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Four-fold torsion theories

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In this note 4-fold torsion theories (for categories of modules) are classified by means of orthogonal pairs of comaximal ideals. Among the applications are results of Kurata concerning lengths of n-fold torsion theories and an upper bound for the number of 4-fold torsion theories over a semiperfect ring.

Introduction

Let R denote a ring with identity. Kurata [2] has introduced the notion of an *n-fold torsion theory* for the category R-mod of left R-modules. For an integer $n > 1$, this is an n-tuple

$$\left(T_1, \ldots, T_n\right)$$

of classes of R-modules such that each successive pair $\left(T_i, T_{i+1}\right)$, for $i = 1, \ldots, n-1$, forms a torsion theory. The T_i-torsion submodule of an R-module M is denoted by $t_i(M)$. (For an excellent account of torsion theories and their relation to topologies and radicals see Chapter 1 of Stenström [5].)

If there exists an integer $1 < i < n$ such that $T_1 = T_{i+1}$, then the smallest such integer is the *length* of $\left(T_1, \ldots, T_n\right)$. If not, $\left(T_1, \ldots, T_n\right)$ has *length* n.

Kurata [2] proved that there are essentially only four types of n-fold torsion theories:

(1) 2-fold torsion theories which cannot be extended to 3-fold

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torsion theories;

(2) 3-fold torsion theories with length 2 ;

(3) 3-fold torsion theories which cannot be extended to 4-fold torsion theories; and

(4) 4-fold torsion theories of length 4 .

Furthermore, for commutative rings and semiprime rings only the first three types exist. (Terminology used here without definition can be found in [3] or [5].)

A 3-fold torsion theory is nothing but a TTF-theory defined by Jans [1]. It was shown in [1] that there exists a one-to-one correspondence between TTF-theories and idempotent ideals of R . Furthermore, the TTF-theories of length 2 are precisely those which correspond to ideals generated by a central idempotent. The purpose of this note is to apply the correspondence in [1] to obtain a one-to-one correspondence between 4-fold torsion theories and orthogonal pairs of comaximal ideals of R or, equivalently, ideals of R which are generated as a one sided ideal by an idempotent element. It is possible to determine from such a pair of ideals the length of the corresponding 4-fold torsion theory. This correspondence is used to recover and extend slightly the results of Kurata mentioned previously. New results obtained from this correspondence include a one-to-one length preserving correspondence between 4-fold torsion theories for R-mod and those for mod-R and an upper bound for the number of 4-fold torsion theories over a semiperfect ring.

The correspondence and applications

Since the correspondence in Jans [1] is crucial to all that follows, we begin this section by describing it. An idempotent ideal I of R determines three classes of left R-modules:

$$C_I = \{M \text{ in } R\text{-mod} \mid IM = M\} ;$$

$$T_I = \{M \text{ in } R\text{-mod} \mid IM = 0\} ;$$

and

$$F_I = \{M \text{ in } R\text{-mod} \mid \{x \in M \mid Ix = 0\} = 0\} .$$

THEOREM A. *There exist inverse one-to-one correspondences between 3-fold torsion theories for R-mod and idempotent ideals of R given by* $(T_1, T_2, T_3) \to t_1(R)$ *and* $I \to (C_I, T_I, F_I)$. *Furthermore,* (T_1, T_2, T_3) *has length* 2 *if and only if* $R = t_1(R) \oplus t_2(R)$ *(ring direct sum).*

The information in Theorem A is a summary of Theorem 2.1, Corollary 2.2 and part of Theorem 2.4 of [1].

If (I, K) is a pair of ideals of R, it is an *orthogonal* pair if $IK = 0$ and the ideals are *comaximal* if $I + K = R$.

THEOREM B. *There exist one-to-one correspondences between each pair of the following:*

(1) *4-fold torsion theories for* R-mod ;

(2) *orthogonal pairs of comaximal ideals of* R ;

(3) *ideals* I *of* R *such that* $I = Re$ *with* $e^2 = e \in R$; *and*

(4) *ideals* K *of* R *such that* $K = fR$ *with* $f^2 = f \in R$.

Proof. First we exhibit the correspondence between (1) and (2). Let (T_1, T_2, T_3, T_4) denote a 4-fold torsion theory. Since both $t_2(R)$ and $R/t_1(R)$ belong to T_2 , it follows from Theorem A applied to (T_1, T_2, T_3) and (T_2, T_3, T_4) , respectively, that $t_1(R)t_2(R) = 0$ and $t_2(R)(R/t_1(R)) = R/t_1(R)$.

The second equality implies that $R = t_1(R) + t_2(R)$. Therefore, let

$$(T_1, T_2, T_3, T_4) \to (t_1(R), t_2(R)) .$$

If (I, K) is an orthogonal pair of comaximal ideals, it is clear that I and K are idempotent. Also for any R-module M , $IM = 0$ if and only if $KM = M$. Thus $T_I = C_K$. Similarly, $F_I = T_K$. It follows from Theorem A that (C_I, T_I, F_I, F_K) is a 4-fold torsion theory and that the correspondence

$$(I, K) \to (C_I, T_I, F_I, F_K)$$

and the correspondence given above are inverses.

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Next we exhibit the correspondence between (2) and (3).

If (I, K) is an orthogonal pair of comaximal ideals, $1 = e + f$ with $e \in I$ and $f \in K$. It is immediate from the orthogonality of I and K that e and f are orthogonal idempotents with $I = Re$ and $K = fR$. Let

$$(I, K) \to I = Re .$$

If $I = Re$ with $e^2 = e$ is an ideal, so is $K = (1-e)R$ and $I + K \supset eR + (1-e)R = R$. Let

$$Re \to (Re, (1-e)R) .$$

If (I, K) and (I, L) are orthogonal pairs of comaximal ideals, $K = (I+L)K = IK + LK = LK \subset L$. Similarly $L \subset K$ and so $K = L$. Thus the above correspondences are inverses.

By symmetry there exists a one-to-one correspondence between (2) and (4). The balance of the proof follows by composing the correspondences now in hand.

COROLLARY 1. *Every 4-fold torsion theory has length 2 or 4. Moreover,* (T_1, T_2, T_3, T_4) *has length 2 if and only if* $t_2(R)t_1(R) = 0$.

Proof. Let $I = t_1(R)$ and $K = t_2(R)$. We first prove the second assertion. By Theorem A, (T_1, T_2, T_3, T_4) has length 2 if and only if $R = I \oplus K$. By Theorem B, $R = I + K$ and $IK = 0$. Thus $R = I \oplus K$ if and only if $I \cap K = 0$. But

$$KI \subset I \cap K = (I+K)(I\cap K) = I(I\cap K) + K(I\cap K) \subset IK + KI = KI .$$

Now we establish the first assertion. If (T_1, T_2, T_3, T_4) does not have length 2, $IK \neq 0$. By Theorem B, $K = fR$ with $f^2 = f$, so $KI \in F_K = T_4$. But $I(KI) = 0$, so $KI \notin C_I = T_1$. Thus $T_1 \neq T_4$, and (T_1, T_2, T_3, T_4) has length 4.

The first part of Corollary 1 is due to Kurata [2, Proposition 3.2].

EXAMPLE. If R is a left artinian ring with zero left singular

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ideal, it is readily verified that the left socle of R is an ideal which is faithful as a right ideal and has the form fR with $f^2 = f \in R$. For rings with zero left singular ideal, the Goldie torsion theory and the dense torsion theory coincide. (See [5].) In the present circumstances, they are equal to (T_{fR}, F_{fR}). It is immediate from Theorem B that this torsion theory can be extended to a 4-fold torsion theory. If R is not semisimple, so that $fR \neq R$, it follows from Corollary 1 that the resulting 4-fold torsion theory has length 4.

Numerous other examples of 4-fold torsion theories of length 4 may be obtained by applying the next corollary in conjunction with Corollary 1. A natural source of examples is rings of triangular matrices over, for instance, a division ring.

Let R be a semiperfect ring and $E = \{e_1, \ldots, e_n\}$ be a complete set of primitive orthogonal idempotents of R. A subset S of E is *triangular*, provided $eR(1-e) = 0$ where $e = \sum e_i$ such that $e_i \in S$.

COROLLARY 2. *Let R be a semiperfect ring and E be a complete set of primitive orthogonal idempotents of R. There exists a one-to-one correspondence between 4-fold torsion theories for R-mod and triangular subsets of E.*

Proof. In view of Theorem B, it suffices to exhibit a one-to-one correspondence between triangular subsets S of E and orthogonal pairs of comaximal ideals of R. Let $e = \sum e_i$ such that $e_i \in S$. Clearly Re is an ideal. Thus we can let

$$S \to (Re, (1-e)R) .$$

To obtain the inverse correspondence we first prove that if (I, K) is an orthogonal pair of comaximal ideals of R, then for each $e_i \in E$ either

$$IRe_i = Re_i \text{ and } e_iRK = 0$$

or

$$IRe_i = 0 \text{ and } e_iRK = e_iR .$$

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Clearly $Re_i = (I+K)Re_i = IRe_i + KRe_i$. Thus $IRe_i \not\subseteq Je_i$ or $KRe_i \not\subseteq Je_i$, where J is the Jacobson radical of R , since Je_i is the unique maximal submodule of Re_i . Hence $Re_i = IRe_i$ or $Re_i = KRe_i$. In the first instance, $e_iR \subset I$ and so $e_iRK = 0$. Now suppose $Re_i = KRe_i$. Clearly $IRe_i = 0$. By symmetry, $e_iR = e_iRK$ or $e_iR = e_iRI$. The latter equality would contradict the fact that $IRe_i = 0$. Thus we have established the above dichotomy. It is immediate from this observation that $I = Re$ and $K = (1-e)R$ where $e = \sum e_i$ such that $e_i \in E$ and $IRe_i = Re_i$. It can now be readily verified that the correspondence

$$(I, K) \to S = \{e_i \in E \mid IRe_i = Re_i\}$$

is the inverse of the one described previously.

The preceding corollary implies that for a semiperfect ring there exist only finitely many 4-fold torsion theories. We sharpen this observation somewhat.

COROLLARY 3. *Let R be a semiperfect ring with k isomorphism types of primitive idempotents. There exist at most 2^k 4-fold torsion theories for R-mod , not more than $2^{k-1} - 1$ of which have length 4 .*

Proof. Let E be a complete set of primitive orthogonal idempotents for R . The elements of E can be indexed as e_{ij} , with $i = 1, \ldots, k$ and $j = 1, \ldots, q_i$, so that Re_{ij} is isomorphic to Re_{uv} if and only if $i = u$. If S is a triangular subset of E and $e_{im} \in S$ then $e_{ij} \in S$ for each $j = 1, \ldots, q_i$. Thus distinct triangular subsets of E determine distinct subsets of $\{1, \ldots, k\}$. The first assertion now follows from the preceding corollary. The empty set and E are vacuously triangular subsets of E but the corresponding 4-fold torsion theories have length 2 . Thus the possibilities in the second instance are reduced to $2^k - 2$. However, if $S \neq E$ is a non-empty subset of E with both S and its complement triangular, it is immediate from Corollary 1 that the 4-fold torsion theory associated with S has length 2 . Thus there are at most $(2^k-2)/2 = 2^{k-1} - 1$ 4-fold torsion theories of length 4 .

COROLLARY 4 (Kurata). *If* $n > 4$, *any* n*-fold torsion theory has length* 2 .

Proof. Let $I = t_1(R)$, $K = t_2(R)$ and $L = t_3(R)$. By Theorem B, (I, K) and (K, L) are orthogonal pairs of comaximal ideals of R . Thus $I = I(K+L) = IK + IL = IL \subset L$. Similarly, $L \subset I$ and so $L = I$. Hence $IK = 0$ and the conclusion follows from Corollary 1.

By the uniqueness of the torsion theory corresponding to any torsion class for R-mod (see [5] or [7]), it is clear that for any integer $n \geq 3$ an n-fold torsion theory of length 2 is obtained by alternate repetition of the first two classes. Thus the next corollary is a consequence of Corollaries 1 and 4.

COROLLARY 5 (Kurata). *There exist only four different types of* n*-fold torsion theories:*

(1) 2*-fold torsion theories which cannot be extended to* 3*-fold torsion theories;*

(2) 3*-fold torsion theories of length* 2 ;

(3) 3*-fold torsion theories which cannot be extended to* 4*-fold torsion theories; and*

(4) 4*-fold torsion theories of length* 4 .

The next result also stems from Corollaries 1 and 4.

COROLLARY 6. *For every integer* $n > 3$, *all* n*-fold torsion theories for* R-mod *have length* 2 *if and only if for every orthogonal pair* (I, K) *of comaximal ideals of* R *the pair* (K, I) *is orthogonal.*

COROLLARY 7. *Each of the following classes of rings has the property that for every integer* $n > 3$ *all* n*-fold torsion theories have length* 2 *:*

(1) *commutative rings;*

(2) *semiprime rings; and*

(3) *left or right injective cogenerator rings.*

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Proof. This is an application of the preceding corollary. It clearly applies in the first two instances and the third case follows from [4, Theorem 7].

The first two parts of this corollary are due to Kurata [2, Propositions 4.4 and 4.6].

The final corollary is immediate from Theorem B, Corollary 1, and their right hand analogs.

COROLLARY 8. *There exists a one-to-one correspondence between 4-fold torsion theories for* R-mod *and 4-fold torsion theories for* mod-R *which preserves length.*

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