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Nef Divisors in Codimension One on the Moduli Space of Stable Curves

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Abstract. Let M_g be the moduli space of smooth curves of genus $g \geqslant 3$, and $\bar{M}_g$ the Deligne-Mumford compactification in terms of stable curves. Let $\bar{M}_g^{[1]}$ be an open set of $\bar{M}_g$ consisting of stable curves of genus g with one node at most. In this paper, we determine the necessary and sufficient condition to guarantee that a $\mathbb{Q}$-divisor D on $\bar{M}_g$ is nef over $\bar{M}_g^{[1]}$, that is, $(D \cdot C) \geqslant 0$ for all irreducible curves C on $\bar{M}_g$ with $C \cap \bar{M}_g^{[1]} \neq \emptyset$.

Mathematics Subject Classifications (2000). 14H10, 14C22

Key words. Nef divisor, Moduli space, Stable curve

Introduction

Throughout this paper, we fix an algebraically closed field k, and every algebraic scheme is defined over k. For simplicity, we assume that the characteristic of k is zero in this introduction.

Let X be a normal complete variety and $\mathcal{P}$ a certain kind of positivity of $\mathbb{Q}$-line bundles on X (e.g. ampleness, effectivity, bigness, etc). A problem to describe the cone $\mathrm{Cone}(X; \mathcal{P})$ consisting of $\mathbb{Q}$-line bundles with the positivity $\mathcal{P}$ is usually very hard and interesting. In this paper, as positivity, we consider numerical effectivity over a fixed open set. Namely, let U be a Zariski open set of X. We say a $\mathbb{Q}$-line bundle L is *nef over* U if, for all irreducible curves C with $C \cap U \neq \emptyset$, $(L \cdot C) \geqslant 0$. We define the relative nef cone $\mathrm{Nef}(X; U)$ over U to be the cone of $\mathbb{Q}$-line bundles on X which are nef over U.

Let g and n be nonnegative integers with $2g - 2 + n > 0$. Let $\bar{M}_{g,n}$ (resp. $M_{g,n}$) denote the moduli space of n-pointed stable curves (resp. n-pointed smooth curves) of genus g. For a nonnegative integer t, an irreducible component of the closed subscheme consisting of curves with at least t nodes is called *a t-codimensional stratum* of $\bar{M}_{g,n}$. (For example, a 1-codimensional stratum is a boundary component.) We denote by $S^t(\bar{M}_{g,n})$ the set of all t-codimensional strata of $\bar{M}_{g,n}$. Let $\bar{M}_{g,n}^{[t]}$ be the open set of $\bar{M}_{g,n}$ obtained by subtracting all $(t+1)$-codimensional strata, i.e., $\bar{M}_{g,n}^{[t]}$ is the open set consisting of curves with at most t nodes. (Note that $\bar{M}_{g,n}^{[0]} = M_{g,n}$.) Here we consider the following problem:

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PROBLEM A. Describe the tower of relative nef cones

$$\mathrm{Nef}(\bar{M}_{g,n}; M_{g,n}) \supseteq \mathrm{Nef}(\bar{M}_{g,n}; \bar{M}_{g,n}^{[1]}) \supseteq \cdots \supseteq \mathrm{Nef}(\bar{M}_{g,n}; \bar{M}_{g,n}^{[3g-3+n-1]}) = \mathrm{Nef}(\bar{M}_{g,n}).$$

We say a $\mathbb{Q}$-divisor on $\bar{M}_{g,n}$ is *F-nef* if the intersection number with every one-dimensional stratum is nonnegative. Let $\mathrm{FNef}(\bar{M}_{g,n})$ denote the cone consisting of F-nef $\mathbb{Q}$-divisors. Concerning the top $\mathrm{Nef}(\bar{M}_{g,n})$ of the tower, it is conjectured in [4, 5, 7] that $\mathrm{FNef}(\bar{M}_{g,n}) = \mathrm{Nef}(\bar{M}_{g,n})$. In other words, the Mori cone of $\bar{M}_{g,n}$ is generated by one-dimensional strata, which gives rise to a concrete description of $\mathrm{Nef}(\bar{M}_{g,n})$ (cf. [4, 5, 7]). Moreover, it is closely related to the relative nef cone $\mathrm{Nef}(\bar{M}_{g,n}; M_{g,n})$. Actually, it was shown in [5] that if the weaker assertion $\mathrm{FNef}(\bar{M}_{g,n}) \subseteq \mathrm{Nef}(\bar{M}_{g,n}; M_{g,n})$ holds for all g, n, then $\mathrm{FNef}(\bar{M}_{g,n}) = \mathrm{Nef}(\bar{M}_{g,n})$. Further, as discussed in [5], $\bar{M}_{g,n}$ admits no interesting birational morphism to a projective variety. However, we can expect the rich birational geometry on $\bar{M}_{g,n}$ in terms of rational maps. In this sense, to understand the tower of relative nef cones as above might be a step toward this natural problem.

We assume that $g \geqslant 3$ and $n = 0$. Let λ be the Hodge class on $\bar{M}_g$, and $\delta_{\mathrm{irr}}, \delta_1, \ldots, \delta_{[g/2]}$ the classes of the irreducible components $\Delta_{\mathrm{irr}}, \Delta_1, \ldots, \Delta_{[g/2]}$ of the boundary $\bar{M}_g \setminus M_g$ as in [2]. Let μ be a divisor on $\bar{M}_g$ given by

$$\mu = (8g+4)\lambda - g\delta_{\mathrm{irr}} - \sum_{i=1}^{[g/2]} 4i(g-i)\delta_i.$$

In the paper [11], we proved that $\mathrm{Nef}(\bar{M}_g; M_g)$ is the convex hull spanned by $\mu, \delta_{\mathrm{irr}}, \delta_1, \ldots, \delta_{[g/2]}$, that is,

$$\mathrm{Nef}(\bar{M}_g; M_g) = \mathbb{Q}_+\mu + \mathbb{Q}_+\delta_{\mathrm{irr}} + \sum_{i=1}^{[g/2]} \mathbb{Q}_+\delta_i,$$

where $\mathbb{Q}_+ = \{x \in \mathbb{Q} | x \geqslant 0\}$. The cone $\mathrm{Nef}(\bar{M}_g; M_g)$ is closely related to the Zariski closure $\bar{H}_g$ of the locus H_g consisting of smooth hyperelliptic curves. Indeed, a $\mathbb{Q}$-divisor $D = a\mu + b_{\mathrm{irr}}\delta_{\mathrm{irr}} + \sum_{i=1}^{[g/2]} b_i\delta_i$ is nef over M_g if and only if $D|_{\bar{H}_g}$ is nef over H_g and $a \geqslant 0$, that is, the dual cone of $\mathrm{Nef}(\bar{M}_g; M_g)$ is generated by the classes of curves in $\bar{H}_g$ and the class of a complete irreducible curve in M_g (cf. Remark 6.3). The main purpose of this paper is to generalize the above results to the cone $\mathrm{Nef}(\bar{M}_g; \bar{M}_g^{[1]})$. Namely we have the following theorem:

THEOREM B (cf. Theorem 5.1 and Section 6). (1) *A $\mathbb{Q}$-divisor $a\mu + b_{\mathrm{irr}}\delta_{\mathrm{irr}} + \sum_{i=1}^{[g/2]} b_i\delta_i$ on $\bar{M}_g$ is nef over $\bar{M}_g^{[1]}$ if and only if the following system of inequalities hold:*

$$a \geqslant \max\left\{ \frac{b_i}{4i(g-i)} \middle| i = 1, \ldots, [g/2] \right\}, \quad B_0 \geqslant B_1 \geqslant B_2 \geqslant \cdots \geqslant B_{[g/2]},$$
$$B^*_{[g/2]} \geqslant \cdots \geqslant B^*_2 \geqslant B^*_1 \geqslant B^*_0,$$

where B_0, B^*_0, B_i and B^*_i $(i = 1, \ldots, [g/2])$ are given by

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$$B_0 = 4b_{\mathrm{irr}}, \quad B_0^* = \frac{4b_{\mathrm{irr}}}{g(2g-1)}, \quad B_i = \frac{b_i}{i(2i+1)} \text{ and } B_i^* = \frac{b_i}{(g-i)(2(g-i)+1)}.$$

(2) *We can construct irreducible complete curves*

$$C_1, \ldots, C_{[g/2]}, C_1^*, \ldots, C_{[g/2]}^*, C_1^\dagger, \ldots, C_{[g/2]}^\dagger$$

on $\bar{M}_g$ *with the following properties* (for concrete constructions of curves, see Section 6):

(2.1) $C_i \subseteq \Delta_i$ *and* $C_i \cap \bar{M}_g^{[1]} \neq \emptyset$ *for all* $1 \leqslant i \leqslant [g/2]$.
(2.2) $C_1^* \subseteq \Delta_{\mathrm{irr}}$, $C_i^* \subseteq \Delta_{i-1}$ $(2 \leqslant i \leqslant [g/2])$ *and* $C_i^* \cap \bar{M}_g^{[1]} \neq \emptyset$ $(1 \leqslant i \leqslant [g/2])$.
(2.3) $C_i^\dagger \subseteq \Delta_i$ *and* $C_i^\dagger \cap \bar{M}_g^{[1]} \neq \emptyset$ *for all* $1 \leqslant i \leqslant [g/2]$.
(2.4) *For a* $\mathbb{Q}$*-divisor* $D = a\mu + b_{\mathrm{irr}}\delta_{\mathrm{irr}} + \sum_{i=1}^{[g/2]} b_i\delta_i$ *on* $\bar{M}_g$,

$$(D \cdot C_i) \geqslant 0 \Longleftrightarrow B_{i-1} \geqslant B_i$$
$$(D \cdot C_i^*) \geqslant 0 \Longleftrightarrow B_{i-1}^* \leqslant B_i^*$$
$$(D \cdot C_i^\dagger) \geqslant 0 \Longleftrightarrow 4i(g-i)a \geqslant b_i$$

In particular, the dual cone of $\mathrm{Nef}(\bar{M}_g; \bar{M}_g^{[1]})$ *is generated by the classes of the above curves.*

An interesting point is that (1) of the above theorem shows us that μ is not only nef over M_g but also nef over $\bar{M}_g^{[1]}$. Moreover, (1) tells us that every nef $\mathbb{Q}$-divisor over $\bar{M}_g^{[1]}$ can be obtained in the following way. Namely, we first fix a nonnegative rational number b_{irr}, and take b_1 with

$$\frac{4(g-1)b_{\mathrm{irr}}}{g} \leqslant b_1 \leqslant 12b_{\mathrm{irr}}.$$

Further, we choose $b_2, \ldots, b_{[g/2]}$ inductively by using

$$\frac{(g-1-i)(2(g-i)-1)}{(g-i)(2(g-i)+1)} b_i \leqslant b_{i+1} \leqslant \frac{(i+1)(2i+3)}{i(2i+1)} b_i.$$

Finally, we take a with

$$a \geqslant \max\left\{ \frac{b_i}{4i(g-i)} \,\middle|\, i = 1, \ldots, [g/2] \right\}.$$

Then, a $\mathbb{Q}$-divisor given by $a\mu + b_{\mathrm{irr}}\delta_{\mathrm{irr}} + \sum_{i=1}^{[g/2]} b_i\delta_i$ is nef over $\bar{M}_g^{[1]}$.

Besides the properties (2.1)–(2.4) of curves $C_1, \ldots, C_{[g/2]}, C_1^*, \ldots, C_{[g/2]}^*, C_1^\dagger, \ldots, C_{[g/2]}^\dagger$, surprisingly we can see $C_i, C_i^* \subseteq \bar{H}_g$ for all $i = 1, \ldots, [g/2]$. Thus, a $\mathbb{Q}$-divisor $D = a\mu + b_{\mathrm{irr}}\delta_{\mathrm{irr}} + \sum_{i=1}^{[g/2]} b_i\delta_i$ is nef over $\bar{M}_g^{[1]}$ if and only if $D|_{\bar{H}_g}$ is nef over $\bar{H}_g \cap \bar{M}_g^{[1]}$ and $4i(g-i)a \geqslant b_i$ for all $i = 1, \ldots, [g/2]$. Moreover, as pointed out by Prof. Keel, the inequalities involving B_i and B_i^* in Theorem B are formally similar to those in [7, Lemma 4.8], which suggests to us a certain kind of connection between $\bar{M}_g$ and $\bar{H}_g$ via $\bar{M}_{0,2g+2}/S_{2g+2}$.

Further, as corollaries of the above theorem, we have the following:

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COROLLARY C (cf. Corollary 5.2). *For an irreducible component* Δ *of the boundary* $\bar{M}_g \setminus M_g$*, let* $\widetilde{\Delta}$ *be the normalization of* Δ*, and* $\rho_\Delta\colon \widetilde{\Delta} \to \bar{M}_g$ *the induced morphism. Then, a* $\mathbb{Q}$*-divisor* D *on* $\bar{M}_g$ *is nef over* $\bar{M}_g^{[1]}$ *if and only if the following are satisfied:*

(1) *D is weakly positive at any points of* M_g.
(2) *For every boundary component* Δ*,* $\rho_\Delta^*(D)$ *is weakly positive at any points of* $\rho_\Delta^{-1}(\bar{M}_g^{[1]})$

For the definition of weak positivity, see Section 1.1.

COROLLARY D (cf. Corollary 5.3). *With notation as above, if* $\rho_\Delta^*(D)$ *is nef over* $\rho_\Delta^{-1}(\bar{M}_g^{[1]})$ *for every boundary component* Δ*, then* D *is nef over* $\bar{M}_g^{[1]}$*. In particular, the Mori cone of* $\bar{M}_g$ *is the convex hull spanned by curves lying on the boundary* $\bar{M}_g \setminus M_g$*, which gives rise to a special case of* [5, Proposition 3.1].

Let us go back to the general situation. Similarly, for $\Delta \in S^l(\bar{M}_{g,n})$, let $\widetilde{\Delta}$ be the normalization of Δ, and $\rho_\Delta\colon \widetilde{\Delta} \to \bar{M}_{g,n}$ the induced morphism. Inspired by the above corollaries, we have the following questions:

QUESTION E. For a nonnegative integer t, if a $\mathbb{Q}$-divisor D on $\bar{M}_{g,n}$ is nef over $\bar{M}_{g,n}^{[t]}$, then is $\rho_\Delta^*(D)$ weakly positive at any points of $\rho_\Delta^{-1}(\bar{M}_{g,n}^{[l]})$ for all $0 \leqslant l \leqslant t$ and all $\Delta \in S^l(\bar{M}_{g,n})$? More strongly, if D is nef over $\bar{M}_{g,n}^{[t]}$, then is D weakly positive at any points of $\bar{M}_{g,n}^{[t]}$?

QUESTION F. Fix an integer t with $0 \leqslant t \leqslant 3g-3+n-1$. If $\rho_\Delta^*(D)$ is nef over $\rho_\Delta^{-1}(\bar{M}_{g,n}^{[t]})$ for all $\Delta \in S^t(\bar{M}_{g,n})$, then is D nef over $\bar{M}_{g,n}^{[t]}$?

In the case $t = 3g-3+n-1$, the above question is nothing more than asking $\mathrm{FNef}(\bar{M}_{g,n}) = \mathrm{Nef}(\bar{M}_{g,n})$.

In order to get the above theorem, we need a certain kind of slope inequalities on the moduli space of n-pointed stable curves. The $\mathbb{Q}$-line bundles λ and $\psi_1, \ldots, \psi_n$ on $\bar{M}_{g,n}$ are defined as follows: Let $\pi\colon \bar{M}_{g,n+1} \to \bar{M}_{g,n}$ be the universal curve of $\bar{M}_{g,n}$, and $s_1, \ldots, s_n : \bar{M}_{g,n} \to \bar{M}_{g,n+1}$ the sections of π arising from the n-points of $\bar{M}_{g,n}$. Then, $\lambda = \det(\pi_*(\omega_{\bar{M}_{g,n+1}/\bar{M}_{g,n}}))$ and $\psi_i = s_i^*(\omega_{\bar{M}_{g,n+1}/\bar{M}_{g,n}})$ for $i = 1, \ldots, n$. Here we set

$$[n] = \{1, \ldots, n\} \quad (\text{note that } [0] = \emptyset),$$
$$\Upsilon_{g,n} = \{(i, I) \mid i \in \mathbb{Z},\ 0 \leqslant i \leqslant g \text{ and } I \subseteq [n]\} \setminus \{(0, \emptyset), (0, \{1\}), \ldots, (0, \{n\})\},$$
$$\overline{\Upsilon}_{g,n} = \{\{(i, I), (j, J)\} \mid (i, I), (j, J) \in \Upsilon_{g,n}, i+j = g, I \cap J = \emptyset, I \cup J = [n]\}.$$

Moreover, for a finite set S, we denote the number of it by $|S|$. The boundary $\bar{M}_{g,n} \setminus M_{g,n}$ has the following irreducible decomposition:

$$\bar{M}_{g,n} \setminus M_{g,n} = \Delta_{\mathrm{irr}} \cup \bigcup_{\{(i,I),(j,J)\} \in \overline{\Upsilon}_{g,n}} \Delta_{\{(i,I),(j,J)\}}.$$

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A general point of Δ_{irr} represents an n-pointed irreducible stable curve with one node. A general point of $\Delta_{\{(i,I),(j,J)\}}$ represents an n-pointed stable curve consisting of an $|I|$-pointed smooth curve C_1 of genus i and a $|J|$-pointed smooth curve C_2 of genus j meeting transversally at one point, where $|I|$-points on C_1 (resp. $|J|$-points on C_2) arise from $\{s_t\}_{t\in I}$ (resp. $\{s_l\}_{l\in J}$). Let δ_{irr} and $\delta_{\{(i,I),(j,J)\}}$ be the classes of Δ_{irr} and $\Delta_{\{(i,I),(j,J)\}}$ in $\mathrm{Pic}(\bar{M}_{g,n}) \otimes \mathbb{Q}$, respectively. For a subset L of $[n]$, we define a $\mathbb{Q}$-divisor θ_L on $\bar{M}_{g,n}$ to be

$$\theta_L = 4(g-1+|L|)(g-1)\sum_{t\in L}\psi_t - 12|L|^2\lambda + |L|^2\delta_{\mathrm{irr}} - \sum_{\upsilon\in\overline{\Upsilon}_{g,n}} 4\gamma_L(\upsilon)\delta_\upsilon,$$

where $\gamma_L\colon \overline{\Upsilon}_{g,n} \to \mathbb{Z}$ is given by

$$\gamma_L(\{(i,I),(j,J)\}) = \left(\det\begin{pmatrix} i & |L\cap I| \\ j & |L\cap J| \end{pmatrix} + |L\cap I|\right)\times \\ \times\left(\det\begin{pmatrix} i & |L\cap I| \\ j & |L\cap J| \end{pmatrix} - |L\cap J|\right).$$

Then, we have the following, theorem:

THEOREM G (cf. Theorem 4.1). *For any subset L of $[n]$, the divisor θ_L is weakly positive at any points of $M_{g,n}$. In particular, it is nef over $M_{g,n}$.*

We remark that R. Hain has already announced the above inequality in the case where $n = 1$. (For details, see [6].) Theorem G is a generalization of his inequality.

Here we assume that $g \geqslant 2$. First note that

$$\mu = (8g+4)\lambda - g\delta_{\mathrm{irr}} - \sum_{\{(i,I),(j,J)\}\in\overline{\Upsilon}_{g,n}} 4ij\delta_{\{(i,I),(j,J)\}}$$

is nef over $M_{g,n}$. Thus, as a consequence of Theorem G, we can see that

$$\mathbb{Q}_+\mu + \sum_{L\subseteq[n]}\mathbb{Q}_+\theta_L + \mathbb{Q}_+\delta_{\mathrm{irr}} + \sum_{\upsilon\in\overline{\Upsilon}_{g,n}}\mathbb{Q}_+\delta_\upsilon \subseteq \mathrm{Nef}(\bar{M}_{g,n}; M_{g,n}),$$

so that we may ask the following question:

QUESTION H. Is $\mathrm{Nef}(\bar{M}_{g,n}; M_{g,n})$ the convex hull spanned by $\mathbb{Q}$-divisors μ, θ_L ($\forall L \subseteq [n]$), δ_{irr} and δ_υ ($\forall \upsilon \in \overline{\Upsilon}_{g,n}$).

Corollaries 4.2 and 4.3 are partial answers for the above question. If the above question is true, then it gives an affirmative answer of Question E for $t = 0$.

1. Notations, Conventions, Terminology and Preliminaries

Throughout this paper, we fix an algebraically closed field k, and every algebraic scheme is defined over k.

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1.1. THE POSITIVITY OF WEIL DIVISORS

Let X be a normal variety. Let denote $Z^1(X)$ (resp. $\mathrm{Div}(X)$) the group of Weil divisors (resp. Cartier divisors) on X, and $\sim$ the linear equivalence on $Z^1(X)$. We set $A^1(X) = Z^1(X)/\!\sim$ and $\mathrm{Pic}(X) = \mathrm{Div}(X)/\!\sim$. Note that $\mathrm{Pic}(X)$ is canonically isomorphic to the Picard group (the group of isomorphism classes of line bundles). Moreover, we denote by $\mathrm{Ref}(X)$ the set of isomorphism classes of reflexive sheaves of rank 1 on X. For a Weil divisor D, the sheaf $\mathcal{O}_X(D)$ is given by

$$\mathcal{O}_X(D)(U) = \{\phi \in \mathrm{Rat}(X)^\times \mid (\phi) + D \text{ is effective over } U\} \cup \{0\}$$

for each Zariski open set U of X. Then, we can see $\mathcal{O}_X(D) \in \mathrm{Ref}(X)$. Conversely, let L be a reflexive sheaf of rank 1 on X. For a nonzero rational section s of L, $\mathrm{div}(s)$ is defined as follows: Let X_0 be the maximal Zariski open set of X over which L is locally free. Note that $\mathrm{codim}(X \setminus X_0) \geqslant 2$. Then, $\mathrm{div}(s) \in Z^1(X)$ is defined by the Zariski closure of $\mathrm{div}(s|_{X_0})$. By our definition, we can see that $\mathcal{O}_X(\mathrm{div}(s)) \simeq L$. Thus, the correspondence $D \mapsto \mathcal{O}_X(D)$ gives rise to an isomorphism $A^1(X) \simeq \mathrm{Ref}(X)$. Here we remark that if $x \notin \mathrm{Supp}(\mathrm{div}(s))$, then L is free at x because $\mathcal{O}_X(\mathrm{div}(s))_x = \mathcal{O}_{X,x}$ for $x \notin \mathrm{Supp}(\mathrm{div}(s))$.

An element of $Z^1(X) \otimes \mathbb{Q}$ (resp. $\mathrm{Div}(X) \otimes \mathbb{Q}$) is celled a *$\mathbb{Q}$-divisor* (resp. *$\mathbb{Q}$-Cartier divisor*). For $\mathbb{Q}$-divisors D_1 and D_2, we say D_1 is $\mathbb{Q}$-linearly equivalent to D_2, denoted by $D_1 \sim_{\mathbb{Q}} D_2$, if there is a positive integer n such that $nD_1, nD_2 \in Z^1(X)$ and $nD_1 \sim nD_2$, i.e., D_1 coincides with D_2 in $A^1(X) \otimes \mathbb{Q}$.

Fix a subset S of X. For $D \in Z^1(X) \otimes \mathbb{Q}$, we say D is *semi-ample over S* if, for any $s \in S$, there is an effective $\mathbb{Q}$-divisor E on X with $s \notin \mathrm{Supp}(E)$ and $D \sim_{\mathbb{Q}} E$. Moreover, D is said to be *weakly positive over S* if there are $\mathbb{Q}$-divisors $Z_1, \ldots, Z_l$, a sequence $\{D_m\}_{m=1}^{\infty}$ of $\mathbb{Q}$-divisors, and sequences $\{a_{1,m}\}_{m=1}^{\infty}, \ldots, \{a_{l,m}\}_{m=1}^{\infty}$ of rational numbers such that

(1) l does not depend on m,
(2) D_m is semi-ample over S for all $m \gg 0$,
(3) $D \sim_{\mathbb{Q}} D_m + \sum_{i=1}^{l} a_{i,m} Z_i$ for all $m \gg 0$, and
(4) $\lim_{m\to\infty} a_{i,m} = 0$ for all $i = 1, \ldots, l$.

In the above definition, if D, D_m and Z_i's are $\mathbb{Q}$-Cartier divisors, then D is said to be *weakly positive over S in terms of Cartier divisors* (for short, C-weakly positive over S). Further, if D is semi-ample over $\{x\}$ for some $x \in X$, then we say D is semi-ample at x. Similarly, we define the weak positivity of D at x and the C-weak positivity of D at x. We remark that weak positivity in [11] is nothing more than C-weak positivity. Moreover, note that if a $\mathbb{Q}$-divisor D is semi-ample at x, then D is a $\mathbb{Q}$-Cartier divisor around x, i.e., there is a Zariski open set U of X such that $x \in U$ and $D|_U$ is a $\mathbb{Q}$-Cartier divisor on U.

A normal variety X is said to be *$\mathbb{Q}$-factorial* if $Z^1(X) \otimes \mathbb{Q} = \mathrm{Div}(X) \otimes \mathbb{Q}$, i.e., any Weil divisors are $\mathbb{Q}$-Cartier divisors. It is well known that if $Y \to X$ is a finite and surjective morphism of normal varieties and Y is $\mathbb{Q}$-factorial, then X is also

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$\mathbb{Q}$-factorial (cf. [8, Lemma 5.16]). Thus the moduli space $\bar{M}_{g,n}$ of n-pointed stable curves of genus g is $\mathbb{Q}$-factorial because $\bar{M}_{g,n}$ is an orbifold. If X is $\mathbb{Q}$-factorial, then the weak positivity of D over S coincides with the C-weak positivity of D over S.

We assume that X is complete and D is a $\mathbb{Q}$-Cartier divisor. We say D is *nef over* S if $(D \cdot C) \geqslant 0$ for any complete irreducible curves C with $S \cap C \neq \emptyset$. Moreover, for a point x of X, we say D *is nef at* x if D is nef over $\{x\}$. Note that

'D is semi-ample at x' $\Longrightarrow$ 'D is C-weakly positive at x' $\Longrightarrow$ 'D is nef at x'

LEMMA 1.1.1 (char$(k) \geqslant 0$). *Let D be a $\mathbb{Q}$-divisor on X, and $x_1, \dots, x_n \in X$. If D is semi-ample at x_i for each i, then there is an effective $\mathbb{Q}$-divisor E on X such that $E \sim_{\mathbb{Q}} D$ and $x_i \notin \mathrm{Supp}(E)$ for all i.*

Proof. By our assumption, there is an effective $\mathbb{Q}$-divisor E_i on X such that $E_i \sim_{\mathbb{Q}} D$ and $x_i \notin \mathrm{Supp}(E_i)$. Take a sufficiently large integer m such that mD, $mE_1, \dots, mE_n \in Z^1(X)$ and $mD \sim mE_i$ for all i. Thus, there is a section s_i of $H^0(X, \mathcal{O}_X(mD))$ with $\mathrm{div}(s_i) = mE_i$. Here since $x_i \notin \mathrm{Supp}(mE_i)$ and $\mathcal{O}_X(mD) \simeq \mathcal{O}_X(mE_i)$, we can see that $\mathcal{O}_X(mD)$ is free at each x_i.

For $\alpha = (\alpha_1, \dots \alpha_n) \in k^n$, we set $s_\alpha = \alpha_1 s_1 + \cdots + \alpha_n s_n \in H^0(X, \mathcal{O}_X(mD))$. Further, we set $V_i = \{\alpha \in k^n \mid s_\alpha(x_i) = 0\}$. Then, $\dim V_i = n - 1$ for all i. Thus, since $\#(k) = \infty$, there is $\alpha \in k^n$ with $\alpha \notin V_1 \cup \cdots \cup V_r$, i.e., $s_\alpha(x_i) \neq 0$ for all i. Let us consider a divisor $E = \mathrm{div}(s_\alpha)$. Then, $E \sim mD$ and $x_i \notin \mathrm{Supp}(E)$ for all i. □

PROPOSITION 1.1.2 (char$(k) \geqslant 0$). *Let $\pi: X \to Y$ be a surjective, proper and generically finite morphism of normal varieties. Let D be a $\mathbb{Q}$-divisor on X and S a subset of Y such that $\pi^{-1}(S)$ is finite. Then, we have the following.*

(1) *If D is semi-ample over $\pi^{-1}(S)$, then $\pi_*(D)$ is semi-ample over S.*
(2) *If D is weakly positive over $\pi^{-1}(S)$, then $\pi_*(D)$ is weakly positive over S.*

Proof. (1) By Lemma 1.1.1, there is an effective divisor E on X such that $E \sim_{\mathbb{Q}} D$ and $s' \notin \mathrm{Supp}(E)$ for all $s' \in \pi^{-1}(S)$. Then, $\pi_*(E) \sim_{\mathbb{Q}} \pi_*(D)$ and $s \notin \pi(\mathrm{Supp}(E)) = \mathrm{Supp}(\pi_*(E))$ for all $s \in S$.

(2) This is a consequence of (1). □

PROPOSITION 1.1.3 (char$(k) \geqslant 0$). *Let $\pi: X \to Y$ be a surjective, proper morphism of normal varieties. We assume that Y is $\mathbb{Q}$-factorial. Let D be a $\mathbb{Q}$-divisor on Y, and S a subset of Y. Then, we have the following.*

(1) *If D is semi-ample over S, then $\pi^*(D)$ is semi-ample over $f^{-1}(S)$.*
(2) *If D is weakly positive over S, then $\pi^*(D)$ is C-weakly positive over S.*

Proof. (1) Let s' be a point in $\pi^{-1}(S)$. Then, there is an effective $\mathbb{Q}$-divisor E on Y with $D \sim_{\mathbb{Q}} E$ and $\pi(s') \notin \mathrm{Supp}(E)$. Thus, $\pi^*(D) \sim_{\mathbb{Q}} \pi^*(E)$ and $s' \notin \mathrm{Supp}(\pi^*(E))$. Therefore, $\pi^*(D)$ is semiample over $\pi^{-1}(S)$.

(2) This is a consequence of (1). □

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LEMMA 1.1.4 (char$(k) \geqslant 0$). *Let X and Y be complete varieties, and let D and E be $\mathbb{Q}$-Cartier divisors on X and Y respectively. Let $p\colon X \times Y$ and $q\colon X \times Y \to Y$ be the projections to the first factor and the second factor, respectively. For $(x, y) \in X \times Y$, $p^*(D) + q^*(E)$ is nef at (x, y) if and only if D and E are nef at x and y respectively.*

Proof. First we assume that $p^*(D) + q^*(E)$ is nef at (x, y). Let C be a complete irreducible curve on X with $x \in C$. Then, $C_y = C \times \{y\}$ is a complete curve on $X \times Y$ with $(x, y) \in C_y$. Moreover, $(p^*(D) + q^*(E) \cdot C_y) = (D \cdot C)$. Thus, $(D \cdot C) \geqslant 0$, which says us that D is nef at x. In the same way, we can see that E is nef at y.

Next we assume that D and E are nef at x and y, respectively. In order to see that $p^*(D) + q^*(E)$ is nef at (x, y), it is sufficient to check that $(p^*(D) \cdot C) \geqslant 0$ and $(q^*(E) \cdot C) \geqslant 0$ for any complete irreducible curves C on $X \times Y$ with $(x, y) \in C$. Here, $p(C)$ is either $\{x\}$, or a complete irreducible curve passing through x. Thus, by virtue of the projection formula, $(p^*(D) \cdot C) \geqslant 0$. In the same way, $(q^*(E) \cdot C) \geqslant 0$. □

1.2. THE FIRST CHERN CLASS OF COHERENT SHEAVES

Let X be a normal variety, and F a coherent $\mathcal{O}_X$-module on X. Here we define $c_1(F) \in A^1(X)$ in the following way.

Case 1. F is a torsion sheaf. In this case, we set

$$D = \sum_{\substack{P \in X, \\ \mathrm{depth}(P)=1}} \mathrm{length}(F_P)\overline{\{P\}},$$

where $\overline{\{P\}}$ is the Zariski closure of $\{P\}$ in X. Then, $c_1(F)$ is defined by the class of D.

Case 2. F is a torsion free sheaf. Let r be the rank of F. Then, $(\bigwedge^r F)^{\vee\vee}$ is a reflexive sheaf of rank 1, where ${}^{\vee\vee}$ means the double dual of sheaves. Thus, we define $c_1(F)$ to be the class of $(\bigwedge^r F)^{\vee\vee}$.

Case 3. F is general. Let T be the torsion part of F. Then, $c_1(F) = c_1(T) + c_1(F/T)$.

Note that if $0 \to F_1 \to F_2 \to F_3 \to 0$ is an exact sequence of coherent $\mathcal{O}_X$-modules, then $c_1(F_2) = c_1(F_1) + c_1(F_3)$. Moreover, let L be a reflexive sheaf of rank 1 on X, and s a nonzero section of L. Then

$$c_1(L) = c_1(\mathrm{Coker}(\mathcal{O}_X \xrightarrow{\times s} L)) = \text{the class of } \mathrm{div}(s).$$

PROPOSITION 1.2.1 (char$(k) \geqslant 0$). *Let X be a normal algebraic variety, F a coherent $\mathcal{O}_X$-module, and x a point of X. If F is generated by global sections at x and F is free at x, then $c_1(F)$ is semi-ample at x.*

Proof. Let T be the torsion part of F. Then, $c_1(F) = c_1(F/T) + c_1(T)$. Here since F is free at x, $c_1(T)$ is semi-ample at x. Moreover, it is easy to see that F/T is generated by global sections at x. Therefore, to prove our proposition, we may assume that F is a torsion free sheaf.

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Let r be the rank of F and $\kappa(x)$ the residue field of x. Then, by our assumption, there are sections $s_1, \dots, s_r$ of F such that $\{s_i(x)\}$ forms a basis of $F \otimes \kappa(x)$. Since we can view s_i as an injection $s_i : \mathcal{O}_X \to F$, $s = s_1 \wedge \cdots \wedge s_r$ gives rise to an injection $s : \mathcal{O}_X \to \left(\bigwedge^r F\right)^{\vee\vee}$, which is bijective at x. Thus, $x \notin \mathrm{div}(s)$. □

1.3. THE DISCRIMINANT DIVISOR OF VECTOR BUNDLES

Let $f: X \to Y$ be a proper surjective morphism of algebraic varieties of the relative dimension one, and let E be a locally free sheaf on X. We define the *discriminant divisor of E with respect to f* to be

$$\mathrm{dis}_{X/Y}(E) = f_*(2\mathrm{rk}(E)c_2(E) - (\mathrm{rk}(E) - 1)c_1(E)^2).$$

LEMMA 1.3.1 ($\mathrm{char}(k) \geqslant 0$). *Let $f: X \to Y$ be a flat, surjective and projective morphism of varieties with* $\dim f = 1$. *Let E be a vector bundle of rank r on X. Then, we have the following.*

(1) $\mathrm{dis}_{X/Y}(E)$ *is a Cartier divisor.*

(2) *Let $u : Y' \to Y$ be a morphism of varieties, and let*

$$\begin{array}{ccc} X & \xleftarrow{u'} & X \times_Y Y' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ Y & \xleftarrow{u} & Y' \end{array}$$

be the induced diagram of the fiber product. If $X \times_Y Y'$ is integral, then $\mathrm{dis}_{X \times_Y Y'/Y'}(u'^*(E)) = u^*(\mathrm{dis}_{X/Y}(E))$.

Proof. (1) We set $F = \mathcal{E}nd(E)$. Let $p: P = \mathbb{P}(F) \to X$ be the projective bundle of F, and $\mathcal{O}_P(1)$ the tautological line bundle on P. Let $g: P \to Y$ be the composition of $P \xrightarrow{p} X \xrightarrow{f} Y$. Then, since

$$p_*(c_1(\mathcal{O}_P(1))^{r^2+1}) = -c_2(F) = -(2rc_2(E) - (r-1)c_1(E)^2),$$

we have $g_*(c_1(\mathcal{O}_P(1))^{r^2+1}) = -\mathrm{dis}_{X/Y}(E)$. Thus,

$$\mathrm{dis}_{X/Y}(E) = -c_1(\langle \mathcal{O}_P(1)^{\cdot r^2+1} \rangle (P/Y)),$$

where

$$\langle\, , \dots ,\, \rangle(P/Y) : \overbrace{\mathrm{Pic}(P) \times \cdots \times \mathrm{Pic}(P)}^{\dim g + 1} \to \mathrm{Pic}(Y)$$

is Deligne's pairing for the flat morphism $g: P \to Y$. Therefore, $\mathrm{dis}_{X/Y}(E)$ is a Cartier divisor.

(2) This follows from the compatibility of Deligne's pairing by base changes. □

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Remark 1.3.2. In (2) of Lemma 1.3.1, $X \times_Y Y'$ is integral if the generic fiber of $X \times_Y Y' \to Y'$ is integral by virtue of [12, Lemma 4.2].

1.4. THE MODULI SPACE OF T-POINTED STABLE CURVES OF GENUS g

Let g be a non-negative integer and T a finite set with $2g - 2 + |T| > 0$, where $|T|$ is the number of T. Recall that $[n] = \{1, \dots, n\}$ and $[0] = \emptyset$. Usually, we use $[n]$ as T. Let $\bar{M}_{g,T}$ (resp. $M_{g,T}$) denote the moduli space of T-pointed stable curves (resp. T-pointed smooth curves) of genus g, namely, $\bar{M}_{g,T}$ (resp. $M_{g,T}$) is the moduli space of $|T|$-pointed stable curves (resp. $|T|$-pointed smooth curves) of genus g, whose marked points are labeled by the index set T.

Roughly speaking, the $\mathbb{Q}$-line bundles λ and $\{\psi_t\}_{t \in T}$ on $\bar{M}_{g,T}$ are defined as follows: Let $\pi : \mathcal{C} \to \bar{M}_{g,T}$ be the universal curve of $\bar{M}_{g,T}$, and $s_t : \bar{M}_{g,T} \to \mathcal{C}$ ($t \in T$) the sections of π arising from the T-points of $\bar{M}_{g,T}$. Then, $\lambda = \det(\pi_*(\omega_{\mathcal{C}/\bar{M}_{g,T}}))$ and $\psi_t = s_t^*(\omega_{\mathcal{C}/\bar{M}_{g,T}})$ for $t \in T$.

For $x \in \bar{M}_{g,T}$, let denote C_x the nodal curve corresponding to x (here we forget the T-points). Let $S^l(\bar{M}_{g,T})$ be the set of all irreducible components of the closed set

$$\{x \in \bar{M}_{g,T} \mid \#(\mathrm{Sing}(C_x)) \geqslant l\}.$$

Then, every element of $S^l(\bar{M}_{g,T})$ is of codimension l, so that it is called an *l-codimensional stratum* of $\bar{M}_{g,T}$. Note that $\bar{M}_{g,T} \setminus M_{g,T}$ is a normal crossing divisor in the sense of orbifolds. Thus the normalization of an element of $S^l(\bar{M}_{g,T})$ is $\mathbb{Q}$-factorial. Moreover, we set

$$\bar{M}_{g,T}^{[l]} = \bar{M}_{g,T} \setminus \bigcup_{\Delta \in S^{l+1}(\bar{M}_{g,T})} \Delta,$$

i.e.,

$$\bar{M}_{g,T}^{[l]} = \{x \in \bar{M}_{g,T} \mid \#(\mathrm{Sing}(C_x)) \leqslant l\}.$$

Note that $\bar{M}_{g,T}^{[0]} = M_{g,T}$.

To describe the boundary of $\bar{M}_{g,T}$, we set

$$\Upsilon_{g,T} = \{(i, I) \mid i \in \mathbb{Z},\ 0 \leqslant i \leqslant g \text{ and } I \subseteq T\} \setminus (\{(0, \emptyset)\} \cup \{(0, \{t\})\}_{t \in T}),$$
$$\overline{\Upsilon}_{g,T} = \{\{(i, I), (j, J)\} \mid (i, I), (j, J) \in \Upsilon_{g,T}, i + j = g, I \cap J = \emptyset, I \cup J = T\}.$$

Then, the boundary $\Delta = \bar{M}_{g,T} \setminus M_{g,T}$ has the following irreducible decomposition:

$$\Delta = \Delta_{\mathrm{irr}} \cup \bigcup_{\{(i,I),(j,J)\} \in \overline{\Upsilon}_{g,T}} \Delta_{\{(i,I),(j,J)\}}.$$

A general point of Δ_{irr} represents a T-pointed irreducible stable curve with one node. A general point of $\Delta_{\{(i,I),(j,J)\}}$ represents a T-pointed stable curve consisting of an I-pointed smooth curve of genus i and a J-pointed smooth curve of genus j meeting transversally at one point.

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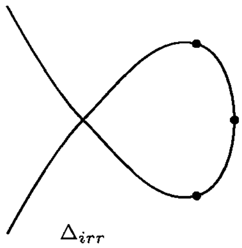
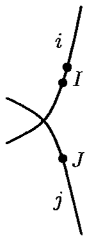


Δ_{irr}



$\Delta_{\{(i,I),(j,J)\}}$

Let δ_{irr} and $\delta_{\{(i,I),(j,J)\}}$ be the classes of Δ_{irr} and $\Delta_{\{(i,I),(j,J)\}}$ in $\text{Pic}(\bar{M}_{g,T}) \otimes \mathbb{Q}$ respectively. Strictly speaking, $\delta_{\text{irr}} = c_1(\mathcal{O}_{\bar{M}_{g,T}}(\Delta_{\text{irr}}))$ and

$$\delta_\upsilon = \begin{cases} \frac{1}{2} c_1(\mathcal{O}_{\bar{M}_{g,T}}(\Delta_\upsilon)), & \text{if} \quad \upsilon = \{\ (1,\emptyset),\ (\text{g}-1,\ \text{T})\ \}, \\ c_1(\mathcal{O}_{\bar{M}_{g,T}}(\Delta_\upsilon)), & \text{otherwise.} \end{cases}$$

In the case where $T = \emptyset$, we denote $\delta_{\{(i,\emptyset),(j,\emptyset)\}}$ by $\delta_{\{i,j\}}$ or $\delta_{\min\{i,j\}}$, i.e.,

$$\delta_i = \delta_{\{i,g-i\}} = \delta_{\{(i,\emptyset),(g-i,\emptyset)\}} \qquad (i = 1, \ldots, [g/2])$$

on $\bar{M}_g$.

Let $(Z; \{P_t\}_{t \in T})$ be a T-pointed stable curve of genus g over k. Let Q be a node of Z, and Z_Q the partial normalization of Z at Q. Then, the type of Q is defined as follows:

- The case where Z_Q is connected. Then, Q is of type 0.
- The case where Z_Q is not connected. Let Z_1 and Z_2 be two connected components of Z_Q. Let i (resp. j) be the arithmetic genus of Z_1 (resp. Z_2). Let $I = \{t \in T \mid P_t \in Z_1\}$ and $J = \{t \in T \mid P_t \in Z_2\}$. Then, we say Q is of type $\{(i, I), (j, J)\}$.

In the case where $T = \emptyset$, for simplicity, a node of type $\{(i, \emptyset), (j, \emptyset)\}$ is said to be of type i, where $i \leqslant j$.

Let Y be a normal variety, and let $f\colon X \to Y$ be a T-pointed stable curve of genus g over Y. Let Y_0 be the maximal open set over which f is smooth. Assume that $Y_0 \neq \emptyset$. For $x \in X$, we define $\text{mult}_x(X)$ to be $\text{length}_{\mathcal{O}_{X,x}}(\omega_{X/Y}/\Omega_{X/Y})$. If x is the generic point of a subvariety T, then we denote $\text{mult}_x(X)$ by $\text{mult}_T(X)$. If x is closed, Y is smooth at $f(x)$ and $Y \setminus Y_0$ is smooth at $f(x)$, then X is locally given by $\{xy = t^{\text{mult}_x(X)}\}$ around x, where t is a defining equation of $Y \setminus Y_0$ around $f(x)$. Thus, if Y is a curve, then the type of singularity at x is $A_{\text{mult}_x(X)-1}$.

Here, for $\upsilon \in \Upsilon_{g,T}$, let $S(X/Y)_\upsilon$ (resp. $S(X/Y)_{\text{irr}}$) be the set of irreducible components of $\text{Sing}(f)$ such that the type of s in $f^{-1}(f(s))$ for a general $s \in S(X/Y)_\upsilon$ (resp. $S(X/Y)_{\text{irr}}$) is υ (resp. 0). We set

$$\delta_\upsilon(X/Y) = \sum_{S \in S(X/Y)_\upsilon} \text{mult}_S(X) f_*(S)$$

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and

$$\delta_{\mathrm{irr}}(X/Y) = \sum_{S \in S(X/Y)_{\mathrm{irr}}} \mathrm{mult}_S(X) f_*(S).$$

Then, δ_{irr} and δ_υ are normalized to guarantee the following formula:

$$\delta_{\mathrm{irr}}(X/Y) = \varphi^*(\delta_{\mathrm{irr}}) \quad \text{and} \quad \delta_\upsilon(X/Y) = \varphi^*(\delta_\upsilon)$$

in $A^1(Y) \otimes \mathbb{Q}$, where $\varphi : Y \to \bar{M}_{g,T}$ is the induced morphism by $X \to Y$.

1.5. THE CLUTCHING MAPS

Here let us consider the clutching maps and their properties.

Let $\pi : X \to Y$ be a prestable curve, i.e., $\pi : X \to Y$ is a flat and proper morphism such that the geometric fibers of π are reduced curves with at most ordinary double points. We don't assume the connectedness of fibers. Let $s_1, s_2 : Y \to X$ be two non-crossing sections such that π is smooth at points $s_1(y)$ and $s_2(y)$ $(\forall y \in Y)$. Then, by virtue of [9, Theorem 3.4], we have the clutching diagram:

$$\begin{tikzcd} X \arrow[rr, "p"] \arrow[dr, "\pi"] & & X' \arrow[dl, "\pi'"'] \\ & Y \arrow[ul, bend left, "s_1"] \arrow[ul, bend left=60, "s_2"] \arrow[ur, bend right, "s"'] & \end{tikzcd}$$

Roughly speaking, X' is a prestable curve over Y obtained by identifying $s_1(Y)$ with $s_2(Y)$, and s is a section of $X' \to Y$ with $p \cdot s_1 = p \cdot s_2 = s$. For details, see [9, Theorem 3.4].

We assume that $\pi' : X' \to Y$ is a T-pointed stable curve of genus g, and s is one of sections of $\pi' : X' \to Y$ arising from T-points of $\pi' : X' \to Y$. Let $\varphi : Y \to \bar{M}_{g,T}$ be the induced morphism. Here we set $\Lambda = \det(R\pi_*(\omega_{X/Y}))$, $\Delta = \det(R\pi_*(\omega_{X/Y}/\Omega_{X/Y}))$ and $\Psi = s_1^*(\omega_{X/Y}) \otimes s_2^*(\omega_{X/Y})$. Then, we have the following.

PROPOSITION 1.5.1. *For simplicity, the divisor* δ_{irr} *on* $\bar{M}_{g,T}$ *is denoted by* δ_0.

(1) $\varphi^*(\lambda) = \Lambda$ *and* $\varphi^*(\delta) = -\Psi + \Delta$, *where* $\delta = \delta_0 + \sum_{\upsilon \in \overline{\Upsilon}_{g,T}} \delta_\upsilon$.

(2) *We assume that* $\pi(\mathrm{Sing}(\pi)) \neq Y$ *and every geometric fiber of* π *has one node at most. Let*

$$\Delta = \Delta_0 + \sum_{\upsilon \in \overline{\Upsilon}_{g,T}} \Delta_\upsilon$$

be the decomposition such that the node of $\pi^{-1}(x)$ $(x \in (\Delta_t)_{\mathrm{red}})$ *gives rise to a node of type* t *in* $\pi'^{-1}(x)$. *Moreover, let* a *be the type of* $s(y)$ *in* $\pi'^{-1}(y)$ $(y \in Y)$. *Then,*

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$$\varphi^*(\delta_t) = \begin{cases} -\Psi + \Delta_a & \textit{if } t = a, \\ \Delta_t & \textit{if } t \neq a. \end{cases}$$

Proof. (1) Since $\det(R\pi'_*(\omega_{X'/Y})) = \det(R\pi_*(\omega_{X/Y}))$, the first statement is obvious. Thus, we can see that

$$\begin{aligned} \varphi^*(\delta) &= \det(R\pi'_*(\omega_{X'/Y}/\Omega_{X'/Y})) \\ &= \det(R\pi'_*(\omega_{X'/Y})) - \det(R\pi'_*(\Omega_{X'/Y})) \\ &= \Lambda - \det(R\pi'_*(\Omega_{X'/Y})). \end{aligned}$$

On the other hand, by [9, Theorem 3.5], there is an exact sequence

$$0 \to s^*(\Psi) \to \Omega_{X'/Y} \to p_*(\Omega_{X/Y}) \to 0.$$

Therefore, we get (1).

(2) This is a consequence of (1). □

As a corollary, we have the following.

COROLLARY 1.5.2. (1) *Let a and b be nonnegative integers, and T and S non-empty finite sets with $T \cap S = \emptyset$, $2a - 2 + |T| > 0$ and $2b - 2 + |S| > 0$. Let us fix $s \in S$ and $t \in T$, and set $T' = T \setminus \{t\}$ and $S' = S \setminus \{s\}$. Let α: $\bar{M}_{a,T} \times \bar{M}_{b,B} \to \bar{M}_{a+b,T' \cup S'}$ be the clutching map, and p: $\bar{M}_{a,T} \times \bar{M}_{b,S} \to \bar{M}_{a,T}$ and q: $\bar{M}_{a,T} \times \bar{M}_{b,S} \to \bar{M}_{b,S}$ the projection to the first factor and the projection to the second factor respectively. We set divisors $D \in \mathrm{Pic}(\bar{M}_{a+b,T' \cup S'}) \otimes \mathbb{Q}$, $E \in \mathrm{Pic}(\bar{M}_{a,T}) \otimes \mathbb{Q}$ and $F \in \mathrm{Pic}(\bar{M}_{b,S}) \otimes \mathbb{Q}$ as follows:*

$$D = c\lambda + \sum_{l \in T' \cup S'} d_l \psi_l + e_{\mathrm{irr}} \delta_{\mathrm{irr}} + \sum_{\{(i,I),(j,J)\} \in \bar{\Upsilon}_{a+b,T' \cup S'}} e_{\{(i,I),(j,J)\}} \delta_{\{(i,I),(j,J)\}},$$

$$\begin{aligned} E = c\lambda - e_{\{(a,T'),(b,S')\}} \psi_t + \sum_{l \in T'} d_l \psi_l + e_{\mathrm{irr}} \delta_{\mathrm{irr}} + \\ + \sum_{\substack{\{(i',I'),(j',J')\} \in \bar{\Upsilon}_{a,T} \\ t \in J'}} e_{\{(i',I'),(j'+b,J' \cup S' \setminus \{t\})\}} \delta_{\{(i',I'),(j',J')\}}, \end{aligned}$$

$$\begin{aligned} F = c\lambda - e_{\{(a,T'),(b,S')\}} \psi_s + \sum_{l \in S'} d_l \psi_l + e_{\mathrm{irr}} \delta_{\mathrm{irr}} + \\ + \sum_{\substack{\{(i'',I''),(j'',J'')\} \in \bar{\Upsilon}_{b,S} \\ s \in J''}} e_{\{(i'',I''),(j''+a,J'' \cup T' \setminus \{s\})\}} \delta_{\{(i'',I''),(j'',J'')\}}. \end{aligned}$$

Then $\alpha^*(D) = p^*(E) + q^*(F)$.

(2) *Let g be a nonnegative integer and T a finite set with $|T| \geqslant 2$ and $2g - 2 + |T| > 0$. Let us fix two elements $t, t' \in T$, and set $T' = T \setminus \{t, t'\}$. Let*

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$\beta\colon \bar{M}_{g,T} \to \bar{M}_{g+1,T'}$ *be the clutching map. We set* $D \in \mathrm{Pic}(\bar{M}_{g+1,T'}) \otimes \mathbb{Q}$ *and* $E \in \mathrm{Pic}(\bar{M}_{g,T}) \otimes \mathbb{Q}$ *as follows:*

$$D = c\lambda + \sum_{l \in T'} d_l \psi_l + e_{\mathrm{irr}}\delta_{\mathrm{irr}} + \sum_{\{(i,I),(j,J)\} \in \overline{\Upsilon}_{g+1,T'}} e_{\{(i,I),(j,J)\}}\delta_{\{(i,I),(j,J)\}},$$

$$E = c\lambda - e_{\mathrm{irr}}(\psi_t + \psi_{t'}) + \sum_{l \in T'} d_l\psi_l + e_{\mathrm{irr}}\delta_{\mathrm{irr}} + \sum_{\substack{\{(i',I'),(j',J')\} \in \overline{\Upsilon}_{g,T} \\ t \in I', t' \in J'}} e_{\mathrm{irr}}\delta_{\{(i',I'),(j',J')\}} +$$
$$+ \sum_{\substack{\{(i',I'),(j',J')\} \in \overline{\Upsilon}_{g,T} \\ t,t' \in J'}} e_{\{(i',I'),(j'+1,J' \setminus \{t,t'\})\}}\delta_{\{(i',I'),(j',J')\}}.$$

Then $\beta^*(D) = E$.

Proof. In the following, for $x \in \bar{M}_{*,*}$, we denote by C_x the corresponding nodal curve to x.

(1) If $C_{\alpha(x,y)}$ has two nodes, then we denote by $ty(x,y)$ the type of the node different from the node arising from the clutching map. Then,

$$ty(x,y) = \begin{cases} \{(i',I'),(j'+b, J' \cup S' \setminus \{t\})\}, & \text{if } x \in \Delta_{\{(i',I'),(j',J')\}} \cap \bar{M}^{[1]}_{a,T}, y \in M_{b,S} \text{ and } t \in J', \\ \{(i'',I''),(j''+a, J'' \cup T' \setminus \{s\})\}, & \text{if } x \in M_{a,T}, y \in \Delta_{\{(i'',I''),(j'',J'')\}} \cap \bar{M}^{[1]}_{b,S} \text{ and } s \in J''. \end{cases}$$

Thus, we get (1) by the above proposition.

(2) In the same way as above, if $C_{\beta(x)}$ has two nodes, then we denote by $ty'(x)$ the type of the node different from the node arising from the clutching map. Then,

$$ty'(x) = \begin{cases} 0, & \text{if } x \in \left(\Delta_{\mathrm{irr}} \cup \bigcup_{t \in I', t' \in J'} \Delta_{\{(i',I'),(j',J')\}}\right) \cap \bar{M}^{[1]}_{g,T}, \\ \{(i',I'),(j'+1, J' \setminus \{t,t'\})\}, & \text{if } x \in \Delta_{\{(i',I'),(j',J')\}} \cap \bar{M}^{[1]}_{g,T} \\ & \text{and } t, t' \in J', \end{cases}$$

which implies (2) by the above proposition. □

2. A Generalization of Relative Bogomolov's Inequality

Let $f\colon X \to Y$ be a projective morphism of quasi-projective varieties of the relative dimension one, and let E be a locally free sheaf on X. Let us fix a point $y \in Y$. Assume that f is smooth over y and $E|_{f^{-1}(y)}$ is strongly semistable. In the paper [11], we proved that $\mathrm{dis}_{X/Y}(E)$ is weakly positive at y under the assumption that Y is smooth. In this section, we generalize it to the case where Y is normal.

PROPOSITION 2.1 (char$(k) \geqslant 0$). *Let X and Y be algebraic varieties, and $f\colon X \to Y$ a surjective and projective morphism of* $\dim f = d$. *Let L and A be line bundles on X. If Y is normal, then there are $\mathbb{Q}$-divisors $Z_0, \ldots, Z_d$ on Y such that*

$$c_1\left(Rf_*(L^{\otimes n} \otimes A)\right) \sim_{\mathbb{Q}} \frac{f_*\left(c_1(L)^{d+1}\right)}{(d+1)!} n^{d+1} + \sum_{i=0}^{d} Z_i n^i$$

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for all $n > 0$.

Proof. We set

$$Y^0 = Y \setminus \mathrm{Sing}(Y), \quad X^0 = f^{-1}(Y^0) \quad \text{and} \quad f^0 = f\big|_{X^0}.$$

Then, we have

$$c_1(Rf^0_*((L^{\otimes n} \otimes A)\big|_{X^0})) = c_1(Rf_*(L^{\otimes n} \otimes A))\big|_{Y^0}$$

and

$$f^0_*(c_1(L|_{X^0})^{d+1}) = f_*(c_1(L)^{d+1})\Big|_{Y^0}.$$

Thus, by virtue of [11, Lemma 2.3], there are $\mathbb{Q}$-divisors $Z^0_0, \ldots, Z^0_d$ on Y^0 such that

$$c_1(Rf_*(L^{\otimes n} \otimes A))\big|_{Y^0} \sim_{\mathbb{Q}} \frac{f_*(c_1(L)^{d+1})\Big|_{Y^0}}{(d+1)!} n^{d+1} + \sum_{i=0}^{d} Z^0_i n^i$$

for all $n > 0$. Let Z_i be the Zariski closure of Z^0_i in Y. Then, since $\mathrm{codim}(\mathrm{Sing}(Y)) \geqslant 2$,

$$c_1(Rf_*(L^{\otimes n} \otimes A)) \sim_{\mathbb{Q}} \frac{f_*(c_1(L)^{d+1})}{(d+1)!} n^{d+1} + \sum_{i=0}^{d} Z_i n^i$$

for all $n > 0$. □

THEOREM 2.2. ($\mathrm{char}(k) \geqslant 0$). *Let X be a quasi-projective variety, Y a normal quasi-projective variety, and $f\colon X \to Y$ a surjective and projective morphism of* $\dim f = 1$. *Let F be a locally free sheaf on X with $f_*(c_1(F)) = 0$, and S a finite subset of Y. We assume that f is flat over any points of S, and that, for all $s \in S$, there are line bundles $L_{\bar{s}}$ and $M_{\bar{s}}$ on the geometric fiber $X_{\bar{s}}$ over s such that*

$$H^0(X_{\bar{s}}, \mathrm{Sym}^m(F_{\bar{s}}) \otimes L_{\bar{s}}) = H^1(X_{\bar{s}}, \mathrm{Sym}^m(F_{\bar{s}}) \otimes M_{\bar{s}}) = 0$$

for $m \gg 0$. *Then, $f_*\big(c_2(F) - c_1(F)^2\big)$ is weakly positive over S.*

Proof. The proof of this theorem is exactly the same as [11, Theorem 2.4] using Proposition 2.1, Proposition 1.2.1 and [11, Proposition 2.2]. For reader's convenience, we give the sketch of the proof of it.

Let A be a very ample line bundle on X such that $A_{\bar{s}} \otimes L_{\bar{s}}$ and $A_{\bar{s}} \otimes M_{\bar{s}}^{\otimes -1}$ are very ample on $X_{\bar{s}}$ for all $s \in S$. Then, we can see the following claim in the same way as in [11, Claim 2.4.1]

CLAIM 2.2.1. $H^0(X_s, \mathrm{Sym}^m(F_s) \otimes A_s^{\otimes -1}) = H^1(X_s, \mathrm{Sym}^m(F_s) \otimes A_s) = 0$ *for all* $s \in S$ *and* $m \gg 0$.

Since X is an integral scheme of dimension greater than or equal to 2, and X_s ($s \in S$) is a one-dimensional scheme over $\kappa(s)$, there is $B \in |A^{\otimes 2}|$ such that B is

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integral, and that $B \cap X_s$ is finite for all $s \in S$, i.e., B is finite over any points of S. Let π: $B \to Y$ be the morphism induced by f. Let H be an ample line bundle on Y such that $\pi_*(F_B) \otimes H$ and $\pi_*(A_B) \otimes H$ are generated by global sections at any points of S, where $F_B = F|_B$ and $A_B = A|_B$.

By using Proposition 2.1, there are $\mathbb{Q}$-divisors $Z_0, \dots, Z_r$ on Y such that

$$\sum_{i \geqslant 0}(-1)^i c_1(R^i f_*(\mathrm{Sym}^m(F \otimes f^*(H)) \otimes A^{\otimes -1} \otimes f^*(H)))$$
$$\sim_{\mathbb{Q}} -\frac{1}{(r+1)!} f_*(c_2(F) - c_1(F)^2) m^{r+1} + \sum_{i=0}^{r} Z_i m^i$$

in the same way as in the proof of [11, Theorem 2.4]. The following claim can also be proved in the same way as in [11, Claim 2.4.2].

CLAIM 2.2.2. *If* $m \gg 0$, *then we have the following.*

(a) $c_1(R^i f_*(\mathrm{Sym}^m(F \otimes f^*(H)) \otimes A^{\otimes -1} \otimes f^*(H))) = 0$ *for all* $i \geqslant 2$.
(b) $f_*(\mathrm{Sym}^m(F \otimes f^*(H)) \otimes A^{\otimes -1} \otimes f^*(H)) = 0$.
(c) $R^1 f_*\big(\mathrm{Sym}^m(F \otimes f^*(H)) \otimes A^{\otimes -1} \otimes f^*(H)\big)$ *is free at any points of* S.
(d) $R^1 f_*(\mathrm{Sym}^m(F \otimes f^*(H)) \otimes A \otimes f^*(H)) = 0$ *around any points of* S.

By (a) and (b) of Claim 2.2.2,

$$\frac{f_*(c_2(F) - c_1(F)^2)}{(r+1)!} \sim_{\mathbb{Q}} \frac{c_1(R^1 f_*(\mathrm{Sym}^m(F \otimes f^*(H)) \otimes A^{\otimes -1} \otimes f^*(H)))}{m^{r+1}} + \sum_{i=0}^{r} \frac{Z_i}{m^{r+1-i}}.$$

Hence, it is sufficient to show that

$$c_1(R^1 f_*(\mathrm{Sym}^m(F \otimes f^*(H)) \otimes A^{\otimes -1} \otimes f^*(H)))$$

is semi-ample over S. This can be proved in the same way as in the proof of [11, Theorem 2.4] by using [11, Proposition 2.2], Claim 2.2.2 and Proposition 1.2.1. □

Let C be a smooth projective curve and E a vector bundle on C. We say E is *strongly semistable* if, for any finite morphisms $\phi: C' \to C$ of smooth projective curves, $\phi^*(E)$ is semistable. Note that if $\mathrm{char}(k) = 0$ and E is semistable, then E is strongly semistable. As a corollary, we have the following, which can be proved in the exactly same way as [11, Corollary 2.5].

COROLLARY 2.3 ($\mathrm{char}(k) \geqslant 0$). *Let* X *be a quasi-projective variety,* Y *a normal quasi-projective variety, and* $f: X \to Y$ *a surjective and projective morphism of* $\dim f = 1$. *Let* E *be a locally free sheaf on* X *and* S *a finite subset of* Y. *If, for all* $s \in S$, f *is flat over* s, *the geometric fiber* $X_{\bar{s}}$ *over* s *is reduced and Gorenstein, and* E *is strongly semistable on each connected component of the normalization of* $X_{\bar{s}}$, *then* $\mathrm{dis}_{X/Y}(E)$ *is weakly positive over* S.

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Remark 2.4. (char$(k) = 0$). In [11], we proved that the divisor

$$(8g+4)\lambda - g\delta_{\mathrm{irr}} - \sum_{i=1}^{[g/2]} 4i(g-i)\delta_i$$

on $\bar{M}_g$ is weakly positive over any finite subsets of M_g. Here we give an alternative proof of this inequality.

Fix a polynomial $P_g(m) = (6m-1)(g-1)$. Let $H_g \subset \mathrm{Hilb}^{P_g}_{\mathbb{P}^{5g-6}}$ be a subscheme of all tricanonically embedded stable curves, $Z_g \subset H_g \times \mathbb{P}^{5g-6}$ the universal tricanonically embedded stable curves, and $f_g\colon Z_g \to H_g$ the natural projection. Then, $G = \mathrm{PGL}\,(5g-5)$ acts on Z_g and H_g, and f_g is a G-morphism. Let $\phi\colon H_g \to \bar{M}_g$ be the natural morphism of the geometric quotient. Then, by Seshadri's theorem [13, Theorem 6.1], there is a finite morphism $h: Y \to \bar{M}_g$ of normal varieties with the following properties. Let W_g be the normalization of $H_g \times_{\bar{M}_g} Y$, and let $\pi\colon W_g \to H_g$ and $\phi'\colon W_g \to Y$ be the induced morphisms by the projections of $H_g \times_{\bar{M}_g} Y \to H_g$ and $H_g \times_{\bar{M}_g} Y \to Y$ respectively. Then, we have the following.

(1) G acts on W_g, and π is a G-morphism.
(2) $\phi'\colon W_g \to Y$ is a principal G-bundle.

Thus, $f'_g\colon U_g = Z_g \times_{H_g} W_g \to W_g$ is a stable curve, G acts on U_g and f'_g is a G-morphism. Since $\phi'\colon W_g \to Y$ is a principal G-bundle, we can easily see that U_g is also a principal G-bundle and the geometric quotient $X = U_g/G$ gives rise to a stable curve $f\colon X \to Y$ over Y. Moreover, $U_g = W_g \times_Y X$. Then, we have the following commutative diagram:

$$\begin{tikzcd}
 & Z_g \arrow[dl, "f_g"'] & & U_g \arrow[ll, "\pi'"'] \arrow[dl, "f'_g"'] \arrow[dd, "\phi''"] \\
H_g \arrow[dd, "\phi"'] & & W_g \arrow[ll, "\pi"] \arrow[dd, "\phi'"'] & \\
 & & & X \arrow[dl, "f"] \\
\bar{M}_g & & Y \arrow[ll, "h"] &
\end{tikzcd}$$

Let Δ be the minimal closed subset of H_g such that f_g is not smooth over a point of Δ. Then, by [2, Theorem (1.6) and Corollary (1.9)], Z_g and H_g are quasi-projective and smooth, and Δ is a divisor with only normal crossings. Let $\Delta = \Delta_{\mathrm{irr}} \cup \Delta_1 \cup \cdots \cup \Delta_{[g/2]}$ be the irreducible decomposition of Δ such that, if $x \in \Delta_i \setminus \mathrm{Sing}(\Delta)$ (resp. $x \in \Delta_{\mathrm{irr}} \setminus \mathrm{Sing}(\Delta)$), then $f_g^{-1}(x)$ is a stable curve with one node of type i (resp. irreducible stable curve with one node).

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Form now on, we consider everything over $\bar{M}_g^{[1]}$. (Recall that $\bar{M}_g^{[1]}$ is the set of stable curves with one node at most.) In the following, the superscript '0' means the objects over $\bar{M}_g^{[1]}$.

In [10, § 3], we constructed a locally sheaf F on Z_g^0 with the following properties.

(a) F is invariant by the action of G.

(b) For each $y \in H_g^0 \setminus (\Delta_1 \cup \cdots \cup \Delta_{[g/2]})$,

$$F|_{f_g^{-1}(y)} = \mathrm{Ker}(H^0(\omega_{f_g^{-1}(y)}) \otimes \mathcal{O}_{f_g^{-1}(y)} \to \omega_{f_g^{-1}(y)}),$$

which is semistable on $f_g^{-1}(y)$.

(c) $\mathrm{dis}_{Z_g^0/H_g^0}(F) = (8g+4)\det(\pi_*(\omega_{Z_g^0/H_g^0})) - g\Delta_{\mathrm{irr}}^0 - \sum_{i=1}^{[\frac{g}{2}]} 4i(g-i)\Delta_i^0$.

Then, $\pi'^*(F)$ is a G-invariant locally free sheaf on U_g^0, so that $\pi'^*(F)$ can be descended to X^0 because $U_g \to X$ is a principal G-bundle. Namely, there is a locally free sheaf F' on X^0 such that $\phi''^*(F') = \pi'^*(F)$. Therefore, by Lemma 1.3.1, $\phi'^*(\mathrm{dis}_{X^0/Y^0}(F')) = \pi^*(\mathrm{dis}_{Z_g^0/H_g^0}(F))$. On the other hand, if we set

$$D = (8g+4)\lambda - g\delta_{\mathrm{irr}} - \sum_{i=1}^{[\frac{g}{2}]} 4i(g-i)\delta_i,$$

then $\phi^*(D^0) = \mathrm{dis}_{Z_g^0/H_g^0}(F)$. Therefore, we get $\phi'^*(h^*(D^0)) = \phi'^*(\mathrm{dis}_{X^0/Y^0}(F'))$, which implies that $h^*(D^0) = \mathrm{dis}_{X^0/Y^0}(F')$ because $\mathrm{Pic}(W_g)^G = \mathrm{Pic}(Y)$. Moreover, by Corollary 2.3, $\mathrm{dis}_{X^0/Y^0}(F')$ is weakly positive over any finite subsets of $h^{-1}(M_g)$. Thus, $h_*(\mathrm{dis}_{X^0/Y^0}(F')) = \deg(h)D^0$ is weakly positive over any finite subsets of M_g by (2) of Proposition 1.1.2. Hence, D is weakly positive over any finite subsets of M_g because $\mathrm{codim}(\bar{M}_g \setminus \bar{M}_g^{[1]}) \geqslant 2$.

3. A Certain Kind of Hyperelliptic Fibrations

We say $f: X \to Y$ is a hyperelliptic fibered surface of genus g if X is a smooth projective surface, Y is a smooth projective curve, the generic fiber of f is a smooth hyperelliptic curve of genus g. Let Y_0 be the maximal open set of Y such that f is smooth over Y_0. Then, the hyperelliptic involution of the generic fiber extends to an automorphism of $X_0 = f^{-1}(Y_0)$ over Y_0. We denote this automorphism by j. Clearly, the order of j is 2, namely, $j \neq \mathrm{id}_{X_0}$ and $j^2 = \mathrm{id}_{X_0}$. Let Γ be a section of $f: X \to Y$ and $\Gamma_0 = \Gamma \cap X_0$. By abuse of notation, we denote by $j(\Gamma)$ the Zariski closure of $j(\Gamma_0)$. The purpose of this section is to show the existence of a special kind of hyperelliptic fibered surfaces as described in the following propositions.

PROPOSITION 3.1 ($\mathrm{char}(k) = 0$). *For fixed integers g and i with $g \geqslant 2$ and $0 \leqslant i \leqslant g-1$, there is a hyperelliptic fibered surface $f: X \to Y$ of genus g, and a section Γ of f such that*

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(1) $\mathrm{Sing}(f) \neq \emptyset$, $j(\Gamma) = \Gamma$,
(2) *every singular fiber of f is a reduced curve consisting of a smooth projective curve of genus i and a smooth projective curve of genus $g - i$ meeting transversally at one point, and that*
(3) Γ *intersects with the component of genus $g - i$ on every singular fiber.*

PROPOSITION 3.2 ($\mathrm{char}(k) = 0$). *For fixed integers g and i with $g \geqslant 2$ and $0 \leqslant i \leqslant g$, there is a hyperelliptic fibered surface $f\colon X \to Y$ of genus g, and a section Γ of f such that*

(1) $\mathrm{Sing}(f) \neq \emptyset$, $j(\Gamma) \cap \Gamma = \emptyset$,
(2) *every singular fiber of f is a reduced curve consisting of a smooth projective curve of genus i and a smooth projective curve of genus $g - i$ meeting transversally at one point, and that*
(3) Γ *intersects with the component of genus $g - i$ on every singular fiber.*

PROPOSITION 3.3 ($\mathrm{char}(k) = 0$). *For fixed integers g and i with $g \geqslant 2$ and $0 \leqslant i \leqslant g - 1$, there is a hyperelliptic fibered surface $f : X \to Y$ of genus g, and a section Γ of f such that*

(1) $\mathrm{Sing}(f) \neq \emptyset$, $j(\Gamma) = \Gamma$,
(2) *every singular fiber of f is a reduced curve consisting of a smooth projective curve of genus i and a smooth projective curve of genus $g - i - 1$ meeting transversally at two points, and that*
(3) Γ *intersects with the component of genus $g - i - 1$ on every singular fiber.*

PROPOSITION 3.4 ($\mathrm{char}(k) = 0$). *For fixed integers g and i with $g \geqslant 2$ and $0 \leqslant i \leqslant g - 1$, there is a hyperelliptic fibered surface $f : X \to Y$ of genus g, and a section Γ of f such that*

(1) $\mathrm{Sing}(f) \neq \emptyset$, $j(\Gamma) \cap \Gamma = \emptyset$,
(2) *every singular fiber of f is a reduced curve consisting of a smooth projective curve of genus i and a smooth projective curve of genus $g - i - 1$ meeting transversally at two points, and that*
(3) Γ *intersects with the component of genus $g - i - 1$ on every singular fiber.*

PROPOSITION 3.5 ($\mathrm{char}(k) = 0$). *For fixed integers g and i with $g \geqslant 2$ and $1 \leqslant i \leqslant g - 1$, there is a hyperelliptic fibered surface $f\colon X \to Y$ of genus g, and non-crossing sections Γ_1 and Γ_2 of f such that*

(1) $\mathrm{Sing}(f) \neq \emptyset$, $j(\Gamma_1) = \Gamma_1$, $j(\Gamma_2) = \Gamma_2$,
(2) *every singular fiber of f is a reduced curve consisting of a smooth projective curve of genus i and a smooth projective curve of genus $g - i$ meeting transversally at one point,*

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(3) Γ_1 *and* Γ_2 *gives rise to a* 2-*pointed stable curve* $(f\colon X \to Y, \Gamma_1, \Gamma_2)$, *and that*
(4) *the type of* x *in* $f^{-1}(f(x))$ *as* 2-*pointed stable curve is* $\{(i, \{1\}), (g-i, \{2\})\}$ *for all* $x \in \mathrm{Sing}(f)$.

Let us begin with the following lemma.

LEMMA 3.6 ($\mathrm{char}(k) = 0$). *For nonnegative integers* a_1 *and* a_2, *there are a morphism* $f_1 : X_1 \to Y_1$ *of smooth projective varieties, an effective divisor* D_1 *on* X_1, *a line bundle* L_1 *on* X_1, *a line bundle* M_1 *on* Y_1, *and noncrossing sections* Γ_1 *and* Γ_2 *of* $f_1 : X_1 \to Y_1$ *with the following properties.*

(1) $\dim X_1 = 2$ *and* $\dim Y_1 = 1$.
(2) *Let* Σ_1 *be the set of all critical values of* f_1, *i.e.,* $P \in \Sigma_1$ *if and only if* $f_1^{-1}(P)$ *is a singular variety. Then, for any* $P \in Y_1 \setminus \Sigma_1$, $f_1^{-1}(P)$ *is a smooth rational curve.*
(3) $\Sigma_1 \neq \emptyset$, *and for any* $P \in \Sigma_1$, $f_1^{-1}(P)$ *is a reduced curve consisting of two smooth rational curves* E_P^1 *and* E_P^2 *joined at one point transversally.*
(4) D_1 *is smooth and* $f_1\big|_{D_1} : D_1 \to Y_1$ *is etale.*
(5) $(E_P^1 \cdot D_1) = a_1 + 1$ *and* $(E_P^2 \cdot D_1) = a_2 + 1$ *for any* $P \in \Sigma_1$. *Moreover,* D_1 *does not pass through* $E_P^1 \cap E_P^2$.
(6) *There is a map* $\sigma\colon \Sigma_1 \to \{1, 2\}$ *such that*

$$D_1 \in \left| L_1^{\otimes a_1 + a_2 + 2} \otimes f_1^*(M_1) \otimes \mathcal{O}_{X_1}\left(-\sum_{P \in \Sigma_1} (a_{\sigma(P)} + 1) E_P^{\sigma(P)} \right) \right|.$$

(7) $\deg(M_1)$ *is divisible by* $(a_1 + 1)(a_2 + 1)$.
(8)

$$\Gamma_1 \in \left| L_1 \otimes \mathcal{O}_{X_1}\left(-\sum_{\substack{P \in \Sigma_1 \\ \sigma(P) = 1}} E_P^1 \right) \right| \quad \textit{and} \quad \Gamma_2 \in \left| L_1 \otimes \mathcal{O}_{X_1}\left(-\sum_{\substack{P \in \Sigma_1 \\ \sigma(P) = 2}} E_P^2 \right) \right|.$$

Moreover,

$$(D_1 \cdot \Gamma_1) = (D_1 \cdot \Gamma_2) = 0 \quad \textit{and} \quad (E_P^i \cdot \Gamma_j) = \begin{cases} 0 & \textit{if } i \neq j, \\ 1 & \textit{if } i = j. \end{cases}$$

Proof. We can prove this lemma in the exactly same way as in [11, Lemma A.1] with a slight effort. We use the notation in [11, Lemma A.1]. Let F_1 and F_2 be curves in $\mathbb{P}^1_{(X,Y)} \times \mathbb{P}^1_{(S,T)}$ defined by $\{X = 0\}$ and $\{X = Y\}$ respectively. Note that

$$F_1 = p^{-1}((0:1)), \quad F_2 = p^{-1}((1:1)),$$

$$D'' = p^{-1}((1:0)), \qquad (D' \cdot F_1) = (D' \cdot F_2) = 1,$$

$$D' \cap F_1 = \{Q_1\} \quad \text{and} \quad D' \cap F_2 = \{Q_2\}.$$

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Then, since

$$u_1^*(F_1) = p_1^{-1}((0:1)) \quad \text{and} \quad u_1^*(F_2) = p_1^{-1}((1:1)),$$

in [11, Claim A.1.3], we can see that each tangent of $u_1^*(D)$ at $Q_{i,j}$ $(i = 1, 2)$ is different from $u_1^*(F_i)$.

Let $\overline{\Gamma}_i$ be the strict transform of $u_1^*(F_i)$ by $\mu_1\colon Z_1 \to \mathbb{P}^1_{(X,Y)} \times Y$. Then,

$$\overline{\Gamma}_i \in \left| \mu_1^*(p_1^*(\mathcal{O}_{\mathbb{P}^1}(1))) \otimes \mathcal{O}_{Z_1}\left(-\sum_j E_{i,j} \right) \right|.$$

Thus, if we set $\Gamma_i = (v_1)_*(\overline{\Gamma}_i)$, then we get our lemma. □

In the following proofs, we use the notation in [11, Proposition A.2 and Proposition A.3].

The proof of Proposition 3.1. We apply Lemma 3.6 to the case where $a_1 = 2i$ and $a_2 = 2g - 2i - 1$. We replace D_2 by $D_2 + \Gamma_2$ and a_2 by $a_2 + 1$. Then, (4), (5) and (6) hold for the new D_2 and a_2. Thus, we can construct $f\colon X \to Y$ in exactly same way as in [11, Proposition A.2]. Since $u_2^*(\Gamma_2)$ is the ramification locus of μ_3, $\overline{\Gamma} = h^*(u_2^*(\Gamma_2))_{\mathrm{red}}$ is a section of f_3. Thus, if we set $\Gamma = v_3(\overline{\Gamma})$, then we have our desired example.

The proof of Proposition 3.2. Applying Lemma 3.6 to the case where $a_1 = 2i$ and $a_2 = 2g - 2i$, we can construct $f\colon X \to Y$ in exactly same way as in [11, Proposition A.2]. Here let us consider $u_2^*(\Gamma_2)$. Then, $u_2^*(\Gamma_2)$ is a section of f_2 such that $u_2^*(\Gamma_2) \cap (D_2 + B) = \emptyset$, $(u_2^*(\Gamma_2) \cdot \overline{E}_Q^1) = 0$ and $(u_2^*(\Gamma_2) \cdot \overline{E}_Q^2) = 1$ for all $Q \in \Sigma_2$. Here we set $\Gamma' = v_3(\mu_3^*(u_2^*(\Gamma_2)))$. Then, since $\mu_3^*(u_2^*(\Gamma_2))$ does not intersect with the ramification locus of μ_3, Γ' is etale over Y. Moreover, we can see $(\Gamma' \cdot C_Q^1) = 0$ and $(\Gamma' \cdot C_Q^2) = 2$ for all $Q \in \Sigma_2$. If Γ' is not irreducible, then we choose Γ as one of irreducible component of Γ'. If Γ' is irreducible, then we consider $X \times_Y \Gamma \to \Gamma$ and the natural section of $X \times_Y \Gamma \to \Gamma$. Then we get our desired example.

The proof of Proposition 3.3. We apply Lemma 3.6 to the case where $a_1 = 2i + 1$ and $a_2 = 2g - 2i - 2$. We replace D_2 by $D_2 + \Gamma_2$ and a_2 by $a_2 + 1$. Then, (4), (5) and (6) hold for the new D_2 and a_2. Note that $\deg(M_1)$ is even. Thus, we can get a double covering $\mu\colon X \to X_1$ in exactly same way as in [11, Proposition A.3]. Let $f\colon X \to Y_1$ be the induced morphism, and $\Gamma = \mu^*(\Gamma_2)_{\mathrm{red}}$. Then, we have our desired example.

The proof of Proposition 3.4. Applying Lemma 3.6 to the case where $a_1 = 2i + 1$ and $a_2 = 2g - 2i - 1$, we can get a double covering $\mu\colon X \to X_1$ in exactly same way as in [11, Proposition A.3]. Let $f\colon X \to Y_1$ be the induced morphism and $\Gamma' = \mu^*(\Gamma_2)$. Then, Γ' is etale over Y_1. If Γ' is not irreducible, then we choose Γ as one of irreducible component of Γ'. If Γ' is irreducible, then we consider $X \times_{Y_1} \Gamma \to \Gamma$ and the natural section of $X \times_{Y_1} \Gamma \to \Gamma$. Then we get our desired example.

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The proof of Proposition 3.5. We apply Lemma 3.6 to the case where $a_1 = 2i - 1$ and $a_2 = 2g - 2i - 1$. We replace D_1 by $D_1 + \Gamma_1$, D_2 by $D_2 + \Gamma_2$, a_1 by $a_1 + 1$, and a_2 by $a_2 + 1$. Then, (4), (5) and (6) hold for the new D_1, D_2, a_1 and a_2. Thus, we can construct $f\colon X \to Y$ in exactly same way as in [11, Proposition A.2]. Since $u_2^*(\Gamma_1)$ and $u_2^*(\Gamma_2)$ are the ramification locus of μ_3, $\overline{\Gamma}_1 = h^*(u_2^*(\Gamma_1))_{\mathrm{red}}$ and $\overline{\Gamma}_2 = h^*(u_2^*(\Gamma_2))_{\mathrm{red}}$ are sections of f_3. Thus, if we set $\Gamma_1 = v_3(\overline{\Gamma}_1)$ and $\Gamma_2 = v_3(\overline{\Gamma}_2)$, then we have our desired example.

Remark 3.7. As a variant of [11, Lemma A.1], we have the following: For nonnegative integers a_1 and a_2, there are a morphism $f_1 : X_1 \to Y_1$ of smooth projective varieties, and noncrossing sections $\Gamma_1, \ldots, \Gamma_{a_1+a_2+2}$ of $f_1 : X_1 \to Y_1$ with the following properties:

(1) $\dim X_1 = 2$ and $\dim Y_1 = 1$.
(2) Let Σ_1 be the set of all critical values of f_1, i.e., $P \in \Sigma_1$ if and only if $f_1^{-1}(P)$ is a singular variety. Then, for any $P \in Y_1 \setminus \Sigma_1$, $f_1^{-1}(P)$ is a smooth rational curve.
(3) $\Sigma_1 \neq \emptyset$, and for any $P \in \Sigma_1$, $f_1^{-1}(P)$ is a reduced curve consisting of two smooth rational curves E_P^1 and E_P^2 joined at one point transversally.
(4) If we set $D_1 = \Gamma_1 + \cdots + \Gamma_{a_1+a_2+2}$, then $(E_P^1 \cdot D_1) = a_1 + 1$ and $(E_P^2 \cdot D_1) = a_2 + 1$ for any $P \in \Sigma_1$.

This can be proved by taking an etale pull-back of Y_1 in [11, Lemma A.1]. Prof. Keel pointed out that the above implies the following: Let S_n be the nth symmetric group, and $\bar{M}_{0,n}/S_n$ the quotient of $\bar{M}_{0,n}$ by the natural action of S_n. Let D be a $\mathbb{Q}$-divisor on $\bar{M}_{0,n}/S_n$. Then D is nef over $M_{0,n}/S_n$ if and only if D is $\mathbb{Q}$-linearly equivalent to an effective sum of boundary components.

Finally, let us consider the following two lemmas, which will be used in the later section.

LEMMA 3.8 (char$(k) \geqslant 0$). *Let X be a smooth projective surface and Y a smooth projective curve. Let $f\colon X \to Y$ be a surjective morphism with connected fibers, and let L be a line bundle on X. If $L|_{X_\eta}$ gives rise to a torsion element of* $\mathrm{Pic}(X_\eta)$ *on the generic fiber X_η of f and* $\deg(L|_F) = 0$ *for every irreducible component F of fibers, then we have* $(L^2) = 0$.

Proof. Replacing L by $L^{\otimes n}$ $(n \neq 0)$, we may assume that $L|_{X_\eta} \simeq \mathcal{O}_{X_\eta}$. Thus, $f_*(L)$ is a line bundle on Y, and the natural homomorphism $f^*f_*(L) \to L$ is injective. Hence, there is an effective divisor E on X such that $f^*f_*(L) \otimes \mathcal{O}_X(E) \simeq L$. Since $f^*f_*(L) \to L$ is surjective on the generic fiber, E is a vertical divisor. Moreover, $(E \cdot F) = 0$ for every irreducible component F of fibers. Therefore, by Zariski's lemma, $(E^2) = 0$. Hence, $(L^2) = (E^2) = 0$. □

LEMMA 3.9 (char$(k) \geqslant 0$). *Let C be a smooth projective curve of genus $g \geqslant 2$. Let ϑ be a line bundle on C with $\vartheta^{\otimes 2} = \omega_C$. Let Δ be the diagonal of $C \times C$, and*

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$p\colon C\times C\to C$ and $q\colon C\times C\to C$ the projection to the first factor and the projection to the second factor respectively. Then, $p^*(\vartheta^{\otimes n})\otimes q^*(\vartheta^{\otimes n})\otimes \mathcal{O}_{C\times C}((n-1)\Delta)$ is generated by global sections for all $n\geqslant 3$.

Proof. Since $p^*(\vartheta^{\otimes n})\otimes q^*(\vartheta^{\otimes n})$ is generated by global sections, the base locus of $p^*(\vartheta^{\otimes n})\otimes q^*(\vartheta^{\otimes n})\otimes \mathcal{O}_{C\times C}((n-1)\Delta)$ is contained in Δ. Moreover,

$$p^*(\vartheta^{\otimes n})\otimes q^*(\vartheta^{\otimes n})\otimes \mathcal{O}_{C\times C}((n-1)\Delta)\big|_{\Delta}\simeq \omega_C.$$

Thus, it is sufficient to see that

$$\begin{aligned}&H^0(p^*(\vartheta^{\otimes n})\otimes q^*(\vartheta^{\otimes n})\otimes \mathcal{O}_{C\times C}((n-1)\Delta))\\ &\quad\to H^0(p^*(\vartheta^{\otimes n})\otimes q^*(\vartheta^{\otimes n})\otimes \mathcal{O}_{C\times C}((n-1)\Delta)\big|_{\Delta})\end{aligned}$$

is surjective.

We define $L_{n,i}$ to be

$$L_{n,i}=p^*(\vartheta^{\otimes n})\otimes q^*(\vartheta^{\otimes n})\otimes \mathcal{O}_{C\times C}(i\Delta).$$

Then, it suffices to check $H^1(L_{n,n-2})=0$ for the above assertion. By induction on i, we will see that $H^1(L_{n,i})=0$ for $0\leqslant i\leqslant n-2$.

First of all, note that $H^1(\vartheta^{\otimes n})=0$ for $n\geqslant 3$. Thus,

$$H^1(p^*(\vartheta^{\otimes n})\otimes q^*(\vartheta^{\otimes n}))=H^1(p_*(p^*(\vartheta^{\otimes n})\otimes q^*(\vartheta^{\otimes n})))=H^1(\vartheta^{\otimes n})\otimes H^1(\vartheta^{\otimes n})=0.$$

Moreover, let us consider the exact sequence

$$0\to L_{n,i-1}\to L_{n,i}\to L_{n,i}\big|_{\Delta}\to 0.$$

Here since $L_{n,i}\big|_{\Delta}\simeq \omega_C^{\otimes n-i}$, $H^1(L_{n,i}\big|_{\Delta})=0$ if $i\leqslant n-2$. Thus, by the hypothesis of induction, we can see $H^1(L_{n,i})=0$. $\square$

4. Slope Inequalities on $\bar{M}_{g,T}$

Let g be a nonnegative integer and T a finite set with $2g-2+|T|>0$. Recall that

$$\begin{aligned}\Upsilon_{g,T}&=\{(i,I)\mid i\in\mathbb{Z},\ 0\leqslant i\leqslant g \text{ and } I\subseteq T\}\setminus(\{(0,\emptyset)\}\cup\{(0,\{t\})\}_{t\in T}),\\ \overline{\Upsilon}_{g,T}&=\{\{(i,I),(j,J)\}\mid (i,I),(j,J)\in\Upsilon_{g,T},\ i+j=g,\ I\cap J=\emptyset,\ I\cup J=T\}.\end{aligned}$$

For a subset L of T, let us introduce a function $\gamma_L\colon \Upsilon_{g,T}\times\Upsilon_{g,T}\to\mathbb{Z}$ given by

$$\begin{aligned}\gamma_L((i,I),(j,J))=&\left(\det\begin{pmatrix} i & |L\cap I|\\ j & |L\cap J|\end{pmatrix}+|L\cap I|\right)\times\\ &\times\left(\det\begin{pmatrix} i & |L\cap I|\\ j & |L\cap J|\end{pmatrix}-|L\cap J|\right).\end{aligned}$$

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Note that $\gamma_L((i, I), (j, J)) = \gamma_L((j, J), (i, I))$, so that γ_L gives rise to a function on $\overline{\Upsilon}_{g,T}$. Further, a $\mathbb{Q}$-divisor θ_L on $\bar{M}_{g,T}$ is defined to be

$$\theta_L = 4(g - 1 + |L|)(g - 1)\sum_{t\in L}\psi_t - 12|L|^2\lambda + |L|^2\delta_{\text{irr}} - \sum_{\upsilon\in\overline{\Upsilon}_{g,T}} 4\gamma_L(\upsilon)\delta_\upsilon.$$

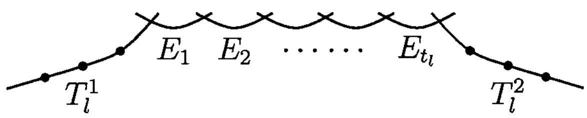
Then, we have the following.

THEOREM 4.1 (char$(k) \geqslant 0$). *For any subset L of T, the divisor θ_L is weakly positive over any finite subsets of $M_{g,T}$.*

Proof. Clearly, we may assume $T = [n]$ for some nonnegative integer n. Let us take an n-pointed stable curve $f\colon X \to Y$ such that the induced morphism $h\colon Y \to \bar{M}_{g,[n]}$ is a finite and surjective morphism of normal varieties. Let Y_0 be the maximal Zariski open set of Y over which f is smooth. Let $Y \setminus Y_0 = B_1 \cup \cdots \cup B_s$ be the irreducible decomposition of $Y \setminus Y_0$. By using [3, Lemma 3.2], we can take a Zariski open set Y_1 with the following properties.

(1) $\operatorname{codim}(Y \setminus Y_1) \geqslant 2$ and $Y_0 \subseteq Y_1$.
(2) Y_1 is smooth at any points of $Y_1 \cap (Y \setminus Y_0)$.
(3) $f\colon \operatorname{Sing}(f) \cap f^{-1}(Y_1) \to f(\operatorname{Sing}(f)) \cap Y_1$ is an isomorphism, so that for all $y \in Y_1$, the number of nodes of $f^{-1}(y)$ is one at most.
(4) There is a projective birational morphism $\phi\colon Z_1 \to X_1 = f^{-1}(Y_1)$ such that if we set $f_1 = f \cdot \phi$, then Z_1 is smooth at any points of $\operatorname{Sing}(f_1) \cap f_1^{-1}(Y \setminus Y_0)$ and $f_1\colon Z_1 \to Y_1$ is an n-pointed semi-stable curve. Moreover, ϕ is an isomorphism over $X_1 \setminus \operatorname{Sing}(f)$.
(5) For each $l = 1, \ldots, s$, there is a t_l such that $\operatorname{mult}_x(X) = t_l + 1$ for all $x \in \operatorname{Sing}(f)$ with $f(x) \in B_l \cap Y_1$.

Let K_0 be a subset of $\{1, \ldots, s\}$ such that $f^{-1}(x)$ is irreducible for all $x \in B_l \cap Y_1$, and let $K_1 = \{1, \ldots, s\} \setminus K_0$. For each $l \in K_1$, there is a $(g_l, I_l), (h_l, J_l) \in \Upsilon_{g,[n]}$ such that the type of x is $\{(g_l, I_l), (h_l, J_l)\}$ for all $x \in \operatorname{Sing}(f)$ with $f(x) \in B_l \cap Y_1$. From now on, by abuse of notation, we denote $B_l \cap Y_1$ by B_l. For $l \in K_1$, $f_1^{-1}(B_l)$ has two essential components T_l^1 and T_l^2, and the components of (-2)-curves $E_1, \ldots, E_{t_l}$ such that $T_l^1 \to B_l$ is an I_l-pointed smooth curve of genus g_l and $T_l^2 \to B_l$ is a J_l-pointed smooth curve of genus h_l. Moreover, the numbering of $E_1, \ldots, E_{t_l}$ is arranged as the following figure:



Let $\Gamma_1, \ldots, \Gamma_n$ be the sections of the n-pointed stable curve of $f\colon X \to Y$. By abuse of notation, the lifting of Γ_a to Z_1 is also denoted by Γ_a. Here we consider a line

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bundle L on Z_1 given by

$$L = \omega_{Z_1/Y_1}^{\otimes|L|} \otimes \mathcal{O}_{Z_1}\left(-(2g-2)\sum_{a\in L}\Gamma_a + \sum_{l\in K_1}(|L|(2g_l-1)-(2g-2)|L\cap I_l|)\tilde{T}_l^1\right),$$

where

$$\tilde{T}_l^1 = (t_l+1)T_l^1 + \sum_{a=1}^{t_l}(t_l+1-a)E_a.$$

We set $E = \mathcal{O}_{X_1} \oplus L$. Then, $\mathrm{dis}_{X_1/Y_1}(E) = -(f_1)_*(c_1(L)^2)$. Here, we know the following formulae:

$$f_*(c_1(\omega_{Z_1/Y_1})\cdot \tilde{T}_l^1) = (t_l+1)(2g_l-1)B_l,$$
$$f_*(\tilde{T}_l^1\cdot\tilde{T}_{l'}^1) = \begin{cases} 0, & \text{if } l\neq l',\\ -(t_l+1)B_l, & \text{if } l=l' \end{cases}$$
$$f_*\left(\sum_{a\in L}\Gamma_a\cdot\tilde{T}_l^1\right) = (t_l+1)|L\cap I_l|B_l,$$
$$f_*(c_1(\omega_{Z_1/Y_1})\cdot\Gamma_a) = -f_*(\Gamma_a\cdot\Gamma_a) \quad \text{(adjunction formula)},$$
$$12\det(f_*(\omega_{Z_1/Y_1})) - \sum_{l=1}^{s}(t_l+1)B_l = f_*(c_1(\omega_{Z_1/Y_1})^2) \quad \text{(Noether's formula)}.$$

Thus, we can see that

$$\begin{aligned}\mathrm{dis}_{Z_1/Y_1}(E) = {} & 4(g-1+|L|)(g-1)f_*\left(c_1(\omega_{Z_1/Y_1})\cdot\sum_{a\in L}\Gamma_a\right) - 12|L|^2\det(f_*(\omega_{Z_1/Y_1})) + \\ & + \sum_{l\in K_0}|L|^2(t_l+1)B_l - \sum_{l\in K_1}4(t_l+1)\gamma_L(\{(g_l,I_l),(h_l,J_l)\})B_l.\end{aligned}$$

On the other hand, for $y\in Y_0$, let $\phi: C'\to f^{-1}(y)$ be a finite morphism of smooth projective curves. Then, $\phi^*(E|_{f^{-1}(y)}) = \mathcal{O}_{C'}\oplus\phi^*(L|_{f^{-1}(y)})$ and

$$\deg(\phi^*(L|_{f^{-1}(y)})) = \deg(\phi)\deg(L|_{f^{-1}(y)}) = 0.$$

Therefore, $\phi^*(E|_{f^{-1}(y)})$ is semistable, which means that $E|_{f^{-1}(y)}$ is strongly semistable for all $y\in Y_0$. Thus, by Corollary 2.3, $\mathrm{dis}_{Z_1/Y_1}(E)$ is weakly positive over any finite subsets of Y_0 as a divisor on Y_1. Therefore, if we set

$$\begin{aligned}\theta'_L = {} & 4(g-1+|L|)(g-1)f_*\left(c_1(\omega_{Z_1/Y_1})\cdot\sum_{a\in L}\Gamma_a\right) - 12|L|^2\det(f_*(\omega_{Z_1/Y_1})) + \\ & + \sum_{l\in K_0}|L|^2(t_l+1)B_l - \sum_{l\in K_1}4(t_l+1)\gamma_L(\{(g_l,I_l),(h_l,J_l)\})B_l.\end{aligned}$$

on Y, then θ'_L is weakly positive over any finite subsets of Y_0 as a divisor on Y. Here $h^*(\theta_L) = \theta'_L$, so that $h_*(\theta'_L) = \deg(h)\theta_L$ by the projection formula. Hence, we have our theorem by (2) of Proposition 1.1.2. □

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Let us apply Theorem 4.1 to the cases $\bar{M}_{g,1}$ and $\bar{M}_{g,2}$.

COROLLARY 4.2 ($\mathrm{char}(k) = 0$). *Let $\bar{M}_{g,1} = \bar{M}_{g,\{1\}}$ be the moduli space of one-pointed stable curves of genus $g \geqslant 1$. We set $\delta_i, \mu, \theta_1 \in \mathrm{Pic}(\bar{M}_{g,1}) \otimes \mathbb{Q}$ as follows:*

$$\delta_i = \delta_{\{(i,\emptyset),(g-i,\{1\})\}} \quad (1 \leqslant i \leqslant g-1),$$

$$\mu = (8g+4)\lambda - g\delta_{\mathrm{irr}} - \sum_{i=1}^{g-1} 4i(g-i)\delta_i,$$

$$\theta_1 = 4g(g-1)\psi_1 - 12\lambda + \delta_{\mathrm{irr}} - \sum_{i=1}^{g-1} 4i(i-1)\delta_i.$$

Then, we have the following:

(1) *μ and θ_1 are weakly positive over any finite subsets of $M_{g,1}$. In particular,*

$$\mathbb{Q}_+\mu + \mathbb{Q}_+\theta_1 + \mathbb{Q}_+\delta_{\mathrm{irr}} + \sum_{i=1}^{g-1} \mathbb{Q}_+\delta_i \subseteq \mathrm{Nef}(\bar{M}_{g,1}; M_{g,1}),$$

where $\mathbb{Q}_+ = \{x \in \mathbb{Q} \mid x \geqslant 0\}$. (Note that $\mu = \theta_1 = 0$ if $g = 1$, and $\mu = 0$ if $g = 2$.)

(2) *We assume $g = 1$. Then, $a\mu + b\theta_1 + c_{\mathrm{irr}}\delta_{\mathrm{irr}}$ is nef over $M_{1,1}$ if and only if $c_{\mathrm{irr}} \geqslant 0$.*

(3) *We assume $g \geqslant 2$. If a $\mathbb{Q}$-divisor*

$$D = a\mu + b\theta_1 + c_{\mathrm{irr}}\delta_{\mathrm{irr}} + \sum_{i=1}^{g-1} c_i\delta_i$$

is nef over $M_{g,1}$, then $b, c_{\mathrm{irr}}, c_1, \ldots, c_{g-1}$ are nonnegative.

Proof. (1) μ is weakly positive over any finite subsets of $M_{g,1}$ by [11, Theorem B] or Remark 2.4, and (2) of Proposition 1.1.3. Moreover, θ_1 is weakly positive over any finite subsets of $M_{g,1}$ by virtue of the case $T = L = \{1\}$ in Theorem 4.1.

(2) This is obvious because $\mu = \theta_1 = 0$.

(3) We assume that D is nef over $M_{g,1}$. Let C be a smooth curve of genus g, and Δ the diagonal of $C \times C$. Let $p\colon C \times C \to C$ be the projection to the first factor. Then, Δ gives rise to a section of p. Hence, we get a morphism $\varphi_1\colon C \to \bar{M}_{g,1}$ with $\varphi_1(C) \subseteq M_{g,1}$. By our assumption, $\deg(\varphi_1^*(D)) \geqslant 0$. On the other hand,

$$\deg(\varphi_1^*(\mu)) = \deg(\varphi_1^*(\delta_{\mathrm{irr}})) = \deg(\varphi_1^*(\delta_1)) = \cdots = \deg(\varphi_1^*(\delta_{g-1})) = 0$$

and $\deg(\varphi_1^*(\theta_1)) = 8g(g-1)^2$. Thus, $b \geqslant 0$.

Let $f_2\colon X_2 \to Y_2$ be a hyperelliptic fibered surface and Γ_2 a section as in Proposition 3.3 for $i = 0$. Let $\varphi_2\colon Y_2 \to \bar{M}_{g,1}$ be the induced morphism. Then, $\varphi_2(Y_2) \cap M_{g,1} \neq \emptyset$,

$$\deg(\varphi_2^*(\mu)) = \deg(\varphi_2^*(\delta_1)) = \cdots = \deg(\varphi_2^*(\delta_{g-1})) = 0$$

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and $\deg(\varphi_2^*(\delta_{\mathrm{irr}})) = \deg(\delta_{\mathrm{irr}}(X_2/Y_2)) > 0$. On the generic fiber, Γ_2 is a ramification point of the hyperelliptic covering. Thus,

$$L_2 = \omega_{X_2/Y_2} \otimes \mathcal{O}_{X_2}(-(2g-2)\Gamma_2)$$

satisfies the conditions of Lemma 3.8. Thus, $(L_2^2) = 0$, which says us that $\deg(\varphi_2^*(\theta_1)) = 0$. Therefore, we get $c_{\mathrm{irr}} \geqslant 0$.

Finally we fix i with $1 \leqslant i \leqslant g-1$. Let $f_3 : X_3 \to Y_3$ be a hyperelliptic fibered surface and Γ_3 a section as in Proposition 3.1. Let $\varphi_3 : Y_3 \to \bar{M}_{g,1}$ be the induced morphism. Then, $\varphi_3(Y_3) \cap M_{g,1} \neq \emptyset$, $\deg(\varphi_3^*(\mu)) = 0$, $\deg(\varphi_3^*(\delta_l)) = 0$ $(l \neq i)$ and $\deg(\varphi_3^*(\delta_i)) = \deg(\delta_i(X_3/Y_3)) > 0$. Let Σ_3 be the set of critical values of f_3. For each $P \in \Sigma_3$, let E_P be the component of genus i in $f_3^{-1}(P)$. On the generic fiber, Γ_2 is a ramification point of the hyperelliptic covering. Thus,

$$L_3 = \omega_{X_3/Y_3} \otimes \mathcal{O}_{X_3}\left(-(2g-2)\Gamma_3 + \sum_{P \in \Sigma_3}(2i-1)E_P\right)$$

satisfies the conditions of Lemma 3.8. Therefore, $(L_3^2) = 0$, which says us that $\deg(\varphi_3^*(\theta_1)) = 0$. Hence, we get $c_i \geqslant 0$. □

COROLLARY 4.3 ($\mathrm{char}(k) = 0$). *Let $\bar{M}_{g,2} = \bar{M}_{g,\{1,2\}}$ be the moduli space of two-pointed stable curves of genus $g \geqslant 2$. We set $\delta_i, \sigma_i, \mu, \theta_{1,2} \in \mathrm{Pic}(\bar{M}_{g,2}) \otimes \mathbb{Q}$ as follows:*

$$\delta_i = \delta_{\{(i,\emptyset),(g-i,\{1,2\})\}} \quad (1 \leqslant i \leqslant g),$$
$$\sigma_i = \delta_{\{(i,\{1\}),(g-i,\{2\})\}} \quad (1 \leqslant i \leqslant g-1),$$
$$\mu = (8g+4)\lambda - g\delta_{\mathrm{irr}} - \sum_{i=1}^{g-1} 4i(g-i)\sigma_i - \sum_{i=1}^{g} 4i(g-i)\delta_i,$$
$$\theta_{1,2} = (g-1)(g+1)(\psi_1 + \psi_2) - 12\lambda + \delta_{\mathrm{irr}} - {} - \sum_{i=1}^{g-1}(2i-g-1)(2i-g+1)\sigma_i - \sum_{i=1}^{g} 4i(i-1)\delta_i.$$

Then, we have the following:

(1) *μ and $\theta_{1,2}$ are weakly positive over any finite subsets of $M_{g,2}$. In particular,*

$$\mathbb{Q}_+\mu + \mathbb{Q}_+\theta_{1,2} + \mathbb{Q}_+\delta_{\mathrm{irr}} + \sum_{i=1}^{g-1}\mathbb{Q}_+\sigma_i + \sum_{i=1}^{g}\mathbb{Q}_+\delta_i \subseteq \mathrm{Nef}(\bar{M}_{g,2}; M_{g,2}).$$

(2) *If a $\mathbb{Q}$-divisor*

$$D = a\mu + b\theta_{1,2} + c_{\mathrm{irr}}\delta_{\mathrm{irr}} + \sum_{i=1}^{g-1} c_i\sigma_i + \sum_{i=1}^{g} d_i\delta_i$$

on $\bar{M}_{g,2}$ is nef over $M_{g,2}$, then

$$b \geqslant 0, \quad c_{\mathrm{irr}} \geqslant 0, \quad c_i \geqslant 0\ (\forall i = 1, \ldots, g-1), \quad d_i \geqslant 0\ (\forall i = 1, \ldots, g).$$

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(3) *Here we set σ, μ' and $\theta'_{1,2}$ as follows:*

$$\sigma = \delta_{\mathrm{irr}} + \sum_{i=1}^{g-1} \sigma_i,$$

$$\mu' = (8g+4)\lambda - g\sigma - \sum_{i=1}^{g} 4i(g-i)\delta_i,$$

$$\theta'_{1,2} = (g-1)(g+1)(\psi_1+\psi_2) - 12\lambda + \sigma - \sum_{i=1}^{g} 4i(i-1)\delta_i.$$

Then, we have

$$\mathbb{Q}_+\mu' + \mathbb{Q}_+\theta'_{1,2} + \mathbb{Q}_+\sigma + \sum_{i=1}^{g} \mathbb{Q}_+\delta_i \subseteq \mathrm{Nef}(\bar{M}_{g,2}; M_{g,2}).$$

Moreover, if a $\mathbb{Q}$-divisor $a\mu' + b\theta'_{1,2} + c\sigma + \sum_{i=1}^{g} d_i\delta_i$ on $\bar{M}_{g,2}$ is nef over $M_{g,2}$, then $b, c, d_1, \dots, d_g$ are nonnegative.

Proof. (1) By [11, Theorem B] or Remark 2.4, and (2) of Proposition 1.1.3, μ is weakly positive over any finite subsets of $M_{g,2}$. Further, $\theta_{1,2}$ is weakly positive over any finite subsets of $M_{g,2}$ by the case $T = L = \{1, 2\}$ in Theorem 4.1.

(2) We assume that D is nef over $M_{g,2}$. Let C be a smooth curve of genus g, and Δ the diagonal of $C \times C$. Let $p : C \times C \to C$ and $q : C \times C \to C$ be the projection to the first factor and the second factor respectively. Moreover, let ϑ be a line bundle on C with $\vartheta^{\otimes 2} = \omega_C$ and $L_n = p^*(\vartheta^{\otimes n}) \otimes q^*(\vartheta^{\otimes n}) \otimes \mathcal{O}_{C\times C}((n-1)\Delta)$. For $n \geqslant 3$, let T_n be a general member of $|L_n|$. Then, since $(L_n^2) > 0$, by Lemma 3.9, T_n is smooth and irreducible. Moreover, T_n meets Δ transversally. Then, we have two morphisms $p_n : T_n \to C$ and $q_n : T_n \to C$ given by $T_n \hookrightarrow C \times C \xrightarrow{p} C$ and $T_n \hookrightarrow C \times C \xrightarrow{q} C$ respectively. Let Γ_{p_n} and Γ_{q_n} be the graph of p_n and q_n in $C \times T_n$ respectively. Then, it is easy to see that Γ_{p_n} and Γ_{q_n} meet transversally, and $(\Gamma_{p_n} \cdot \Gamma_{q_n}) = (T_n \cdot \Delta) = 2g-2$. Let $X \to C \times T_n$ be the blowing-ups at points in $\Gamma_{p_n} \cap \Gamma_{q_n}$, and let $\overline{\Gamma}_{p_n}$ and $\overline{\Gamma}_{q_n}$ be the strict transform of Γ_{p_n} and Γ_{q_n} respectively. Then, $\overline{\Gamma}_{p_n}$ and $\overline{\Gamma}_{q_n}$ give rise to two noncrossing sections of $X \to T_n$. Moreover,

$$(\omega_{X/T_n} \cdot \overline{\Gamma}_{p_n}) = (\omega_{C\times T_n/T_n} \cdot \Gamma_{p_n}) = 2(g-1)\deg(\Gamma_{p_n} \to C) = 2(g-1)(ng-1).$$

In the same way, $(\omega_{X/T_n} \cdot \overline{\Gamma}_{q_n}) = 2(g-1)(ng-1)$. Let $\pi_n : T_n \to \bar{M}_{g,2}$ be the induced morphism. Then, we can see that

$$\deg(\pi_n^*(\lambda)) = \deg(\pi_n^*(\sigma_i)) = \deg(\pi_n^*(\delta_i)) = 0, \quad \text{for all } i = 1, \dots, g-1.$$

Moreover,

$$\deg(\pi_n^*(\psi_1+\psi_2)) = 4(g-1)(ng-1) \quad \text{and} \quad \deg(\pi_n^*(\delta_g)) = 2(g-1).$$

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Thus,

$$\deg(\pi_n^*(D)) = 4(g+1)(g-1)^2(ng-1)b - 8g(g-1)^2 d_g \geqslant 0$$

for all $n \geqslant 3$. Therefore, we get $b \geqslant 0$.

Let $f_2 : X_2 \to Y_2$ be a hyperelliptic fibered surface and Γ_2 a section as in Proposition for $i = 0$. Then, Γ_2 and $j(\Gamma_2)$ gives two points of X_2 over Y_2. Let $\varphi_2 : Y_2 \to \bar{M}_{g,2}$ be the induced morphism. Then, $\varphi_2(Y_2) \cap M_{g,2} \neq \emptyset$, $\deg(\varphi_2^*(\mu)) = 0$, $\deg(\varphi_2^*(\sigma_i)) = 0$ for all $i = 1, \ldots, g-1$, and $\deg(\varphi_2^*(\delta_i)) = 0$ for all $i = 1, \ldots, g$. Moreover, $\deg(\varphi_2(\delta_{\mathrm{irr}})) > 0$. On the generic fiber, two points arising from Γ_2 and $j(\Gamma_2)$ are invariant under the action of the hyperelliptic involution. Thus,

$$L_2 = \omega_{X_2/Y_2} \otimes \mathcal{O}_{X_2}(-(g-1)(\Gamma_2 + j(\Gamma_2))$$

satisfies the conditions of Lemma 3.8. Thus, $(L_2^2) = 0$, which says us that $\deg(\varphi_2^*(\theta_{1,2})) = 0$. Thus, we get $c_{\mathrm{irr}} \geqslant 0$.

We fix i with $1 \leqslant i \leqslant g$. Let $f_3 : X_3 \to Y_3$ be a hyperelliptic fibered surface and Γ_3 a section as in Proposition 3.2. Let $\varphi_3 : Y_3 \to \bar{M}_{g,2}$ be the induced morphism arising from the 2-pointed curve $\{f_3 : X_3 \to Y_3; \Gamma_3, j(\Gamma_3)\}$. Then, $\varphi_3(Y_3) \cap M_{g,2} \neq \emptyset$, $\deg(\varphi_3^*(\mu)) = 0$, $\deg(\varphi_3^*(\sigma_s)) = 0$ $(\forall s = 1, \ldots, g-1)$, $\deg(\varphi_3^*(\delta_s)) = 0$ $(\forall s \neq i)$ and $\deg(\varphi_3^*(\delta_i)) = \deg(\delta_i(X_3/Y_3)) > 0$. Let Σ_3 be the set of critical values of f_3. For each $P \in \Sigma_3$, let E_P be the component of genus i in $f_3^{-1}(P)$. On the generic fiber, two points arising from Γ_2 and $j(\Gamma_2)$ are invariant under the action of the hyperelliptic involution. Thus,

$$L_3 = \omega_{X_3/Y_3} \otimes \mathcal{O}_{X_3}\left(-(g-1)(\Gamma_3 + j(\Gamma_3)) + \sum_{P \in \Sigma_3} (2i-1)E_P\right)$$

satisfies the conditions of Lemma 3.8. Therefore, $(L_3^2) = 0$, which says us that $\deg(\varphi_3^*(\theta_{1,2})) = 0$. Hence, we get $d_i \geqslant 0$.

Finally we fix i with $1 \leqslant i \leqslant g-1$. Let $f_4 : X_4 \to Y_4$ be a hyperelliptic fibered surface and Γ_4, Γ_4' sections as in Proposition 3.5. Let $\varphi_4 : Y_4 \to \bar{M}_{g,2}$ be the induced morphism. Then, $\varphi_4(Y_4) \cap M_{g,2} \neq \emptyset$, $\deg(\varphi_4^*(\mu)) = 0$, $\deg(\varphi_4^*(\delta_s)) = 0$ $(\forall s)$, $\deg(\varphi_3^*(\sigma_s)) = 0$ $(\forall s \neq i)$, and $\deg(\varphi_4^*(\sigma_i)) > 0$. Let Σ_4 be the set of critical values of f_4. For each $P \in \Sigma_4$, let E_P be the component of genus i in $f_4^{-1}(P)$. On the generic fiber, Γ_4 and Γ_4' are a ramification point of the hyperelliptic covering. Thus,

$$L_4 = \omega_{X_4/Y_4} \otimes \mathcal{O}_{X_4}\left(-(g-1)(\Gamma_4 + \Gamma_4') + \sum_{P \in \Sigma_4} ((2i-1) - (g-1))E_P\right)$$

satisfies the conditions of Lemma 3.8. Therefore, $(L_4^2) = 0$, which says us that $\deg(\varphi_4^*(\theta_{1,2})) = 0$. Hence, we get $c_i \geqslant 0$.

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(3) There are nonnegative integers e_i and f_i $(1 \leqslant i \leqslant g-1)$ with

$$\mu' = \mu + \sum_{i=1}^{g-1} e_i\sigma_i \quad \text{and} \quad \theta'_{1,2} = \theta_{1,2} + \sum_{i=1}^{g-1} f_i\sigma_i.$$

Thus, (3) is a consequence of (1) and (2). □

5. The Proof of the Main Result

Throughout this section, we fix an integer $g \geqslant 3$. The purpose of this section is to prove the following theorem.

THEOREM 5.1 (char$(k) = 0$). *A $\mathbb{Q}$-divisor $a\mu + b_{\mathrm{irr}}\delta_{\mathrm{irr}} + \sum_{i=1}^{[g/2]} b_i\delta_i$ on $\bar{M}_g$ is nef over $\bar{M}_g^{[1]}$ if and only if the following system of inequalities hold:*

$$a \geqslant \max\left\{\frac{b_i}{4i(g-i)} \mid i = 1, \ldots, [g/2]\right\},$$
$$B_0 \geqslant B_1 \geqslant B_2 \geqslant \cdots \geqslant B_{[g/2]},$$
$$B^*_{[g/2]} \geqslant \cdots \geqslant B^*_2 \geqslant B^*_1 \geqslant B^*_0,$$

where B_0, B^*_0, B_i and B^*_i $(i = 1, \ldots, [g/2])$ are given by

$$B_0 = 4b_{\mathrm{irr}}, \quad B^*_0 = \frac{4b_{\mathrm{irr}}}{g(2g-1)}, \quad B_i = \frac{b_i}{i(2i+1)} \ \textit{and}\ B^*_i = \frac{b_i}{(g-i)(2(g-i)+1)}.$$

Proof. In the following proof, we denote δ_i by $\delta_{\{i,g-i\}}$. Moreover, we set

$$\bar{\upsilon}_g = \{\{i,j\} \mid 1 \leqslant i,j \leqslant g,\ i+j = g\}.$$

For a $\mathbb{Q}$-divisor $D = a\mu + b_{\mathrm{irr}}\delta_{\mathrm{irr}} + \sum_{\{i,j\}\in\bar{\upsilon}_g} b_{\{i,j\}}\delta_{\{i,j\}}$, let us consider the following inequalities:

$$a \geqslant \frac{b_{\{s,t\}}}{4st} \quad (\forall\{s,t\} \in \bar{\upsilon}_g), \tag{5.1.1}$$

$$4b_{\mathrm{irr}} \geqslant \frac{b_{\{s,t\}}}{s(2s+1)}, \quad \frac{b_{\{s,t\}}}{t(2t+1)} \geqslant \frac{4b_{\mathrm{irr}}}{g(2g-1)} \quad (\forall\{s,t\} \in \bar{\upsilon}_g \text{ with } s \leqslant t), \tag{5.1.2}$$

$$\frac{b_{\{l,k\}}}{l(2l+1)} \geqslant \frac{b_{\{s,t\}}}{s(2s+1)}, \quad \frac{b_{\{s,t\}}}{t(2t+1)} \geqslant \frac{b_{\{l,k\}}}{k(2k+1)}$$
$$(\forall\{s,t\}, \{l,k\} \in \bar{\upsilon}_g \text{ with } l < s \leqslant t < k), \tag{5.1.3}$$

$$a \geqslant 0, \quad b_{\mathrm{irr}} \geqslant 0, \quad b_{\{s,t\}} \geqslant 0 \quad (\forall\{s,t\} \in \bar{\upsilon}_g). \tag{5.1.4}$$

Let $\beta\colon \bar{M}_{g-1,2} \to \bar{M}_g$ and $\alpha_{s,t}\colon \bar{M}_{s,1} \times \bar{M}_{t,1} \to \bar{M}_g$ $(\{s,t\} \in \bar{\upsilon}_g)$ be the clutching maps. First, we claim the following:

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CLAIM 5.1.5. The following are equivalent.

(1) $\beta^*(D)$ is nef over $M_{g-1,2}$ and $\alpha^*_{s,t}(D)$ is nef over $M_{s,1} \times M_{t,1}$ for all $\{s, t\} \in \overline{\upsilon}_g$
(2) (5.1.1), (5.1.2), (5.1.3) and (5.1.4) hold.

On $\bar{M}_{g-1,2}$, we define σ and δ_i ($i = 1, \ldots, g-1$) as in Corollary 4.3. Moreover, we set

$$\mu' = (8g-4)\lambda - (g-1)\sigma - \sum_{i=1}^{g-1} 4i(g-1-i)\delta_i,$$

$$\theta' = (g-2)g(\psi_1 + \psi_2) - 12\lambda + \sigma - \sum_{i=1}^{g-1} 4i(i-1)\delta_i.$$

Then, by using (2) of Corollary 1.5.2, we can see

$$\beta^*(D) = \frac{(g-1)(g-2)(2g-1)a - 3b_{\mathrm{irr}}}{g(g-2)(2g-1)}\mu' + \frac{ag - b_{\mathrm{irr}}}{g(g-2)}\theta' + \frac{(g-1)(2g+1)b_{\mathrm{irr}}}{g(2g-1)}\sigma + \sum_{i=1}^{g-1}\left(b_{\{i,g-i\}} - \frac{4i(2i+1)}{g(2g-1)}b_{\mathrm{irr}}\right)\delta_i. \tag{5.1.a}$$

Thus, by Corollary 4.3, if $\beta^*(D)$ is nef over $\bar{M}_{g-1,2}$, then

$$ag \geqslant b_{\mathrm{irr}} \geqslant 0, \tag{5.1.6}$$

$$b_{\{i,g-i\}} \geqslant \frac{4i(2i+1)}{g(2g-1)}b_{\mathrm{irr}}, \quad (i = 1, \ldots, g-1). \tag{5.1.7}$$

Here we set $\mu'_1 = \theta'_1 = 0$ on $\bar{M}_{1,1}$, and

$$\mu'_e = \frac{1}{e-1}\left((8e+4)\lambda - e\delta_{\mathrm{irr}} - \sum_{l=1}^{e-1} 4l(e-l)\delta_l\right),$$

$$\theta'_e = \frac{1}{e-1}\left(4e(e-1)\psi_1 - 12\lambda + \delta_{\mathrm{irr}} - \sum_{l=1}^{e-1} 4l(l-1)\delta_l\right)$$

on $\bar{M}_{e,1}$ ($e \geqslant 2$), where δ_l's are defined as in Corollary 4.2. Let us fix $\{s, t\} \in \overline{\upsilon}_g$. Then, by using (1) of Corollary 1.5.2, we can see

$$\alpha^*_{s,t}(D) = p^*(D_s) + q^*(D_t),$$

where $p\colon \bar{M}_{s,1} \times \bar{M}_{t,1} \to \bar{M}_{s,1}$ and $q\colon \bar{M}_{s,1} \times \bar{M}_{t,1} \to \bar{M}_{t,1}$ are the projections, and $D_s \in \mathrm{Pic}(\bar{M}_{s,1}) \otimes \mathbb{Q}$ and $D_t \in \mathrm{Pic}(\bar{M}_{t,1}) \otimes \mathbb{Q}$ are given by

$$D_s = \frac{4(g-1)s(2s+1)a - 3b_{\{s,t\}}}{4s(2s+1)}\mu'_s + \frac{4sta - b_{\{s,t\}}}{4s}\theta'_s + \left(b_{\mathrm{irr}} - \frac{b_{\{s,t\}}}{4s(2s+1)}\right)\delta_{\mathrm{irr}} + \sum_{l=1}^{s-1}\left(b_{\{l,g-l\}} - \frac{l(2l+1)}{s(2s+1)}b_{\{s,t\}}\right)\delta_l \tag{5.1.b}$$

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and

$$D_t = \frac{4(g-1)t(2t+1)a - 3b_{\{s,t\}}}{4t(2t+1)}\mu'_t + \frac{4sta - b_{\{s,t\}}}{4t}\theta'_s \qquad (5.1.c)$$
$$+ \left(b_{\mathrm{irr}} - \frac{b_{\{s,t\}}}{4t(2t+1)}\right)\delta_{\mathrm{irr}} + \sum_{l=1}^{t-1}\left(b_{\{l,g-l\}} - \frac{l(2l+1)}{t(2t+1)}b_{\{s,t\}}\right)\delta_l.$$

Thus, by using Corollary 4.2 and Lemma 1.1.4, if $\alpha^*_{s,t}(D)$ is nef over $M_{s,1} \times M_{t,1}$, then

$$4sta \geqslant b_{\{s,t\}}, \qquad (5.1.8)$$

$$b_{\mathrm{irr}} \geqslant \frac{b_{\{s,t\}}}{4s(2s+1)}, \quad b_{\mathrm{irr}} \geqslant \frac{b_{\{s,t\}}}{4t(2t+1)}, \qquad (5.1.9)$$

$$b_{\{l,g-l\}} \geqslant \frac{l(2l+1)}{s(2s+1)}b_{\{s,t\}} \quad (l = 1, \dots, s-1), \qquad (5.1.10)$$

$$b_{\{l,g-l\}} \geqslant \frac{l(2l+1)}{t(2t+1)}b_{\{s,t\}} \quad (l = 1, \dots, t-1). \qquad (5.1.11)$$

Therefore, (1) implies (5.1.6)–(5.1.11). Conversely, we assume (5.1.6)–(5.1.11). Then by using (5.1.6) and (5.1.7), we can see (5.1.4). Thus, we have

$$ag - b_{\mathrm{irr}} \geqslant 0 \Longrightarrow (g-1)(g-2)(2g-1)a - 3b_{\mathrm{irr}} \geqslant 0$$
$$4sta \geqslant b_{\{s,t\}} \Longrightarrow 4(g-1)s(2s+1)a \geqslant 3b_{\{s,t\}} \text{ and } 4(g-1)t(2t+1)a \geqslant 3b_{\{s,t\}}.$$

Therefore, by Corollary 4.2, Corollary 4.3 and Lemma 1.1.4, we can see that $\beta^*(D)$ is nef over $M_{g-1,2}$ and $\alpha^*_{s,t}(D)$ is nef over $M_{s,1} \times M_{t,1}$ for all $\{s,t\} \in \overline{v}_g$. Hence it is sufficient to see that the system of inequalities (5.1.6)–(5.1.11) is equivalent to (5.1.1)–(5.1.3) under the assumption (5.1.4).

The case $s = 1$, $t = g-1$ in (5.1.8) and the case $i = g-1$ in (5.1.7) produce inequalities

$$4(g-1)a \geqslant b_{\{1,g-1\}} \quad \text{and} \quad b_{\{1,g-1\}} \geqslant \frac{4(g-1)}{g}b_{\mathrm{irr}}$$

respectively, which gives rise to (5.1.6). Moreover, it is easy to see that (5.1.7) and (5.1.9) are equivalent to (5.1.2), so that it is sufficient to see that (5.1.10) and (5.1.11) are equivalent to (5.1.3).

From now on, we assume $s \leqslant t$. Since $s(2s+1) \leqslant t(2t+1)$, (5.1.10) and (5.1.11) are equivalent to saying that

$$\frac{b_{\{l,k\}}}{l(2l+1)} \geqslant \frac{b_{\{s,t\}}}{s(2s+1)} \quad (1 \leqslant l < s) \qquad (5.1.12)$$

$$\frac{b_{\{l,k\}}}{l(2l+1)} \geqslant \frac{b_{\{s,t\}}}{t(2t+1)} \quad (s < l < t), \qquad (5.1.13)$$

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where $k = g - l$. In (5.1.12), $t < k \leqslant g - 1$, Thus, (5.1.12) is nothing more than

$$\frac{b_{\{l,k\}}}{l(2l+1)} \geqslant \frac{b_{\{s,t\}}}{s(2s+1)} \quad (1 \leqslant l < s \leqslant t < k \leqslant g-1).$$

Moreover, in (5.1.13), $s < k < t$. Thus, (5.1.13) is nothing more than

$$\frac{b_{\{l,k\}}}{k(2k+1)} \geqslant \frac{b_{\{s,t\}}}{t(2t+1)} \quad (1 \leqslant s < l \leqslant k < t \leqslant g-1).$$

Therefore, replacing $\{s, t\}$ and $\{l, k\}$, we have

$$\frac{b_{\{s,t\}}}{t(2t+1)} \geqslant \frac{b_{\{l,k\}}}{k(2k+1)} \quad (1 \leqslant l < s \leqslant t < k \leqslant g-1).$$

Thus, we get Claim 5.1.5.

By Claim 5.1.5, it is sufficient to show the following claim to complete the proof of Theorem 5.1.

CLAIM 5.1.14. (1) D is nef over $\bar{M}_g^{[1]}$ if and only if D is nef over M_g, $\beta^*(D)$ is nef over $M_{g-1,2}$, and $\alpha_{s,t}^*(D)$ is nef over $M_{s,1} \times M_{t,1}$ for all $\{s,t\} \in \bar{\upsilon}_g$.
(2) D is nef over M_g if and only if (5.1.4) holds.
(3) (5.1.1), (5.1.2) and (5.1.3) imply (5.1.4)

(1) is obvious because

$$\bar{M}_g^{[1]} = M_g \cup \beta_g(M_{g-1,2}) \cup \bigcup_{\{s,t\} \in \bar{\upsilon}_g} \alpha_{s,t}(M_{s,1} \times M_{t,1}).$$

(2) is a consequence of [11, Theorem C]. For (3), let us consider the case $s = 1$, $t = g - 1$ in (5.1.2). Then, we have

$$12 b_{\mathrm{irr}} \geqslant b_{\{1,g-1\}} \quad \text{and} \quad b_{\{1,g-1\}} \geqslant \frac{4(g-1)}{g} b_{\mathrm{irr}},$$

which imply $b_{\mathrm{irr}} \geqslant 0$. Thus, we can see (5.1.4) using (5.1.1) and (5.1.2). □

COROLLARY 5.2 (char$(k) = 0$). *Let $\widetilde{\Delta}_{\mathrm{irr}}$ and $\widetilde{\Delta}_i$ $(i = 1, \ldots, [g/2])$ be the normalizations of the boundary components Δ_{irr} and Δ_i on $\bar{M}_g$, and $\rho_{\mathrm{irr}} \colon \widetilde{\Delta}_{\mathrm{irr}} \to \bar{M}_g$ and $\rho_i \colon \widetilde{\Delta}_i \to \bar{M}_g$ the induced morphisms. Then, a $\mathbb{Q}$-divisor D on $\bar{M}_g$ is nef over $\bar{M}_g^{[1]}$ if and only if the following are satisfied:*

(1) *D is weakly positive at any points of M_g.*
(2) *$\rho_{irr}^*(D)$ is weakly positive at any points of $\rho_{irr}^{-1}(\bar{M}_g^{[1]})$.*

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(3) *$\rho_i^*(D)$ is weakly positive at any points of $\rho_i^{-1}(\bar{M}_g^{[1]})$ for all i.*

Proof. Let $\beta : \bar{M}_{g-1,2} \to \bar{M}_g$ be the clutching map. Then, there is a finite and surjective morphism $\beta' : \bar{M}_{g-1,2} \to \widetilde{\Delta}_{\mathrm{irr}}$ with $\beta = \rho_{\mathrm{irr}} \cdot \beta'$. Further, for $1 \leqslant i \leqslant [g/2]$, let $\alpha_{i,g-i} : \bar{M}_{i,1} \times \bar{M}_{g-i,1} \to \bar{M}_g$ be the clutching map. Then, there is a finite and surjective morphism $\alpha'_{i,g-i} : \bar{M}_{i,1} \times \bar{M}_{g-i,1} \to \widetilde{\Delta}_i$ with $\alpha_{i,g-i} = \rho_i \cdot \alpha'_{i,g-i}$. In particular, $\widetilde{\Delta}_{\mathrm{irr}}$ and $\widetilde{\Delta}_i$'s are $\mathbb{Q}$-factorial. Therefore, if D satisfies (1), (2) and (3), then D is nef over $\bar{M}_g^{[1]}$.

Conversely, we assume that D is nef over $\bar{M}_g^{[1]}$. (1) is nothing more than [11, Theorem C]. As in Theorem 5.1, we set $D = a\mu + b_{\mathrm{irr}}\delta_{\mathrm{irr}} + \sum_{i=1}^{[g/2]} b_i\delta_i$ on $\bar{M}_g$. If we trace-back the proof of Theorem 5.1, we can see that

$$\beta^*(D) \in \mathbb{Q}_+\mu' + \mathbb{Q}_+\theta' + \mathbb{Q}_+\sigma + \sum_i \mathbb{Q}_+\delta_i.$$

Here μ' and θ' are weakly positive at any points of $M_{g-1,2}$ by (1) of Corollary 4.3. Thus, so is $\beta^*(D) = \beta'^*(\rho_{\mathrm{irr}}^*(D))$. Therefore, by virtue of (2) of Proposition 1.1.2, $\beta'_*(\beta^*(D)) = \deg(\beta')\rho_{\mathrm{irr}}^*(D)$ is weakly positive at any points of $\rho_{\mathrm{irr}}^{-1}(\bar{M}_g^{[1]})$. Finally, let us consider (3). As in the proof of Theorem 5.1, there are $D_i \in \mathrm{Pic}(\bar{M}_{i,1}) \otimes \mathbb{Q}$ and $D_{g-i} \in \mathrm{Pic}(\bar{M}_{g-i,1}) \otimes \mathbb{Q}$ with $\alpha_{i,g-i}^*(D) = p^*(D_i) + q^*(D_{g-i})$, where $p : \bar{M}_{i,1} \times \bar{M}_{g-i,1} \to \bar{M}_{i,1}$ and $q : \bar{M}_{i,1} \times \bar{M}_{g-i,1} \to \bar{M}_{g-i,1}$ are the projections to the first factor and the second factor respectively. In the same way as for $\beta^*(D)$, we can see that D_i (resp. D_{g-i}) is weakly positive at any points of $M_{i,1}$ (resp. $M_{g-i,1}$) by virtue of (1) of Corollary 4.2. Thus, by using (2) of Proposition 1.1.3, $\alpha_{i,g-i}^*(D)$ is weakly positive at any points of $M_{i,1} \times M_{g-i,1}$. Therefore, we get (3) by (2) of Proposition 1.1.2. □

COROLLARY 5.3 ($\mathrm{char}(k) = 0$). *With notation as in Corollary 5.2, if $\rho_{\mathrm{irr}}^*(D)$ is nef over $\rho_{\mathrm{irr}}^{-1}(\bar{M}_g^{[1]})$ and $\rho_i^*(D)$ is nef over $\rho_i^{-1}(\bar{M}_g^{[1]})$ for all i, then D is nef over $\bar{M}_g^{[1]}$. In particular, the Mori cone of $\bar{M}_g$ is the convex hull spanned by curves lying on the boundary $\bar{M}_g \setminus M_g$, which gives rise to a special case of* [5, Proposition 3.1].

Proof. Let $\beta' : \bar{M}_{g-1,2} \to \widetilde{\Delta}_{\mathrm{irr}}$ and $\alpha'_{i,g-i} : \bar{M}_{i,1} \times \bar{M}_{g-i,1} \to \widetilde{\Delta}_i$ be the same as in Corollary 5.2. By our assumption, $\beta^*(D) = \beta'^*(\rho_{\mathrm{irr}}^*(D))$ is nef over $M_{g-1,2}$ and $\alpha_{i,g-i}^*(D) = \alpha'^*_{i,g-i}(\rho_i^*(D))$ is nef over $M_{i,1} \times M_{g-i,1}$ for every i. Therefore, by Claim 5.1.5 in Theorem 5.1, we can see that D is nef over $\bar{M}_g^{[1]}$.

Let $\mathrm{Nef}_\Delta(\bar{M}_g)$ be the dual cone of the convex hull spanned by curves on the boundary $\Delta = \bar{M}_g \setminus M_g$. In order to see the last assertion of this corollary, it is sufficient to check $\mathrm{Nef}_\Delta(\bar{M}_g) = \mathrm{Nef}(\bar{M}_g)$, which is a consequence of the first assertion. □

EXAMPLE 5.4. For example, the area of (b_0, b_1) (resp. (b_0, b_1, b_2)) with $\lambda - b_0\delta_0 - b_1\delta_1$ (resp. $\lambda - b_0\delta_0 - b_1\delta_1 - b_2\delta_2$) nef over $\bar{M}_3^{[1]}$ (resp. $\bar{M}_4^{[1]}$) is the inside of the following triangle (resp. polyhedron):

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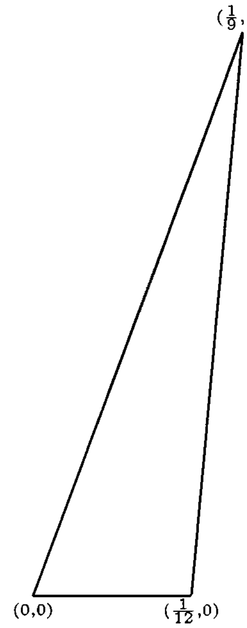
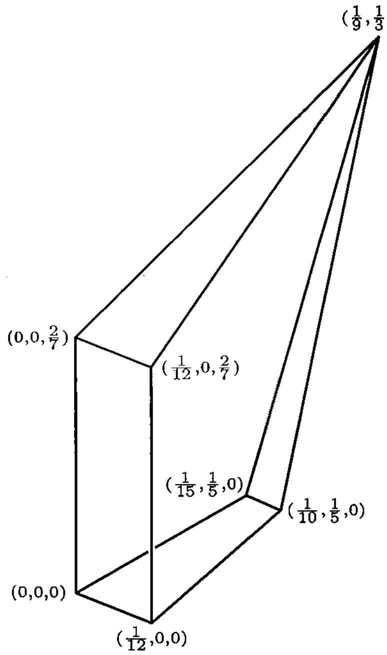


The area of (b_0, b_1) with
$\lambda - b_0\delta_0 - b_1\delta_1$ nef over $\bar{M}_3^{[1]}$



The area of (b_0, b_1, b_2) with
$\lambda - b_0\delta_0 - b_1\delta_1 - b_2\delta_2$ nef over $\bar{M}_4^{[1]}$

6. The Dual Cone of Nef($\bar{M}_g$; $\bar{M}_g^{[1]}$)

Throughout this section, we assume that the characteristic of the base field k is zero. We would like to describe the dual cone of Nef($\bar{M}_g$; $\bar{M}_g^{[1]}$). First of all, let us introduce the following complete irreducible curves

$$C_1, \dots, C_{[g/2]}, \ C_1^*, \dots, C_{[g/2]}^*, \ C_1^\dagger, \dots, C_{[g/2]}^\dagger$$

on $\bar{M}_g$ (Note that we denote by j the hyperelliptic involution in the following contexts):

C_1: Let $f_1 : X_1 \to Y_1$ be a nonisotrivial elliptic surface, and Γ_1 a section of f_1 such that (1) $j(\Gamma_1) = \Gamma_1$, and that (2) every singular fiber of f_1 is an irreducible rational curve with one node. Let $\varphi_1 : Y_1 \to \bar{M}_{1,1}$ be the induced morphism by the one-pointed stable curve $(X_1 \to Y_1, \Gamma_1)$, and $\alpha_{1,g-1} : \bar{M}_{1,1} \times \bar{M}_{g-1,1} \to \bar{M}_g$ the clutching map. We choose $x_1 \in \bar{M}_{g-1,1}$ such that the corresponding curve is a smooth hyperelliptic curve and the marked point is a ramification point of the hyperelliptic curve. Then, C_1 is defined to be $\alpha_{1,g-1}(\varphi_1(Y_1) \times \{x_1\})$.

C_i $(2 \leqslant i \leqslant [g/2])$: As we constructed in Proposition 3.6, let $f_i : X_i \to Y_i$ be a nonisotrivial hyperelliptic fibered surface of genus i, and Γ_i a section of f_i such that (1) $j(\Gamma_i) = \Gamma_i$, (2) every singular fiber of f_i is a stable curve consisting of a smooth projective curve of genus $i-1$ and an elliptic curve meeting transversally at one

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point, and that (3) Γ_i intersects with the elliptic curve on every singular fiber. Let $\varphi_i\colon Y_i \to \bar{M}_{i,1}$ be the induced morphism by the one-pointed stable curve $(X_i \to Y_i, \Gamma_i)$, and $\alpha_{i,g-i}\colon \bar{M}_{i,1} \times \bar{M}_{g-i,1} \to \bar{M}_g$ the clutching map. We choose $x_i \in \bar{M}_{g-i,1}$ such that the corresponding curve is a smooth hyperelliptic curve and the marked point is a ramification point of the hyperelliptic curve. Then, C_i is defined to be $\alpha_{i,g-i}(\varphi_i(Y_i) \times \{x_i\})$.

C_1^*: As we constructed in Proposition 3.2, let $f_1^*\colon X_1^* \to Y_1^*$ be a nonisotrivial hyperelliptic fibered surface of genus $g-1$, and Γ_1^* a section of f_1^* such that (1) $j(\Gamma_1^*) \cap \Gamma_1^* = \emptyset$, (2) every singular fiber of f_1^* is a stable curve consisting of a smooth projective curve of genus $g-2$ and an elliptic curve meeting transversally at one point, and that (3) Γ_1^* intersects with the elliptic curve on every singular fiber. Let $\varphi_1^*\colon Y_1^* \to \bar{M}_{g-1,2}$ be the induced morphism by the two-pointed stable curve $(X_1^* \to Y_1^*, \Gamma_1^*, j(\Gamma_1^*))$, and $\beta\colon \bar{M}_{g-1,2} \to \bar{M}_g$ the clutching map. Then, C_1^* is defined to be $\beta(\varphi_1^*(Y_1^*))$.

C_i^* $(2 \leqslant i \leqslant [g/2])$: As we constructed in Proposition 3.6, let $f_i^*\colon X_i^* \to Y_i^*$ be a nonisotrivial hyperelliptic fibered surface of genus $g-i+1$, and Γ_i^* a section of f_i^* such that (1) $j(\Gamma_i^*) = \Gamma_i^*$, (2) every singular fiber of f_i^* is a stable curve consisting of a smooth projective curve of genus $g-i$ and an elliptic curve meeting transversally at one point, and that (3) Γ_i^* intersects with the elliptic curve on every singular fiber. Let $\varphi_i^*\colon Y_i^* \to \bar{M}_{g-i+1,1}$ be the induced morphism by the one-pointed stable curve $(X_i^* \to Y_i^*, \Gamma_i^*)$, and $\alpha_{g-i+1,i-1}\colon \bar{M}_{g-i+1,1} \times \bar{M}_{i-1,1} \to \bar{M}_g$ the clutching map. We choose $x_i^* \in \bar{M}_{i-1,1}$ such that the corresponding curve is a smooth hyperelliptic curve and the marked point is a ramification point of the hyperelliptic curve. Then, C_i^* is defined to be $\alpha_{g-i-1,i+1}(\varphi_i^*(Y_i^*) \times \{x_i^*\})$.

$C_i^\dagger$ $(1 \leqslant i \leqslant [g/2])$: Let T_i be a smooth projective curve of genus $g-i$, Δ_i the diagonal of $T_i \times T_i$, and $p_i\colon T_i \times T_i \to T_i$ the projection to the first factor. Then, $(p_i\colon T_i \times T_i \to T_i, \Delta_i)$ gives rise to a one-pointed stable curve of genus $g-i$ over T_i. Let $\psi_i\colon T_i \to \bar{M}_{g-i,1}$ be the induced morphism by the one-pointed stable curve $(T_i \times T_i \to T_i, \Delta_i)$, and $\alpha_{g-i,i}\colon \bar{M}_{g-i,1} \times \bar{M}_{i,1} \to \bar{M}_g$ the clutching map. We choose $y_i \in \bar{M}_{i,1}$ such that the corresponding curve is a smooth curve. Then, $C_i^\dagger$ is defined to be $\alpha_{g-i,i}(\psi_i(T_i) \times \{y_i\})$.

PROPOSITION 6.1.

(1) $C_i \subseteq \Delta_i$ *and* $C_i \cap \bar{M}_g^{[1]} \neq \emptyset$ *for all* $1 \leqslant i \leqslant [g/2]$.
(2) $C_1^* \subseteq \Delta_{\mathrm{irr}}$, $C_i^* \subseteq \Delta_{i-1}$ $(2 \leqslant i \leqslant [g/2])$ *and* $C_i^* \cap \bar{M}_g^{[1]} \neq \emptyset$ $(1 \leqslant i \leqslant [g/2])$.
(3) $C_i^\dagger \subseteq \Delta_i$ *and* $C_i^\dagger \cap \bar{M}_g^{[1]} \neq \emptyset$ *for all* $1 \leqslant i \leqslant [g/2]$.
(4) *For a* $\mathbb{Q}$*-divisor* $D = a\mu + b_{\mathrm{irr}}\delta_{\mathrm{irr}} + \sum_{i=1}^{[g/2]} b_i\delta_i$ *on* $\bar{M}_g$,

$$(D \cdot C_i) \geqslant 0 \Longleftrightarrow B_{i-1} \geqslant B_i$$
$$(D \cdot C_i^*) \geqslant 0 \Longleftrightarrow B_{i-1}^* \leqslant B_i^*$$
$$(D \cdot C_i^\dagger) \geqslant 0 \Longleftrightarrow 4i(g-i)a \geqslant b_i$$

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(5) *Let $\bar{H}_g$ be the Zariski closure of the locus of hyperelliptic curves of genus g in $\bar{M}_g$. Then, $C_i, C_i^* \subseteq \bar{H}_g$ for all $i = 1, \dots, [g/2]$.*

Proof. (1), (2) and (3) are obvious by our construction. Using calculations in the proof of Corollary 4.2 and Corollary 4.3 together with formulae (5.1.a), (5.1.b) and (5.1.c) in the proof of Theorem 5.1, we can see (4). (5) is a consequence of the following well-known facts (actually they can be shown by the similar ways as in [11, Lemma A.1, Proposition A.2 and Proposition A.3]):

(i) Let C' and C'' be a smooth hyperelliptic curves of genus i and $g-i$ respectively. Let $P' \in C'$ and $P'' \in C''$ be ramification points of the double covers $C' \to \mathbb{P}^1$ and $C'' \to \mathbb{P}^1$. Let C be a stable curve by gluing C' and C'' at P' and P''. Then, the class of C in $\bar{M}_g$ lies in $\bar{H}_g$.

(ii) Let C' be a smooth hyperelliptic curves of genus $g-1$ and $j\colon C' \to C'$ the hyperelliptic involution. For $P \in C'$ with $j(P) \neq P$, let C be an irreducible stable curve by gluing C' at P and $j(P)$. Then, the class of C in $\bar{M}_g$ lies in $\bar{H}_g$. □

COROLLARY 6.2. *The dual cone of* $\mathrm{Nef}(\bar{M}_g; \bar{M}_g^{[1]})$ *is generated by the classes of the curves*

$$C_1, \dots, C_{[g/2]}, \quad C_1^*, \dots, C_{[g/2]}^*, \quad C_1^\dagger, \dots, C_{[g/2]}^\dagger,$$

that is,

$$\sum_{C \in \mathrm{Curve}(\bar{M}_g^{[1]})} \mathbb{Q}_+[C] = \sum_{i=1}^{[g/2]} \mathbb{Q}_+[C_i] + \sum_{i=1}^{[g/2]} \mathbb{Q}_+[C_i^*] + \sum_{i=1}^{[g/2]} \mathbb{Q}_+[C_i^\dagger],$$

where $\mathrm{Curve}(\bar{M}_g^{[1]})$ *is the set of all complete irreducible curve on $\bar{M}_g$ with $C \cap \bar{M}_g^{[1]} \neq \emptyset$. Moreover, a $\mathbb{Q}$-divisor $D = a\mu + b_{\mathrm{irr}}\delta_{\mathrm{irr}} + \sum_{i=1}^{[g/2]} b_i\delta_i$ is nef over $\bar{M}_g^{[1]}$ if and only if $D|_{\bar{H}_g}$ is nef over $\bar{H}_g \cap \bar{M}_g^{[1]}$ and $4i(g-i)a \geqslant b_i$ for all $i = 1, \dots, [g/2]$.*

Proof. This is a corollary of Theorem 5.1 and Proposition 6.1. □

Remark 6.3. The dual cone of $\mathrm{Nef}(\bar{M}_g; M_g)$ is generated by the following complete irreducible curves $\ell, \ell_0, \ell_1, \dots, \ell_{[g/2]}$ on $\bar{M}_g$.

ℓ: ℓ is a complete irreducible curve in M_g.

ℓ_0: Let $f_0\colon X_0 \to Y_0$ be a nonisotrivial hyperelliptic fibered surface of genus g such that every singular fiber of f_0 is an irreducible stable curve with one node. Let $\varphi_0\colon Y_0 \to \bar{M}_g$ be the induced morphism by the stable curve $X_0 \to Y_0$. Then, ℓ_0 is defined to be $\varphi_0(Y_0)$.

ℓ_i ($1 \leqslant i \leqslant [g/2]$): Let $f_i\colon X_i \to Y_i$ be a nonisotrivial hyperelliptic fibered surface of genus g such that every singular fiber of f_i is a stable curve consisting of a smooth projective curve of genus i and a smooth projective curve of genus $g-i$ meeting transversally at one point. Let $\varphi_i\colon Y_i \to \bar{M}_g$ be the induced morphism by the stable curve $X_i \to Y_i$. Then, ℓ_i is defined to be $\varphi_i(Y_i)$.

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In particular, $D = a\mu + b_{\mathrm{irr}}\delta_{\mathrm{irr}} + \sum_{i=1}^{[g/2]} b_i\delta_i$ is nef over M_g if and only if $D|_{\bar{H}_g}$ is nef over H_g and $a \geqslant 0$.

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References

1. Cornalba, M. and Harris, J.: Divisor classes associated to families of stable varieties, with application to the moduli space of curves, *Ann. Sci. Ecole. Norm. Sup.* **21** (1988), 455–475.
2. Deligne, P. and Mumford, D.: The irreducibility of the space of curves of given genus, *Publ. Math. IHES* **36** (1969), 75–110.
3. De Jong, A. J.: Smoothness, semi-stability and alteration, *Publ. Math. IHES* **83** (1996), 51–93.
4. Faber, C.: Intersection-theoretical computation on $\bar{M}_g$, Banach Center Publ. 36, Polish Acad. Sci., Warsaw, 71–81. (alg-geom/9504005).
5. Gibney, A., Keel, S., and Morrison, I.: Towards the ample cone of $\bar{M}_{g,n}$, (math.AG/0006208).
6. Hain, R.: Moriwaki's inequality and generalizations, talk at Chicago, http://www.math.-duke.edu/~hain/talks/
7. Keel, S. and McKernan, J.: Contractible extremal rays on $\bar{M}_{0,n}$, Preprint, (alg-geom/9607009).
8. Kollár, J. and Mori, S.: *Birational Geometry of Algebraic Varieties*, Cambridge Tracts in Math. 134, Cambridge University Press.
9. Knudsen, F.: The projectivity of the moduli space of stable curves, II and III, *Math. Scand.* **52** (1983), 161–199, 200–212.
10. Moriwaki, A.: A sharp slope inequality for general stable fibrations of curves, *J. reine angew. Math.* **480** (1996), 177–195.
11. Moriwaki, A.: Relative Bogomolov's inequality and the cone of positive divisors on the moduli space of stable curves, *J. Amer. Math. Soc.* **11** (1998), 569–600.
12. Moriwaki, A.: The continuity of Deligne's pairing, *Internat. Math. Res. Notices*, No. 19, (1999), 1057–1066.
13. Seshadri, C. S.: Quotient spaces modulo reductive algebraic groups, *Ann. of Math.* **95** (1972), 511–556.

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