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Global Injectivity of C^1 Maps of the Real Plane, Inseparable Leaves and the Palais–Smale Condition

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Abstract. We study two sufficient conditions that imply global injectivity for a C^1 map $X\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that its Jacobian at any point of $\mathbb{R}^2$ is not zero. One is based on the notion of half-Reeb component and the other on the Palais–Smale condition. We improve the first condition using the notion of inseparable leaves. We provide a new proof of the sufficiency of the second condition. We prove that both conditions are not equivalent, more precisely we show that the Palais–Smale condition implies the nonexistence of inseparable leaves, but the converse is not true. Finally, we show that the Palais–Smale condition it is not a necessary condition for the global injectivity of the map X.

1 Introduction and Statement of the Main Result

In this note we denote by $X = (f, g)\colon \mathbb{R}^2 \to \mathbb{R}^2$ a C^1 map such that its Jacobian at any point of $\mathbb{R}^2$ is non zero. Then by the inverse function theorem, this map is locally injective at any point of $\mathbb{R}^2$. As the map $f(x, y) = (e^x \cos x, e^x \sin y)$ (*i.e.,* the complex map $f(z) = e^z$ as a map from $\mathbb{R}^2$ to $\mathbb{R}^2$) shows that the above local condition is not sufficient to guarantee the global injectivity. Indeed, Pinchuck [13] showed that there is a polynomial map X satisfying the above local condition for injectivity that is not globally injective in $\mathbb{R}^2$ (see also [4, p. 241]). Consequently, the goal is to give sufficient conditions on such a map to insure that it is globally injective. Our concern is to discuss two of these conditions (that appeared in the literature), to compare them, and to establish relationships between them.

We denote by X_f the Hamiltonian planar vector field associated with f, *i.e.,* $X_f = (-\partial f/\partial y, \partial f/\partial x)$. Of course, X_f is, in general, only a C^0 vector field, but we know that its solutions lie in the level curves of f. Similarly we define X_g.

A vector field Y in $\mathbb{R}^2$ defines a *planar foliation* if Y has no singularities. Since the Jacobian of X is never zero, X_f and X_g are planar Hamiltonian foliations as well.

The orbits of a foliation are called *leaves*. It is well known in the general theory of planar foliations that the relevant leaves in the study of the phase portrait of the foliations are the inseparable leaves and their accumulations, if such accumulations exist.

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Two different leaves L_1 and L_2 of a foliation Y are *inseparable* if for any arcs T_1 and T_2, to which Y is nowhere tangent and such that L_i has non-empty intersection with T_i, $i = 1, 2$, there is a third leaf L, distinct from L_1 and L_2, which intersects both T_1 and T_2. In other words, the distinct orbits through the points p and q of $\mathbb{R}^2$ are said to be *inseparable* if there exist one-sided compact transversal sections Σ_p at p and Σ_q at q such that the Poincaré map $\pi\colon \Sigma_p \setminus \{p\} \to \Sigma_q \setminus \{q\}$ may be defined and satisfies $\lim_{x\to p} \pi(x) = q$. The notion of inseparable leaves may be considered in general when $L_1 = L_2$, but it forces the vector field to have a singular point, and hence not a foliation.

Let $X = (f, g)\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a local homeomorphism. Then it is easy to see from the definitions that the surjective map $\widetilde{X}\colon \mathbb{R}^2 \to X(\mathbb{R}^2)$, induced by X, is a finite covering map if and only if it is a proper one. In this way,

(i) if $X(\mathbb{R}^2)$ is simply connected, X is globally injective if and only if $\widetilde{X}$ is a proper map;
(ii) if $X(\mathbb{R}^2) = \mathbb{R}^2$, then X is a proper map $\Leftrightarrow$ X is a homeomorphism $\Leftrightarrow$ every level set of the form $\{f = \text{constant}\}$ and $\{g = \text{constant}\}$ is connected $\Leftrightarrow$ X_f and X_g have no inseparable leaves.

However if X is a (proper) embedding, that is, X is globally injective and $X(\mathbb{R}^2) \subsetneq \mathbb{R}^2$ (in which case $X(R^2)$ is simply connected), it may happen that both X_f and X_g have infinitely many pairs of inseparable leaves. This will depend on the geometry of the set $X(\mathbb{R}^2)$. For instance, if $X(\mathbb{R}^2)$ is vertically (resp., horizontally) convex, then X_f (resp., X_g) has no inseparable leaves. The examples given below will show that if $X(\mathbb{R}^2)$ is far from being convex, then both X_f and X_g have infinitely many pairs of inseparable leaves. More precisely, we have the following result.

Proposition 1 *There exists a smooth embedding $X\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that both X_f and X_g have infinitely many pairs of inseparable leaves.*

This proposition is proved at the beginning of Section 2.

The above definition of a pair of inseparable leaves is equivalent to the notion of *saddle-at-infinity* (SAI) or *half-Reeb component* (hRc), widely used in the literature, specially in the context of the injectivity problem. We just notice that in the definition of (hRc) (see [3] for more details), the homeomorphism does not need to be extended to infinity.

The first main result in this note deals with the sufficiency (but not necessity) of the non existence of inseparable leaves in X_f or X_g in order that X determines a global injective map.

Theorem 2 *Let $X = (f, g)\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 map such that its Jacobian at any point of $\mathbb{R}^2$ is not zero.*

(i) *If X_f or X_g has no inseparable leaves, then X is globally injective.*
(ii) *There are maps X globally injective such that both foliations X_f and X_g have no inseparable leaves.*

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(iii) *There are maps X globally injective such that one and only one of the foliations X_f and X_g has inseparable leaves.*
(iv) *The converse of statement* (i) *does not hold; i.e. there are maps X globally injective such that both X_f and X_g have inseparable leaves.*

Theorem 2(i) was indeed proved by Cobo, Llibre and Gutierrez [3], although we include the proof here for completeness (see also [5]). Statement (ii) is trivial, by considering $X = (f, g)$ equal to the identity map of $\mathbb{R}^2$, while statements (iii) and (iv) are proved in Section 2.

From the result above there is still an open question of finding the sufficient and necessary conditions on X_f and X_g to guarantee global injectivity. Recently, a new paper [7] has appeared in the line of [3], giving sufficient conditions for global injectivity on C^1 maps $X\colon \mathbb{R}^2 \to \mathbb{R}^2$ *i.e.,* using the eigenvalues of the Jacobian of X at any point of $\mathbb{R}^2$.

Given a C^1 map $f\colon \mathbb{R}^2 \to \mathbb{R}$ and $c \in \mathbb{R}$, we say that f satisfies the Palais–Smale condition at level c (see [2,14]) if every sequence $\{p_m\}$ in $\mathbb{R}^2$ satisfying (i) $f(p_m) \to c$ and (ii) $\|(f_x, f_y)(p_m)\| \to 0$, as $m \to \infty$, possesses a converging subsequence. If f satisfies the condition for every $c \in \mathbb{R}$, then we say that f satisfies the *Palais–Smale condition* or the (PS) *condition*. We notice that when f is such that X_f has no singular points, (that is, $(f_x, f_y) \neq (0, 0)$ in all $\mathbb{R}^2$) the (PS) condition can be stated as follows: for any sequence $p_m \to \infty$ in $\mathbb{R}^2$ satisfying $f(p_m) \to c, c \in R$ there exists an $\varepsilon > 0$ such that $\|(f_x, f_y)(p_m)\| > \varepsilon$ for all m.

Theorem 3 *Let $X = (f, g)\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 map such that its Jacobian at any point of $\mathbb{R}^2$ is not zero.*

(i) *If f or g satisfies the (PS) condition, then X is globally injective.*
(ii) *There are maps X globally injective such that both f and g satisfy the (PS) condition.*
(iii) *There are maps X globally injective such that one and only one of the functions f or g satisfies the (PS) condition.*
(iv) *The converse of statement* (i) *does not hold,* i.e., *there are maps X globally injective such that neither f nor g satisfy the (PS) condition.*

Statement (i) of Theorem 3 has been proved by Silva and Teixeira [15]. So the (PS) condition gives an alternative (to the non existence of inseparable leaves condition) sufficient condition for global injectivity. In Section 2, we provide a new proof of statement (i), by using Theorems 2 and 4.

Statements (i) and (iii) of Theorem 3 are proved in Section 2, while statement (ii) is trivial by considering $X = (f, g)$ equal to the identity map of $\mathbb{R}^2$.

A natural question to ask is whether the two sufficient conditions stated in Theorem 2(i) and Theorem 3(i) are equivalent. The answer is that they are not. More precisely, the (PS) condition (an analytic condition), implies the nonexistence of inseparable leaves (a topological condition), but the converse is not true. This is detailed in the next theorem, proved in Section 2.

Theorem 4 *Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a C^1 map with gradient different from zero at any point of $\mathbb{R}^2$.*

(i) *If f satisfies the (PS) condition, then X_f has no inseparable leaves.*
(ii) *The converse of statement* (i) *does not hold.*

We provide two different proofs of Theorem 4(i). The first uses a previous result of Silva and Teixeira [16] called the level surface theorem while the second involves the notion of virtual critical point (or, equivalently, asymptotic value).

2 Proofs of the Results

In order to prove Proposition 1 we need the following two preliminary lemmas. The first one is immediate.

Lemma 5 *Suppose that $X = (f, g)\colon \mathbb{R}^2 \to \mathbb{R}^2$ is an embedding such that the boundary C of $X(\mathbb{R}^2)$ is a topological circle. Suppose that for some $(x_0, y_0) \in C$ and some $\varepsilon > 0$ there is a smooth function $\varphi\colon [y_0 - \varepsilon, y_0 + \varepsilon] \mapsto \mathbb{R}$ such that*

(i) *$\varphi(y) = x_0 + c(y - y_0)^2$ + higher order terms, where $c < 0$ is a constant;*
(ii) *$\{(\varphi(y), y) : y \in [y_0 - \varepsilon, y_0 + \varepsilon]\} \subset C$ and if $(x, y) \in X(\mathbb{R}^2)$ with $|y - y_0| < \varepsilon$ is close to C, then $x > \varphi(y)$.*

Then $\{f = x_0\}$ contains a pair of inseparable leaves.

Lemma 6 *Given a bounded simply connected open subset U of $\mathbb{R}^2$, there exists a smooth embedding $X\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $X(\mathbb{R}^2) = U$.*

Proof It follows directly from the Riemann mapping theorem. ■

Proof of Proposition 1 Let C be a topological circle such that

(i) The curves

$$\{(x, x\sin(1/x)/2) : x \in (0, 1)\} \quad \text{and} \quad \{(y\sin(1/y)/2, y) : y \in (0, 1)\}$$

are part of C.
(ii) Away from $(0, 0)$, the circle C is smooth.

Let U be the topological 2-disc bounded by C. It follows from Lemma 6 that there exists a smooth embedding $X\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $X(\mathbb{R}^2) = U$. Therefore, by Lemma 5, X_f has infinitely many pairs of inseparable leaves. Similarly, it can be seen that X_g has infinitely many pairs of inseparable leaves. ■

Now we prove Theorem 2, Theorem 4 and Theorem 3, in this order.

Theorem 2(iv) follows from Proposition 1. Theorem 2(iii) can be proved in a similar way, although we will prove these items below in a different way by exhibiting explicit examples.

Proof of Theorem 2(i) Let $X = (f, g)\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 map such that its Jacobian at any point of $\mathbb{R}^2$ is non zero. Suppose that X_f has no inseparable leaves, and we shall prove that X is globally injective. If X_g has no inseparable leaves, the proof is similar.

Since the Jacobian of X is never zero, the induced Hamiltonian vector field X_f has no singularities in $\mathbb{R}^2$. Hence, by Neumann's classification of flows on 2-manifolds [12] (see also [10]), the phase portrait of X_f is topologically equivalent to the horizontal foliation of $\mathbb{R}^2$ (*i.e.,* the foliation of $\mathbb{R}^2$ given by the constant vector field $(1, 0)$) or equivalently, there is a global transversal section.

Assume that in two different leaves L_1 and L_2 of the topological horizontal foliation of X_f, the function f takes the same value, and consider the segment S of the transversal segment connecting $p_1 \in L_1$ with $p_2 \in L_2$. At any point of S we consider the basis of $\mathbb{R}^2$ given by the transversal and the orbit passing through that point. Now it is clear that the two partial derivatives at one point must be zero. So f will have a local maximum or minimum, and the gradient of f will be zero, in contradiction with the fact that the Jacobian of X never vanishes. So there is a unique leaf in every level of f.

We claim that g restricted to each leaf of f is strictly monotone. The claim follows easily by noting that if we denote by $(x(t), y(t))$ a solution curve of X_f, then

$$\frac{d}{dt} g(x(t), y(t)) = \frac{\partial g}{\partial x}\dot{x} + \frac{\partial g}{\partial y}\dot{y} = -\frac{\partial g}{\partial x}\frac{\partial f}{\partial y} + \frac{\partial g}{\partial y}\frac{\partial f}{\partial x} \neq 0, \tag{1}$$

since the Jacobian of X is never zero. Then X is globally injective. ■

Proof of Theorem 2(iii) We consider the analytic map $X = (f, g)\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$f(x, y) = e^y(1 - x^2), \qquad g(x, y) = -e^y x.$$

Since $X_f = (e^y(x^2 - 1), -2e^y x)$ and $X_g = (e^y x, -e^y)$, it is clear that X_f and X_g define regular foliations in $\mathbb{R}^2$. Moreover, the Jacobian of X at any point $(x, y) \in \mathbb{R}^2$ is equal to $e^{2y}(1 + x^2) > 0$.

Looking at the phase portrait of the foliation X_f, it follows that the straight lines $x = \pm 1$ are the two inseparable leaves of the foliation X_f. On the other hand, the phase portrait of X_g shows that this foliation has no inseparable leaves. Then by Theorem 2(i), the map X is globally injective. ■

We will need the following lemma in the proof of Theorem 2(iv).

Lemma 7 *Let R be an open, simply connected subset of $\mathbb{R}^2$. Let $Z = (f, g)\colon R \to \mathbb{R}^2$ be a C^1 local diffeomorphism. Suppose that there exist two points $p, q \in \mathbb{R}^2$ such that $Z(p) = Z(q) = (c, d)$. Then p and q belong to different connected components of both $f^{-1}(c)$ and $g^{-1}(d)$.*

Proof Suppose, by contradiction, that p and q belong to the same connected component L of $f^{-1}(c)$ (the case of $g^{-1}(c)$ is similar). Then, as $g|_L$ is strictly monotone (since Z is a local diffeomorphism, see (1)), we should not have $g(p) = g(q) = d$. ■

Proof of Theorem 2(iv) Let $I = (0,1)$ and let $\Theta\colon I \times I \longmapsto \mathbb{R}^2$ be the diffeomorphism given by $\Theta(x,y) = (\tan(\pi x - \pi/2), \tan(\pi y - \pi/2))$. Given $\varepsilon = 0.1$, let $Y = (h,k)\colon I \times I \longmapsto \mathbb{R}^2$ be defined by

$$h(x,y) = (y-\varepsilon)^2 - \varepsilon^2 x^2 \quad \text{and} \quad k(x,y) = (y-1+\varepsilon)^2 - \varepsilon^2 x^2.$$

We claim that the map $X = Y \circ \Theta^{-1} = (f,g)\colon \mathbb{R}^2 \to \mathbb{R}^2$ is an analytic, globally injective, local diffeomorphism such that both X_f and X_g have inseparable leaves. We notice that the only role of Θ is to extend the behaviour from $I \times I$ to $\mathbb{R}^2$ (in particular, the square boundary goes to infinity). Of course, Theorem 2(iv) follows from the claim. We establish the claim in three steps.

First, we compute $\det(DY(x,y)) = 4\varepsilon^2(1-2\varepsilon)x$, which is different from zero in $I \times I$, so Y and then X are local diffeomorphisms.

Second, we must show that Y is globally injective. In the light of Lemma 7, we only need to see that there is no intersection between level sets of h and k with more than one connected component. Let $c \in Y(I,I)$. By dealing with the expressions of h and k, it is not difficult to see that $h^{-1}(c)$ (resp., $k^{-1}(c)$) has at most two connected components, and moreover they have exactly two if and only if $0 \leq c < \varepsilon^2$. Indeed, $c = 0$ corresponds to two pairs of straight lines (one pair for $h^{-1}(0)$ and one pair for $k^{-1}(0)$. Above that value of $c = \varepsilon^2$ each level set is given by one connected component. One can check that $h^{-1}([0,\varepsilon^2)) \cap k^{-1}([0,\varepsilon^2)) = \varnothing$.

Third, we see that in the filled square $I \times I$ the Hamiltonian foliations Y_h and Y_k have both inseparable leaves lying in $A = \{(x,y) \in I \times I : \varepsilon^2 x^2 \geq (y-\varepsilon)^2\}$ and $B = \{(x,y) \in I \times I : \varepsilon^2 x^2 \geq (y-1+\varepsilon)^2\}$, respectively. So the Hamiltonian foliations of the plane defined by X_f and X_g have both inseparable leaves lying in $\Theta(A)$ and $\Theta(B)$, respectively. ■

The next goal is to prove Theorem 4. Indeed, we prove (i) by using two different arguments, one using the "mountain pass theorem" and the other by characterizing the inseparable leaves in terms of sequences.

Given a C^1 map $f\colon \mathbb{R}^2 \to \mathbb{R}$ and $c \in \mathbb{R}$, we define the sets $S_c(f) = \{u \in \mathbb{R}^2 : f(u) = c\}$ and $K_c = \{u \in \mathbb{R}^2 : f(u) = c,\ \nabla f(u) = 0\}$, respectively. Of course, ∇f denotes the gradient of f. We say that c is an admissible level of f if, either c is a regular value of f, or each component of K_c is just a point and c is an isolated critical value of f.

Silva and Teixeira [16] (see also [15]) proved the following version of the mountain pass theorem which they call the "level surface theorem".

Theorem 8 *Suppose that the C^1 map $f\colon \mathbb{R}^2 \to \mathbb{R}$ satisfies the (PS) condition. Assume that $c \in \mathbb{R}$ is an admissible level of f and that u and v are two distinct points of $S_c(f)$. Then, either u and v are in the same path-component of $S_c(f)$, or f has a critical value $d \neq c$.*

First proof of Theorem 4(i) Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a C^1 map with gradient different from zero at any point of $\mathbb{R}^2$, and assume that f satisfies the (PS) condition. Of course, X_f is a Hamiltonian foliation in $\mathbb{R}^2$. Therefore, all the levels of f are admissible, and since f has no critical values, by Theorem 8, every level of f has a unique component, which is not compatible with having two different inseparable leaves, because each of them is a different connected component of a level curve of f. ■

Alternatively, we can prove Theorem 4(i) without using Theorem 8. For that we need some preliminary notions and results.

Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a C^1 map. A point $p \in \mathbb{R}^2$ is a *virtual critical (point) for f* if there exists a sequence of embedded intervals C_i with endpoints $p_i, q_i, i \geq 0$, on each of which f has a constant value distinct from $f(p)$, with $p_i \to p$, $q_i \to q$, but there is a sequence $\{r_i\}$, $r_i \in C_i$, having no accumulation points in $\mathbb{R}^2$.

The next lemma shows that provided that the function f has gradient different from zero at any point of $\mathbb{R}^2$, the existence of virtual critical points and the existence of inseparable leaves are equivalent notions (see also [9], where the result is proved for a general f).

Lemma 9 *Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a C^1 map with gradient different from zero at any point of $\mathbb{R}^2$. If p is a virtual critical point for f, then the level curve through p is an inseparable leaf of X_f. Conversely, if X_f has inseparable leaves, then there exists a point p which is a virtual critical point for the function f.*

Proof The converse implication is straightforward, by taking the points p and q as in the definition of inseparable leaves. The sequence of embedded intervals C_i with endpoints $p_i, q_i, i \geq 0$, correspond to the segment orbits of X_f passing through the points p_i with $p_i \to p$, and the sequence of $r_i \in C_i$ with $r_i \to \infty$ can be easily chosen in these segment orbits.

We assume that p is a virtual critical point. To see that the orbits of p and q are inseparable leaves (both tending to infinity in forward and backward time), we consider the sequence of embedded intervals C_i with endpoints p_i and q_i, where $p_i \to p$ and $q_i \to q$. On each C_i the function f takes a constant value distinct from $f(p)$. Clearly, these orbit segments of X_f can be enlarged so that they cross transversally the cross sections Σ_p and Σ_q of the definition of inseparable leaves, and consequently the orbits of p and q are inseparable leaves. ■

Second proof of Theorem 4(i) From the definition, it is enough to see that if f has a gradient different from zero at any point in $\mathbb{R}^2$ and X_f has inseparable leaves, then there is (at least) a sequence of points $\{p_m\}$ such that conditions (i) and (ii) are satisfied in the definition of (PS).

So assume X_f has inseparable leaves (and $\nabla f(p) \neq 0$ for every $p \in \mathbb{R}^2$). From Lemma 9 this is equivalent to assuming that there exists a virtual critical point p. We denote by Ω_0 the canonical region partially fulfilled by the orbits passing through the two one-sided transversal sections Σ_p at p and Σ_q at q. We denote by Ω the subset of Ω_0 determined by the segments of the two one-sided transversal sections Σ_p at p and Σ_q at q joining p with p_1 and q with q_1, the segment orbit joining p_1 and q_1, and the two orbits passing through p and q (see Figure 1).

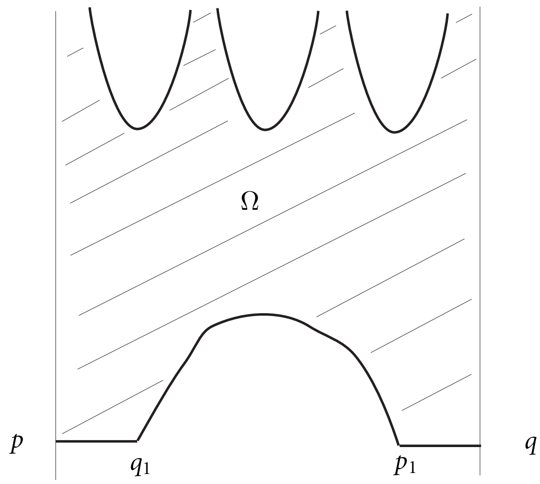

Figure 1: The set Ω as a subset of the canonical region associated with the inseparable leaves through p and q (or equivalently, through the virtual critical point p).

Notice that Ω is an open unbounded set. Moreover, it can be assumed that f takes values strictly less than c in Ω. Observe that f takes the value c only in the orbits passing through p and q, and perhaps in other components different than Ω_0).

We consider an increasing sequence of radius $\{\delta_i\} \in \mathbb{R}$ and the corresponding sequence of balls $B_{\delta_i}(0)$, centered at the origin. It is clear that we can choose this sequence in a such way that for each ball we can choose a point r_i lying in $\Omega \cap (B_{\delta_i}(0) \setminus B_{\delta_i - 1}(0))$ and with $f(r_i)$ being a strictly increasing sequence. Of course, $f(r_i) \to c$ as $i \to \infty$. For each point r_i we consider the gradient solution curve $\phi_t(r_i)$ passing through that point for $t \in (-1/4, 1/4)$. If such a curve does not entirely belong to Ω, we change r_i by the point $\phi_{-1/4}(r_i)$ which surely lies in Ω and by construction the gradient solution curve passing through it is entirely included in Ω (for $t \in (-1/4, 1/4)$). For simplicity we rename all points to be r_i again.

We consider a C^1 parametrizable curve $\mathcal{C}$ inside Ω passing through all of these points (*i.e.,* the r_i) and which coincides with the gradient solution curves around the r_i's. Hence $\mathcal{C}$ can be understood as the image of a function $g\colon \mathbb{R} \to \Omega$ such that $\|g'(t)\| = 1$ (by arc parametrization). Of course, $\mathcal{C}$ must cross all the level curves $f =$ constant inside Ω. Moreover, we also assume that $r_i = g(i), i \geq 0$.

Now we apply the mean value theorem to $h(t) = f(g(t))$ around each i in the intervals $I_i = (i - 1/4, i + 1/4)$, for $i = 1, 2, \ldots$. Consequently, for every i we have that

$$h(i + 1/8) = h(i) + h'(c_i)\frac{1}{8} = h(i) + \nabla f(g(c_i))\overrightarrow{g}'(t_i).$$

Of course, $h'(c_i) \to 0$ as $i \to \infty$. But this forces $\|\nabla f(g(c_i))\| \to 0$, since

$$\nabla f(g(c_i))\overrightarrow{g}'(t_i) = \|\nabla f(g(c_i))\|\|\overrightarrow{g}'(t_i)\| \cos\theta,$$

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and $\|\overrightarrow{g}'(t_i)\| = \cos\theta = 1$. Thus, the sequence $g(c_i)$ for $i > 0$, satisfies $f(g(c_i)) \to c$ (since it still belongs to the unbounded curve C), and $\|\nabla f(g(c_i))\| \to 0$. ■

To show Theorem 4(ii), we first prove the following result.

Lemma 10 *Let* $f(x, y) = \arctan x + \arctan y$ *where we choose the values of* $\arctan x$ *and* $\arctan y$ *in the interval* $(-\pi/2, \pi/2)$*. The following statements hold.*

(i) *The function* f *has gradient different from zero at any point of* $\mathbb{R}^2$*, and does not satisfy the (PS) condition.*

(ii) *The vector field* X_f *has no inseparable leaves.*

Proof Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be the C^1 map defined by $f(x, y) = \arctan x + \arctan y$ where we choose the values of $\arctan x$ and $\arctan y$ in the interval $(-\pi/2, \pi/2)$.

Since its gradient is $(1/(1 + x^2), 1/(1 + y^2))$, clearly it is not zero at any point of $\mathbb{R}^2$. To see that f does not satisfy the (PS) condition, we consider the sequence $\{p_n\}$ with $p_n = (n, -n)$. Then $f(p_n) = 0$ for all n, and $\nabla f(p_n) \to (0, 0)$ when $n \to \infty$. However the sequence $\{p_n\}$ has no any convergent partial subsequence. So f does not satisfy the (PS) condition, and statement (i) follows.

Finally we prove statement (ii). The phase portrait of X_f is topologically equivalent to the phase portrait of the polynomial vector field $Y = (-(1+x^2), 1+y^2)$ (they differ only in time scaling). Of course, Y defines a polynomial foliation. The inseparable leaves of the polynomial foliations correspond to hyperbolic sectors at infinity, see [8]. These hyperbolic sectors can be associated with a unique singular point of Y at infinity (having the two separatrices of the hyperbolic sector outside the infinity), or with two different singular points at infinity, as illustrated in Figure 2(a).

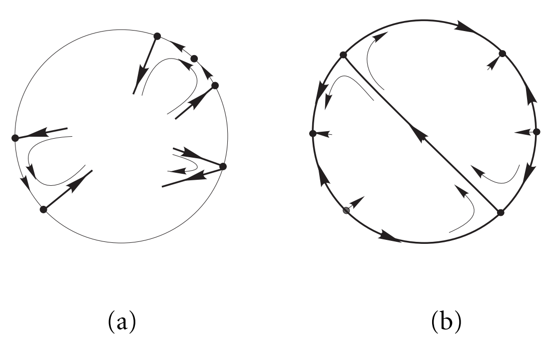

Figure 2: In (a) we illustrate the possible hyperbolic sectors at infinity of the Poincaré sphere corresponding to saddles-at-infinity, while in (b) we show the phase portrait of the vector field Y used in the proof of Lemma 10.

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Now we use the notation introduced in the appendix on the Poincaré compactification. For the vector field Y we have that $F(z_1) = z_1(z_1+1)$ and $G(z_1) = -z_1(z_1+1)$. Therefore, the singular points at infinity are $(0,0)$ and $(-1,0)$ of the local chart U_1, and $(0,0)$ of the local chart U_2, and their diametrally opposite singular points in the local charts V_1 and V_2. The local phase portrait at these infinite singular points can be studied using their linear parts given in (2) and (3). Then we get that the point $(0,0)$ of the local chart U_1 is an unstable node, the point $(-1,0)$ of the local chart U_1 is a saddle having the two unstable separatrices at infinity, and the $(0,0)$ of the local chart U_2 is a stable node. So Y has no inseparable leaves, and the claim is proved. The phase portrait of Y is given in Figure 2(b) where it is easy to see that the sequence in question belongs to the saddle connection between the two saddle points at infinity. ■

Proof of Theorem 4(ii) This follows from the above proposition. ■

We finish this section with the proof of Theorem 3.

Proof of Theorem 3(i) Let $X = (f,g)\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 map such that its Jacobian at any point of $\mathbb{R}^2$ is not zero, and assume that f satisfies the (PS) condition. If g satisfies the (PS) condition, the proof is similar.

Since the Jacobian of X is never zero, the gradient of f is not zero at any point of $\mathbb{R}^2$. So by Theorem 4, f has no inseparable leaves, and consequently, by Theorem 2, the map X is globally injective. ■

Proof of Theorem 3(iii) We consider the analytic map $X = (f,g)\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x,y) = \arctan x + \arctan y$, $g(x,y) = x$, where the values of $\arctan x$ and of $\arctan y$ in the interval $(-\pi/2, \pi/2)$. Since $X_f = (-1/(1+y^2), 1/(1+x^2))$ and $X_g = (0,1)$, it is clear that X_f and X_g define foliations in $\mathbb{R}^2$. Moreover, the Jacobian of X at any point $(x,y) \in \mathbb{R}^2$ is equal to $-1/(1+y^2) < 0$.

By Lemma 10, f does not satisfy the (PS) condition. Clearly, the leaves of the foliation X_g are formed by the straight lines $x =$ constant. On the other hand, since $\|(g_x, g_y)(x,y)\| = 1$ for all $(x,y) \in \mathbb{R}^2$, the (PS) condition is trivially satisfied. Since g satisfies the (PS) condition, applying statement (i) of Theorem 3, the map X (having Jacobian different from zero at any point) is globally injective. ■

Proof of Theorem 3(iv) We consider the analytic map $X = (f,g)\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x,y) = \arctan x + \arctan y$, $g(x,y) = \arctan x$, where we choose the values of $\arctan x$ and of $\arctan y$ in the interval $(-\pi/2, \pi/2)$. Since we have that $X_f = (-1/(1+y^2), 1/(1+x^2))$ and $X_g = (0, 1/(1+x^2))$, it is clear that X_f and X_g define foliations in $\mathbb{R}^2$. Moreover, the Jacobian of X at any point $(x,y) \in \mathbb{R}^2$ is equal to $1/((1+x^2)(1+y^2)) > 0$.

By Lemma 10, f does not satisfy the (PS) condition. On the other hand, the leaves of the foliation X_g are formed by the straight lines $x =$ constant, and if $p_m = (m, -m)$, then $\|g(p_m)\| \to -\pi/2$ and $\|(g_x, g_y)(p_m)\| \to 0$ as $m \to \infty$. Since there is no convergent subsequences of $\{p_m\}$, g does not satisfy the (PS) condition.

To finish the proof, we need to show that X is globally injective. If (a, b) belongs to the image of the map X, then there exists a unique $(x, y) \in \mathbb{R}^2$ such that $X(x, y) = (a, b)$. In fact, $x = \tan b$ and $y = \tan(a - b)$. Therefore, the map X is globally injective. ■

A The Poincaré Compactification

Let $X = (P, Q)$ be a polynomial vector field of degree d. The Poincaré compactified vector field $p(X)$ corresponding to X is a vector field induced in $\mathbb{S}^2$ as follows (see for instance [1,6]).

Let $\mathbb{S}^2 = \{y = (y_1, y_2, y_3) \in \mathbb{R}^3 : y_1^2 + y_2^2 + y_3^2 = 1\}$ (called the *Poincaré sphere*) and $T_y\mathbb{S}^2$ be the tangent space to $\mathbb{S}^2$ at point y. Consider the central projections $f_+ \colon T_{(0,0,1)}\mathbb{S}^2 \to \mathbb{S}^2_+ = \{y \in \mathbb{S}^2 : y_3 > 0\}$ and $f_- \colon T_{(0,0,1)}\mathbb{S}^2 \to \mathbb{S}^2_- = \{y \in \mathbb{S}^2 : y_3 < 0\}$. These maps define two copies of X, one in the northern hemisphere and the other in the southern hemisphere. Denote by X' the vector fields $Df_+ \circ X$ and $Df_- \circ X$ in $\mathbb{S}^2$ except on its equator $\mathbb{S}^1 = \{y \in \mathbb{S}^2 : y_3 = 0\}$. Obviously $\mathbb{S}^1$ is identified with the infinity of $\mathbb{R}^2$. In order to extend X' to an analytic vector field in $\mathbb{S}^2$ (including $\mathbb{S}^1$) it is necessary that X satisfy suitable hypotheses. The *Poincaré compactification* $p(X)$ is the only analytic extension of $y_3^{d-1}X'$ to $\mathbb{S}^2$.

For the flow of the compactified vector field $p(X)$, the equator $\mathbb{S}^1$ is invariant. On $\mathbb{S}^2\backslash\mathbb{S}^1$ there are two symmetric copies of X, and knowing the behaviour of $p(X)$ around $\mathbb{S}^1$, we know the behaviour of X near infinity. The projection of the closed northern hemisphere of $\mathbb{S}^2$ in $y_3 = 0$ under $(y_1, y_2, y_3) \mapsto (y_1, y_2)$ is called the *Poincaré disc*. Due to these two symmetric copies of X on $\mathbb{S}^2$, it follows that the *infinite singular points* (*i.e.,* the singular points on $\mathbb{S}^1$) appear in pairs of diametrally opposite points.

As $\mathbb{S}^2$ is a differentiable manifold for computing the expression of $p(X)$, we can consider the six local charts $U_i = \{y \in \mathbb{S}^2 : y_i > 0\}$, and $V_i = \{y \in \mathbb{S}^2 : y_i < 0\}$ where $i = 1, 2, 3$, and the diffeomorphisms $F_i \colon U_i \to \mathbb{R}^2$ and $G_i \colon V_i \to \mathbb{R}^2$ defined as the inverses of the central projections from the tangent planes at the points $(1, 0, 0)$, $(-1, 0, 0)$, $(0, 1, 0)$, $(0, -1, 0)$, $(0, 0, 1)$ and $(0, 0, -1)$, respectively. If we denote by $z = (z_1, z_2)$ the value of $F_i(y)$ or $G_i(y)$ for any $i = 1, 2, 3$, then z represents different things according to the local charts under consideration. Some straightforward calculations give for $p(X)$ the following expressions:

$$z_2^d \Delta(z)\Big[Q\Big(\frac{1}{z_2}, \frac{z_1}{z_2}\Big) - z_1 P\Big(\frac{1}{z_2}, \frac{z_1}{z_2}\Big), -z_2 P\Big(\frac{1}{z_2}, \frac{z_1}{z_2}\Big)\Big] \quad \text{in} \quad U_1,$$

$$z_2^d \Delta(z)\Big[P\Big(\frac{z_1}{z_2}, \frac{1}{z_2}\Big) - z_1 Q\Big(\frac{z_1}{z_2}, \frac{1}{z_2}\Big), -z_2 Q\Big(\frac{z_1}{z_2}, \frac{1}{z_2}\Big)\Big] \quad \text{in} \quad U_2,$$

$$\Delta(z)[P(z_1, z_2),\ Q(z_1, z_2)] \qquad \text{in} \quad U_3,$$

where $\Delta(z) = (z_1^2 + z_2^2 + 1)^{-\frac{1}{2}}$. The expression for V_i is the same as that for U_i except for the multiplicative factor $(-1)^{d-1}$. In these coordinates for $i = 1, 2$, $z_2 = 0$ always denotes the points of $\mathbb{S}^1$. We omit the factor $\Delta(z)$ by rescaling the vector field $p(X)$.

Since the unique singular point at infinity which cannot be contained in the charts $U_1 \cup V_1$ coincides with the origin $(0, 0)$ in U_2 and V_2, when we study the infinite singular points on the charts $U_2 \cup V_2$, we need only consider whether the $(0, 0)$ of these charts are or are not singular points.

A singular point q of $p(X)$ is called an *infinite* (respectively, *finite*) singular point if $q \in \mathbb{S}^1$ (respectively, $q \in \mathbb{S}^2 \setminus \mathbb{S}^1$).

We want to study the local phase portrait at infinite singular points. For this we choose an infinite singular point $(z_1, 0)$ and start by looking at the expression of the linear part of the field $p(X)$. For $i = 0, 1, 2$ we denote by P_i and Q_i the homogeneous polynomials of degree i of P and Q, respectively. Then $(z_1, 0) \in \mathbb{S}^1 \cap (U_1 \cup V_1)$ is an infinite singular point of $p(X)$ if and only if $F(z_1) = Q_d(1, z_1) - z_1 P_d(1, z_1) = 0$. Similarly $(z_1, 0) \in \mathbb{S}^1 \cap (U_2 \cup V_2)$ is an infinite singular point of $p(X)$ if and only if $G(z_1) = P_d(z_1, 1) - z_1 Q_d(z_1, 1) = 0$.

The Jacobian matrix of the vector field $p(X)$ at an infinite singular point $(z_1, 0)$ is

$$\begin{pmatrix} F'(z_1) & Q_{d-1}(1, z_1) - z_1 P_{d-1}(1, z_1) \\ 0 & -P_d(1, z_1) \end{pmatrix}, \tag{2}$$

or

$$\begin{pmatrix} G'(z_1) & P_{d-1}(z_1, 1) - z_1 Q_{d-1}(z_1, 1) \\ 0 & -Q_d(z_1, 1) \end{pmatrix}, \tag{3}$$

if $(z_1, 0)$ belongs to U_1 or U_2, respectively.

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References

[1] A. A. Andronov, E. A. Leontovich, I. I. Gordon, and A. L. Maier, *Qualitative theory of second–order dynamic systems.* Wiley, New York, 1973.

[2] A. Ambrosetti and P. H. Rabinowitz, *Dual variational methods in critical point theory and applications.* J. Functional Analysis **14**(1973), 349–381.

[3] M. Cobos, C. Gutierrez and J. Llibre, *On the injectivity of $\mathcal{C}^1$ maps on the real plane.* Canad. J. Math. **54**(2001), no. 6, 1187–1201.

[4] A. van den Essen, *Polynomial Automorphisms and the Jacobian Conjecture.* Progress in Mathematics 190, Birkhauser Verlag, Basel, 2000.

[5] A. Fernandes, C. Gutierrez, and R. Rabanal, *Global asymptotic stability for differentiable vector fields of* $\mathbb{R}^2$. J. Differential Equations **206**(2004), no. 2, 470–482.

[6] E. A. Gonzales Velasco, *Generic properties of polynomial vector fields at infinity.* Trans. Amer. Math. Soc. **143**(1969), 201–222.

[7] C. Gutierrez and N. Ch. Nguyen, *A remark on an eigenvalue condition for the global injectivity of differentiable maps of* $\mathbb{R}^2$. Disc. Cont. Dyn. Syst. **17**(2007), no. 2, 397–402.

[8] X. Jarque and J. Llibre, *Polynomial foliations of* $\mathbb{R}^2$. Pacific J. Math. **197**(2001), no. 1, 53–72.

[9] X. Jarque and Z. Nitecki, *Hamiltonian stability in the plane.* Ergodic Theory Dynam. Systems **20**(2000), no. 3, 775–799.

[10] W. Kaplan, *Regular curve–families filling the plane. II.* Duke Math. J. **8**(1941), 11–46.

[11] J. Kotus, M. Krych, and Z. Nitecki, *Global structural stability of flows on open surfaces.* Mem. Amer. Math. Soc. **37**(1982) no. 261.

[12] D. Neumann, *Classification of continuous flows on* 2*-manifolds.* Proc. Amer. Math. Soc. **48**(1975), 73–81.

[13] S. Pinchuck, *A counterexample to the strong Jacobian conjecture.* Math. Z. **217**(1994), no. 1, 1–4.

[14] P. H. Rabinowitz, *Minimax Methods in Critical Point Theory with Applications to Differential Eqautions,* C.B.M.S. Regional Conference Series in Mathematics 65, American Mathematical Society, Providence, RI, 1986.

[15] E. A. de B. e Silva and M. A. Teixeira, *A version of Rolle's Theorem and applications.* Bol. Soc. Brasil. Mat. **29**(1998), no. 2, 301–328.

[16] ———, *Global injectivity and asymptotic stability via minimax method.* In: Progress in Nonlinear Analysis Nankai Ser. Pure Appl. Math. Theoret. Phys. 6, World Sci. Publishing, River Edge, NJ, 2000, pp. 339–358.

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