfrak{F}$ for which there is a bound on the orders of their finite subgroups. The class $\mathbf{LH}\mathfrak{F}$ is the class of groups such that all of its finitely generated subgroups are in $\mathbf{H}\mathfrak{F}$. The class $\mathbf{LH}\mathfrak{F}$ contains, for example, all elementary amenable groups and all linear groups, and it is extension closed, closed under ascending unions and closed under amalgamated free products and HNN extensions.

In [1] Adem and Smith proved Conjecture I under the hypothesis that the periodicity isomorphisms are given by the cup product with a cohomology element. In [31] Talelli combined the following theorem with the result of Adem and Smith to show that Conjecture I holds for the groups of the class $\mathbf{H}\mathfrak{F}$.

Theorem 1.1 [31, Theorem 3.2, Corollary 3.3, Proposition 3.4]. *Let G be a group with periodic cohomology of period q after k-steps. Then the following statements are equivalent:*

(1) $H^i(G, P) = 0$ *for all $i > k$ and every projective $\mathbb{Z}G$-module P;*
(2) *the periodicity isomorphism is induced by the cup product with an element in $H^q(G, \mathbb{Z})$;*
(3) $\text{spli}\, G < \infty$.

For $G \in \mathbf{H}\mathfrak{F}$ the above equivalent conditions hold. Thus if $G \in \mathbf{H}\mathfrak{F}$, Conjecture I holds for G. Here $\text{spli}\, G$ is the supremum of the projective lengths of the injective $\mathbb{Z}G$-modules [12].

For an arbitrary group G, the complete cohomology of G was introduced independently by Benson and Carlson [6], Mislin [23] and Vogel [13] and their approaches turned out to be isomorphic (as shown by Mislin). The projective complete cohomological dimension, $\text{pccd}\, G$, of a group G comes naturally from the complete cohomology of G. It is defined as the least integer $n \geq -1$ for which $H^i(G, -) \cong \widehat{H}^i(G, -)$ for all $i > n$, or ∞ if no such n exists, where $\widehat{H}^i(G, -)$ is the complete

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cohomology of G [17]. The possible values of $\operatorname{pccd} G$ are integers greater than or equal to -1 and ∞. It is known that if $G = *_{n\in\mathbb{N}} G_n$ and G_n is a free abelian group of rank n, then $\operatorname{pccd} G = \infty$, and if G is the Thompson group T, $\oplus_{i=1}^{\infty}\mathbb{Z}$, or $GL_n(K)$, where K is a subfield of the algebraic closure of $\mathbb{Q}$, then $\operatorname{pccd} G = -1$ [17, 18].

Note that condition (1) of Theorem 1.1 is equivalent to the condition $\operatorname{pccd} G \leq k$ (see [17, Proposition 2.3]). It can be seen from [1, Corollary 2.10] and Theorem 1.1 that the validity of Conjecture I depends on the finiteness of the projective cohomological dimension. It was also known from [17, Theorem 3.17] that if $G \in \mathbf{H}\mathfrak{F}$ or $\operatorname{pccd} G < \infty$, then condition (1) of Theorem 1.1 holds. Thus, if G has periodic cohomology of period q after k-steps, then every proper subgroup $H < G$ of finite projective complete cohomological dimension satisfies $\operatorname{pccd} H \leq k$, since H also has periodic cohomology of period q after k-steps. From this viewpoint, we may ask whether Question A has an affirmative answer.

Note that if G has periodic cohomology of period q after k-steps, then G admits a complete projective resolution [24, 27] and so $\operatorname{pccd} G > -1$ [17, Proposition 3.10]. Thus, when we consider Conjecture I, we only need to treat the case that the pccd of a group is greater than -1. But we also consider the possibility that the pccd of a group is -1 in Question A. Analogous questions to Question A were also considered in [19, 25], but there is a crucial difference. The finiteness of the cohomological dimension of a group is a subgroup closed property while the finiteness of the pccd of a group is not a subgroup closed property. As we noted earlier, if G is a free abelian group of infinite rank then $\operatorname{pccd} G = -1$, while for any positive integer k there is a subgroup H of G with $\operatorname{pccd} H > k$ [17]. As noted above, if Question A has an affirmative answer, then Conjecture I is a theorem. One of the purposes of this paper is to present a large class of groups such that if we restrict ourselves to this class of groups we have an affirmative answer to Question A (and therefore a partial answer to Conjecture I). In Theorem 2.9, we show that if a group G belongs to a certain class $\mathfrak{X}$ of groups, then Question A is affirmative for G. If, in addition, G belongs to the class $\mathbf{L}\mathfrak{X}$ and satisfies the $\aleph_n$-condition, then Question A has an affirmative answer for G (see Section 2 for the definitions of $\mathfrak{X}, \mathbf{L}\mathfrak{X}$ and the $\aleph_n$-condition).

In [4], using the notion of flat covers and proper flat resolutions, Asadollahi *et al.*, investigated the notion of periodic homology of period q after k steps, that is, that $H_i(G, -)$ and $H_{i+q}(G, -)$ are naturally equivalent for all $i > k$. They showed that if a group G with the property that every flat $\mathbb{Z}G$-module has finite projective dimension, then G has periodic cohomology of period q after some steps with the periodicity isomorphisms induced by the cup product with an element in $H^q(G, \mathbb{Z})$ if and only if G has periodic homology of period q after some steps with the periodicity isomorphisms induced by the cap product with an element in $H^q(G, C)$, where C is the cotorsion envelope of the trivial $\mathbb{Z}G$-module $\mathbb{Z}$. In [32], Talelli showed that a countable group G has periodic cohomology of period q after some steps with the periodicity isomorphisms induced by the cup product with an element in $H^q(G, \mathbb{Z})$ if and only if G has periodic homology of period q after some steps with the periodicity isomorphisms induced by the cap product with an element in $H^q(G, \mathbb{Z})$. In Theorem 3.3, we show

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that if G is a finite group or an infinite group of cardinality $\aleph$ such that $2^{\aleph} = \aleph_k$ for some $k \in \mathbb{N}$, then G satisfies the statement in Question B.

On the other hand, recall the notion of property $\mathcal{P}_1$ [22, 30] which comes naturally from the notion of periodic cohomology after 1-step [28, 29]: a group G is said to have property P_1 if there exists a $\mathbb{Z}$-free $\mathbb{Z}G$-module A such that $\text{proj.dim}_{\mathbb{Z}G} A \leq 1$ and $H^0(G, A) \neq 0$. In [22, 30] Kropholler and Talelli showed that G has property $\mathcal{P}_1$ if and only if $\text{cd}_{\mathbb{Q}} G \leq 1$ if and only if G is the fundamental group of a graph of finite groups. As noted above, if G admits a finite dimensional free G-CW-complex homotopy equivalent to a sphere, then there exists an exact sequence of $\mathbb{Z}G$-modules $0 \to \mathbb{Z} \to A \to P_{n-1} \to \cdots \to P_0 \to \mathbb{Z} \to 0$, where each P_i is projective and $\text{proj.dim}_{\mathbb{Z}G} A < \infty$. It can be easily seen that $H^0(G, A) \neq 0$. In [5, 31] Bahlekeh *et al.* and Talelli showed that $\text{spli}\, G < \infty$ if and only if there is a $\mathbb{Z}$-split, $\mathbb{Z}G$-exact sequence $0 \to \mathbb{Z} \to A$ with A $\mathbb{Z}$-free and $\text{proj.dim}_{\mathbb{Z}G} A < \infty$. Moreover, in this case $\text{proj.dim}_{\mathbb{Z}G} A = \text{Gcd}\, G$. Here $\text{Gcd}\, G$ is the Gorenstein cohomological dimension of G which is defined as the Gorenstein projective dimension of the trivial $\mathbb{Z}G$-module $\mathbb{Z}$ [14]. It is known from [3, 9, 12, 17] that for any group G,

$$\text{pccd}\, G \leq \underline{\text{cd}}\, G = \text{Gcd}\, G \leq \text{silp}\, G = \text{spli}\, G \leq \underline{\text{cd}}\, G + 1 = \text{Gcd}\, G + 1,$$

where $\underline{\text{cd}}\, G := \sup\{n : \text{Ext}^n_{\mathbb{Z}G}(M, F) \neq 0, M \text{ is } \mathbb{Z}\text{-free and } F \text{ is } \mathbb{Z}G\text{-free}\}$ is Ikenaga's generalised cohomological dimension [15] and $\text{silp}\, G$ is the supremum of the injective lengths of the projective $\mathbb{Z}G$-modules [12]. From this viewpoint, analogous to property $\mathcal{P}_1$, we can naturally consider property $\mathcal{P}_n$. For a positive integer n, a group G is said to have property $\mathcal{P}_n$ if there exists a $\mathbb{Z}$-free $\mathbb{Z}G$-module A such that $\text{proj.dim}_{\mathbb{Z}G} A \leq n$ and $H^0(G, A) \neq 0$. One might expect that G has property $\mathcal{P}_n$ if and only $\text{cd}_{\mathbb{Q}} G \leq n$. Note that for any group G, $\text{Gcd}_{\mathbb{Q}} G \leq \text{Gcd}\, G$, and for an $\mathbf{LH}\mathfrak{F}$-group G, $\text{Gcd}_{\mathbb{Q}} G = \text{cd}_{\mathbb{Q}} G$ [33]. As mentioned in [33], we cannot expect in general that $\text{Gcd}_{\mathbb{Q}} G \leq \text{Gcd}\, G$, because there are torsion-free groups G such that $\text{Gcd}_{\mathbb{Q}} G = \text{cd}_{\mathbb{Q}} G < \text{cd}\, G = \text{Gcd}\, G < \infty$. But, Talelli conjectured in [33] that this is true for elementary amenable groups as in Conjecture II. It can be seen that if an elementary amenable group G is an affirmative answer to Conjecture II, then G has property $\mathcal{P}_n$ if and only if $\text{cd}_{\mathbb{Q}} G \leq n$ (Lemma 3.4). We show in Theorem 3.5 that every elementary amenable group of type FP_∞ satisfies Conjecture II. As a corollary, we also show that if G is an elementary amenable group of type FP_∞, then G has property $\mathcal{P}_n$ if and only if $\text{cd}_{\mathbb{Q}} G \leq n$ (Corollary 3.6).

2. About Conjecture I and Question A

In order to have a class of groups which have an affirmative answer to Question A (and so Conjecture I), we start with the following lemma.

LEMMA 2.1. *Let G admit an n-dimensional contractible G-CW-complex X. If there exists a nonnegative integer k such that for any isotropy subgroup G_σ, $\text{pccd}\, G_\sigma \leq k$, then $\text{pccd}\, G \leq n + k$.*

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Proof. It is well known that for any $\mathbb{Z}G$-module M, there is a spectral sequence

$$E_1^{p,q} = \prod_{\sigma\in\sum_p} H^q(G_\sigma, M) \Rightarrow H^{p+q}(G, M),$$

where $\sum_p$ is a set of representatives for the p-simplices of X mod G. By our assumption, if M is projective, then $E_1^{p,q} = 0$ for $p > n$ or $q > k$ and thereby $E_1^{p,q} \cong E_\infty^{p,q} = 0$ for $p > n$ or $q > k$. Therefore $H^{p+q}(G, P) = 0$ for $p > n$ or $q > k$ and projective P. Hence $\operatorname{pccd} G \le n + k$. □

Recall that every group G is expressed as the direct limit of the direct family of its finitely generated subgroups, that is, $G = \varinjlim_{i\in I} G_i$, where G_i is finitely generated.

Definition 2.2. Let G be an arbitrary group. We say that G satisfies the $\aleph_n$-condition if the cardinality of the directed set I is $\aleph_n$, where n is a nonnegative integer, $G = \varinjlim_{i\in I} G_i$ and each G_i is a finitely generated subgroup of G.

The following can be shown by the method of [15, Proposition 6].

Lemma 2.3. *Let $G = \varinjlim_{i\in I} G_i$, where $G_i < G$ and $|I| = \aleph_n$. Then*

$$\operatorname{pccd} G \le \sup_{i\in I}\{\operatorname{pccd} G_i\} + n + 1.$$

Proof. Notice the following two facts (cf. [16]):

(a) If $\{A_i\}$ is a direct system of R-modules and B is a R-module, then there is a spectral sequence

$$E_2^{p,q} = \varprojlim{}^{(p)} \operatorname{Ext}_R^q(A_i, B) \Rightarrow \operatorname{Ext}_R^{p+q}(\varinjlim A_i, B).$$

(b) Let $\{M_i\}_I$ be an inverse system of modules such that $|I| \le \aleph_n$. Then

$$\varprojlim{}^{(m)} M_i = 0 \quad \text{for } m > n + 1.$$

Since $G = \varinjlim G_i$, we have $\varinjlim(\mathbb{Z} \otimes_{\mathbb{Z}G_i} \mathbb{Z}G) \cong \mathbb{Z}$ and

$$\operatorname{Ext}_{\mathbb{Z}G}^q(\mathbb{Z} \otimes_{\mathbb{Z}G_i} \mathbb{Z}G, B) \cong \operatorname{Ext}_{\mathbb{Z}G_i}^q(\mathbb{Z}, B) \cong H^q(G_i, B).$$

Let P be a projective $\mathbb{Z}G$-module. From the above fact (a), we have the following spectral sequence:

$$\varprojlim{}^{(p)} H^q(G_i, P) \Rightarrow H^{p+q}(G, P).$$

If $\sup\{\operatorname{pccd} G_i\} \le l$, then the spectral sequence only lives in the rectangle $0 \le p \le n + 1, 0 \le q \le l$. Hence if $\operatorname{pccd} G = \infty$, then $\sup\{\operatorname{pccd} G_i\} = \infty$. On the other hand, if $\operatorname{pccd} G < \infty$, then $\sup\{\operatorname{pccd} G_i\} = m < \infty$ and $\operatorname{pccd} G \le m + n + 1$. □

Proposition 2.4. *Let $G = \oplus_{i\in I} G_i$ be a direct sum of groups G_i with $|I| = \aleph_n$. If Question A is affirmative for each G_i, then it is affirmative for G as well.*

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PROOF. Suppose that there exists a nonnegative integer k such that for each proper subgroup $H < G$ of finite projective complete cohomological dimension, $\operatorname{pccd} H \leq k$. Notice that for each $i \in I$ and for any proper subgroup $H_i < G_i$ of finite projective complete cohomological dimension, $\operatorname{pccd} H_i \leq k$, since H_i is a subgroup of G. Thus, for each i, $\operatorname{pccd} G_i < \infty$ and thereby $\operatorname{pccd} G_i \leq k$. Hence, $\operatorname{pccd} G < \infty$ by Lemma 2.3. □

PROPOSITION 2.5. *Let $1 \to N \to G \to Q \to 1$ be an extension of groups such that* $\operatorname{vcd} Q < \infty$. *If Question A is affirmative for N, then it is affirmative for G as well.*

PROOF. Suppose that there exists a nonnegative integer k such that for each proper subgroup $H < G$ of finite projective complete cohomological dimension, $\operatorname{pccd} H \leq k$. Notice that for each proper subgroup $L < N$ of finite projective complete cohomological dimension, $\operatorname{pccd} L \leq k$, since L is a subgroup of G. Since Question A is affirmative for N, $\operatorname{pccd} N < \infty$ and thereby $\operatorname{pccd} N \leq k$. From [17, Proposition 2.5], it follows that $\operatorname{pccd} G \leq k + \operatorname{vcd} Q < \infty$. □

PROPOSITION 2.6. *Suppose that G admits a finite-dimensional contractible G-CW-complex X. If Question A is affirmative for each isotropy group G_σ of X, then it is affirmative for G as well.*

PROOF. Suppose that there exists a nonnegative integer k such that for each proper subgroup $H < G$ of finite projective complete cohomological dimension, $\operatorname{pccd} H \leq k$. Notice that for each isotropy group G_σ, any subgroup $H_\sigma < G_\sigma$ of finite projective complete cohomological dimension satisfies $\operatorname{pccd} H_\sigma \leq k$, since G_σ is a subgroup of G. Since Question A is affirmative for G_σ, $\operatorname{pccd} G_\sigma < \infty$ and thereby $\operatorname{pccd} G_\sigma \leq k$. Hence $\operatorname{pccd} G < \infty$ by Lemma 2.1. □

COROLLARY 2.7. *Let G be a group which belongs to the class $\mathbf{H}\mathfrak{F}$. Then Question A is affirmative for G. If, in addition, G belongs to the class $\mathbf{LH}\mathfrak{F}$ and satisfies the $\aleph_n$-condition, then Question A is affirmative for G.*

PROOF. The corollary follows from Proposition 2.6 and transfinite induction. □

DEFINITION 2.8. Let $\mathfrak{X}$ denote the smallest class of groups which:

(1) contains all groups of type $\mathbf{H}\mathfrak{F}$;
(2) contains all groups G with $\operatorname{spli} G < \infty$;
(3) contains all groups G with $\operatorname{proj.dim}_{\mathbb{Z}G} B(G, \mathbb{Z}) < \infty$;
(4) is closed under direct sums of groups $\oplus_{i \in I} G_i$ with $|I| = \aleph_n$;
(5) is closed under extensions of groups $1 \to N \to G \to Q \to 1$ such that $\operatorname{vcd} Q < \infty$;
(6) is closed under passing to the group G which admits a finite-dimensional contractible G-CW-complex from isotropy groups G_σ.

A group belongs to $\mathbf{L}\mathfrak{X}$ if all of its finitely generated subgroups belong to $\mathfrak{X}$.

THEOREM 2.9. *Let G be a group which belongs to the class $\mathfrak{X}$. Then Question A is affirmative for G. If, in addition, G belongs to the class $\mathbf{L}\mathfrak{X}$ and satisfies the $\aleph_n$-condition, then Question A is affirmative answer for G.*

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Proof. If G belongs to the class $\mathfrak{X}$, then the result follows from Lemma 2.3, Propositions 2.4–2.6, Corollary 2.7 and our preliminaries in Section 2. If G is an $\mathbf{LH}\mathfrak{F}$-group satisfying the $\aleph_n$-condition, then the result follows from Lemma 2.3. □

Corollary 2.10. *Groups belonging to class $\mathfrak{X}$ satisfy Conjecture I. Furthermore, the groups of the class $\mathbf{L}\mathfrak{X}$ satisfying the $\aleph_n$-condition satisfy Conjecture I.*

Remark 2.11. In [8] Dembegioti and Talelli conjectured that for any group G, $\operatorname{fd} G = \operatorname{Gcd} G + 1$, and showed that this holds for some classes of groups. Note that if G has a periodic cohomology after k-steps, then $\operatorname{fd} G \leq k + 1$ [24, Lemma 4.7]. Note also that $\operatorname{pccd} G = \operatorname{Gcd} G$ for a group G with $\operatorname{Gcd} G < \infty$. Thus, if the conjecture of Dembegioti and Talelli is a theorem, then so is Conjecture I.

3. About Conjecture II and Question B

Ikenaga's generalised homological dimension, $\underline{\operatorname{hd}}\, G$, of a group G is defined by $\underline{\operatorname{hd}}\, G := \sup\{n : \operatorname{Tor}_n^G(M, C) \neq 0, M \text{ is } \mathbb{Z}\text{-torsion-free and } C \text{ is cofree}\}$, and $\operatorname{sfli} G$ is the supremum of the flat lengths of injective $\mathbb{Z}G$-modules. We denote by $\operatorname{fd} M$ the flat dimension of a $\mathbb{Z}G$-module M.

Lemma 3.1. *For any group G, the following statements are equivalent:*

(1) *there is a $\mathbb{Z}$-split, $\mathbb{Z}G$-exact sequence $0 \to \mathbb{Z} \to A$ such that A is $\mathbb{Z}$-torsion-free and $\operatorname{fd} A < \infty$;*

(2) $\operatorname{sfli} G < \infty$;

(3) $\underline{\operatorname{hd}}\, G < \infty$.

Proof. (1) ⇔ (2). This was mentioned in [11, Remark 4.4] without a detailed proof. For the convenience of the reader, we give a proof of the implication (1) ⇒ (2), which is a homological analogue of the implication (2) ⇒ (3) of [31, Theorem 2.2].

Let $0 \to \mathbb{Z} \to A \to B \to 0$ be a $\mathbb{Z}$-split, $\mathbb{Z}G$-exact sequence with A $\mathbb{Z}$-torsion-free and $\operatorname{fd} A < \infty$. Since $0 \to \mathbb{Z} \to A \to B \to 0$ is $\mathbb{Z}$-split it follows that for any injective $\mathbb{Z}G$-module I, $0 \to I \to I \otimes A \to I \otimes B \to 0$ is $\mathbb{Z}G$-split exact. Consider a $\mathbb{Z}G$-exact sequence $0 \to K \to F \to I \to 0$ with F a flat $\mathbb{Z}G$-module. Since A is $\mathbb{Z}$-torsion-free, it is $\mathbb{Z}$-flat (cf. [26, Corollary 3.51]) and so the sequence $0 \to K \otimes A \to F \otimes A \to I \otimes A \to 0$ is $\mathbb{Z}G$-exact. Note that $F \otimes A$ is $\mathbb{Z}G$-flat [7, Exercise III.0.1]. Since K is $\mathbb{Z}$-torsion-free and $\operatorname{fd} A < \infty$, it is clear that $\operatorname{fd}(K \otimes A) \leq \operatorname{fd} A$. Thus $\operatorname{fd}(I \otimes A) \leq \operatorname{fd}(K \otimes A) + 1 \leq \operatorname{fd} A + 1$. Since $0 \to I \to I \otimes A \to I \otimes B \to 0$ is $\mathbb{Z}G$-split, it follows that $\operatorname{fd} I \leq \operatorname{fd} A + 1$. Hence, $\operatorname{sfli} G \leq \operatorname{fd} A + 1$.

(2) ⇔ (3). This follows from [18, Proposition 5.4(i)] or [11, Proposition 4.9(iii)]. □

Theorem 3.2. *Let G be an infinite group of cardinality $\aleph$ such that $2^{\aleph} = \aleph_k$ for some $k \in \mathbb{N}$. The the following are equivalent:*

(1) *there is a $\mathbb{Z}$-split, $\mathbb{Z}G$-exact sequence $0 \to \mathbb{Z} \to A$ such that A is $\mathbb{Z}$-torsion-free and $\operatorname{fd} A < \infty$;*

(2) $\operatorname{sfli} G < \infty$;

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(3) $\underline{\mathrm{hd}}\, G < \infty$;
(4) $\underline{\mathrm{cd}}\, G < \infty$;
(5) $\mathrm{spli}\, G < \infty$;
(6) *there is a* $\mathbb{Z}$*-split,* $\mathbb{Z}G$*-exact sequence* $0 \to \mathbb{Z} \to B$ *such that* B *is* $\mathbb{Z}$*-free and* $\mathrm{proj.dim}\, B < \infty$.

Proof. This follows immediately from Lemma 3.1 and [11, Corollary 4.5]. □

Theorem 3.3. *Let G be a countable group or an infinite group of cardinality* $\aleph$ *such that* $2^{\aleph} = \aleph_k$ *for some* $k \in \mathbb{N}$*. Then the following are equivalent:*

(1) *G has periodic homology of period q after some steps with the periodicity isomorphisms induced by the cap product with an element* $g \in H^q(G, \mathbb{Z})$*;*
(2) *G has periodic cohomology of period q after some steps with the periodicity isomorphisms induced by the cup product with an element* $g \in H^q(G, \mathbb{Z})$*.*

Proof. By [32, Theorem], it suffices to consider the case that G is an infinite group of cardinality $\aleph$ such that $2^{\aleph} = \aleph_k$ for some $k \in \mathbb{N}$.

(1) ⇒ (2). From the proof of [32, Proposition] it follows that g is represented by a q-extension of the form $0 \to \mathbb{Z} \to A \to P_{q-2} \to \cdots \to P_0 \to \mathbb{Z} \to 0$ such that each P_i is projective, A is $\mathbb{Z}$-free and $\mathrm{fd}\, A < \infty$. By Theorem 3.1 and [11, Corollary 4.5], we may conclude that $\mathrm{spli}\, G < \infty$ and so $\mathrm{fin.dim}\, G < \infty$ by Theorem 3.2. Thus it follows from [16, Proposition 6] and the argument of dimension shifting that $\mathrm{proj.dim}\, A < \infty$. Hence, the cup product with an element $g \in H^q(G, \mathbb{Z})$ induces periodicity isomorphisms in cohomology after $\mathrm{proj.dim}\, A$-steps.

(2) ⇒ (1). This follows from the argument in the proof of [32, Theorem]. □

By definition, a group G is of type FP_∞ if there exists a projective resolution $P_* \twoheadrightarrow \mathbb{Z}$ in which every P_i is finitely generated (cf. [7]). Note that if G is an elementary amenable group of type FP_∞, then $h(G) = \mathrm{cd}_{\mathbb{Q}}\, G = \mathrm{cd}_{\mathcal{F}}\, G < \infty$, where $h(G)$ is the Hirsch rank of G and $\mathrm{cd}_{\mathcal{F}}\, G$ is the Bredon cohomological dimension of G [21].

Lemma 3.4. *Let G be an elementary amenable group and n a positive integer. If G is an affirmative answer to Conjecture II, then G has property* $\mathcal{P}_n$ *if and only if* $\mathrm{cd}_{\mathbb{Q}}\, G \le n$.

Proof. Since G is an $\mathbf{LH}\mathfrak{F}$-group and satisfies Conjecture II, it follows from [33, Theorem 3.5] that $\mathrm{Gcd}\, G = \mathrm{Gcd}_{\mathbb{Q}}\, G = \mathrm{cd}_{\mathbb{Q}}\, G$. Suppose that G has property $\mathcal{P}_n$. Then there exists a $\mathbb{Z}$-free $\mathbb{Z}G$-module A such that $\mathrm{proj.dim}_{\mathbb{Z}G}\, A \le n$, and so $\mathrm{spli}\, G < \infty$ by [31, Theorem 2.2] and $\mathrm{Gcd}\, G = \mathrm{proj.dim}_{\mathbb{Z}G}\, A$ by [5, Theorem 2.7]. Thus $\mathrm{cd}_{\mathbb{Q}}\, G \le n$. Conversely, suppose that $\mathrm{cd}_{\mathbb{Q}}\, G \le n$. Then $\mathrm{Gcd}\, G \le n$ and there exists a $\mathbb{Z}$-free $\mathbb{Z}G$-module A such that $\mathrm{proj.dim}_{\mathbb{Z}G}\, A = \mathrm{Gcd}\, G$ by [5, Theorem 2.7]. Hence G has property $\mathcal{P}_n$. □

Theorem 3.5. *Let G be an elementary amenable group of type* FP_∞*. Then G satisfies Conjecture II.*

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Proof. By [33, Theorem 3.2], it suffices to show that $\operatorname{Gcd} G \le \operatorname{Gcd}_{\mathbb{Q}} G$. We may assume that $\operatorname{Gcd}_{\mathbb{Q}} G < \infty$. Since G is an $\mathbf{LH}\mathfrak{F}$-group, it follows that $\operatorname{Gcd}_{\mathbb{Q}} G = \operatorname{cd}_{\mathbb{Q}} G$. Let H be a torsion-free subgroup of finite index in G. Then we have

$$\operatorname{vcd} G = \operatorname{cd} H = h(H) = \operatorname{cd}_{\mathbb{Q}} H = \operatorname{cd}_{\mathbb{Q}} G = \operatorname{Gcd}_{\mathbb{Q}} G < \infty.$$

Since $\operatorname{vcd} G < \infty$, it follows that $\underline{\operatorname{cd}}\, G = \operatorname{vcd} G$ [15, Corollary 1]. Hence, we conclude that $\operatorname{Gcd} G = \operatorname{Gcd}_{\mathbb{Q}} G < \infty$. □

Corollary 3.6. *If G is an elementary amenable group of type FP_∞ and n is a positive integer, then G has property $\mathcal{P}_n$ if and only if $\operatorname{cd}_{\mathbb{Q}} G \le n$.*

Proof. This follows immediately from Lemma 3.4 and Theorem 3.5. □

Appendix

In [18, Proposition 5.4] the author claimed that for any group G, $\operatorname{sfli} G = \operatorname{silf} G$ if both are finite. Soon after [18] was published, the author realised that there is an incorrect argument in the proof of [18, Proposition 5.4] even though $\operatorname{sfli} G \le \operatorname{silf} G$ is true. Asadollahi *et al.* showed in [2, Theorem 3.7] that $\operatorname{sfli} G = \operatorname{silf} G$ provided that $\mathbb{Z}G$ is coherent. Thus [18, Question B] cannot be proved by the argument in [18, Question B]. However, Emmanouil showed in [10] that [18, Question B] has a positive answer for any group G.

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