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Symmetric Powers of Galois Modules on Dedekind Schemes

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Abstract. We prove a certain Riemann–Roch-type formula for symmetric powers of Galois modules on Dedekind schemes which, in the number field or function field case, specializes to a formula of Burns and Chinburg for Cassou–Noguès–Taylor operations.

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0. Introduction

Let G be a finite group and E a number field. Let $\mathcal{O}_E$ denote the ring of integers in E, $Y := \mathrm{Spec}(\mathcal{O}_E)$, and

$$\mathrm{Cl}(\mathcal{O}_Y G) := \ker(\mathrm{rank}\colon K_0(\mathcal{O}_E G) \to \mathbb{Z})$$

the locally free classgroup associated with E and G. For any $k \geqslant 1$, Cassou–Noguès and Taylor have constructed a certain endomorphism ψ_k^{CNT} of $\mathrm{Cl}(\mathcal{O}_Y G)$ which, via Fröhlich's Hom-description of $\mathrm{Cl}(\mathcal{O}_Y G)$, is dual to the kth Adams operation on the classical ring of virtual characters of G (see [CT]). Now, let $\gcd(k, \mathrm{ord}(G)) = 1$ and let $k' \in \mathbb{N}$ be an inverse of k modulo $\mathrm{ord}(G)$. In the paper [K 3], we have shown that then the endomorphism $\psi_{k'}^{\mathrm{CNT}}$ is a simply definable symmetric power operation σ^k.

Now, let F/E be a finite tame Galois extension with Galois group G. Let $f\colon X := \mathrm{Spec}(\mathcal{O}_F) \to Y$ denote the corresponding G-morphism and let f_* be the homomorphism

$$f_* : K_0(G, X) \to \mathrm{Cl}(\mathcal{O}_Y G), \quad [\mathcal{E}] \mapsto [f_*(\mathcal{E})] - \mathrm{rank}(\mathcal{E}) \cdot [\mathcal{O}_Y G],$$

from the Grothendieck group $K_0(G, X)$ of all locally free $\mathcal{O}_X$-modules with (semilinear) G-action to $\mathrm{Cl}(\mathcal{O}_Y G)$. Furthermore, let $\mathcal{D}$ denote the different of F/E and ψ^k the kth Adams operation on $K_0(G, X)$. The paper [BC] by Burns and Chinburg together with the identification of $\psi_{k'}^{\mathrm{CNT}}$ with σ^k mentioned above

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then implies the following Riemann–Roch type formula for all $x \in K_0(G, X)$:

$$\sigma^k(f_*(x)) = f_*\left(\sum_{i=0}^{k'-1}[\mathcal{D}^{-ik}] \cdot \psi^k(x)\right) \quad \text{in} \quad \mathrm{Cl}(\mathcal{O}_Y G)/\mathrm{Ind}_1^G \mathrm{Cl}(\mathcal{O}_Y) \tag{1}$$

(see Theorem 5.6 and Theorem 3.7 in [K 3]).

We now assume that Y is an arbitrary Dedekind scheme (i.e., Noetherian, regular, irreducible, and $\dim(Y) = 1$) and that X is the normalization of Y in a finite Galois extension F of the function field E of Y with Galois group G. We again assume that the corresponding G-morphism $f: X \to Y$ is tamely ramified. Similarly to the number field case, we define the locally free classgroup $\mathrm{Cl}(\mathcal{O}_Y G)$ (see Section 2 or [AB]), the symmetric power operation σ^k on $\mathrm{Cl}(\mathcal{O}_Y G)$ (see Sections 1 and 2), and the homomorphism $f_*: K_0(G, X) \to \mathrm{Cl}(\mathcal{O}_Y G)$ (see Section 3). The object of this paper is to study the following natural question. Does the formula (1) still hold in this more general situation?

First of all, we mention that the paper [BC] also implies that the formula (1) holds if Y is a projective smooth curve over a finite field L and the characteristic of L does not divide the order of G (see Theorem 3.5(b)). In this semisimple function field case, a Hom-description of $\mathrm{Cl}(\mathcal{O}_Y G)$ again exists and the operation σ^k is dual to the Adams operation $\psi^{k'}$ as in the number field case (see Theorem 2.10). In particular, Fröhlich's techniques can be applied as in the number field case (see [BC]).

In this paper, we moreover obtain the following results whose proof however requires completely different methods since there is no Hom-description of $\mathrm{Cl}(\mathcal{O}_Y G)$ available in general.

THEOREM A. *The formula (1) holds if one of the following assumptions is satisfied:*

(a) $k = 1$.
(b) *The group G is Abelian and* $f: X \to Y$ *is unramified.*

THEOREM B. *The formula (1) holds after passing from* $\mathrm{Cl}(\mathcal{O}_Y G)/\mathrm{Ind}_1^G \mathrm{Cl}(\mathcal{O}_Y)$ *to* $\hat{K}_0(G, Y)[k^{-1}]/(\mathrm{Ind}_1^G K_0(Y))\hat{K}_0(G, Y)[k^{-1}]$ *via the Cartan homomorphism.*

Here, $K_0(G, Y)$ denotes the Grothendieck group of all locally free $\mathcal{O}_Y$-modules with G-action and $\hat{K}_0(G, Y)[k^{-1}]$ denotes the I-adic completion of $K_0(G, Y)[k^{-1}]$ where I is the augmentation ideal of $K_0(G, Y)[k^{-1}]$.

The proof of Theorem A in the case $k = 1$ relies on the results of the paper [C] by Chase (see Proposition 3.2). Note that, despite the fact $\sigma^k = \mathrm{id}$ for $k = 1$, the formula (1) is nontrivial since k' may be an arbitrary natural number in the coset $1 + \mathrm{ord}(G)\mathbb{Z}$. If G is Abelian and $f: X \to Y$ is unramified, the proof of Theorem A relies on the following two facts (see Theorem 3.5). Firstly, applying the operation σ^k to the element $[\mathcal{Q}] - [\mathcal{P}]$ in $\mathrm{Cl}(\mathcal{O}_Y G)$ is the same as pulling back the G-action on $\mathcal{P}$ and $\mathcal{Q}$ along the automorphism $G \to G$, $g \mapsto g^k$ (see Theorem 2.7). Secondly, the

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map $H^1(Y, G) \to \mathrm{Cl}(\mathcal{O}_Y G)$ which maps a principal G-bundle $f : X \to Y$ to the class $[f_*(\mathcal{O}_X)] - [\mathcal{O}_Y G]$ is a homomorphism (by Theorem 5 in the paper [W] by Waterhouse). Theorem B follows from the equivariant Adams–Riemann–Roch theorem (see [K 2]) and the case $k = 1$ of Theorem A (see Theorem 3.3). Moreover, in the semisimple function field case mentioned above, the formula (1) modulo torsion can be deduced from Theorem B if the order of G is a power of a prime (see Remark 3.6).

1. Symmetric Power Operations on K_0-, K_1-, and Relative Grothendieck Groups

Let X be a Noetherian scheme and G a finite group which, in this section, is assumed to act trivially on X.

First, we introduce the category of locally projective modules over the group ring $\mathcal{O}_X G$. Then, we (purely algebraically) construct symmetric power operations on the Grothendieck group $K_0(\mathcal{O}_X G)$ and the Bass group $K_1^{\det}(\mathcal{O}_X G)$ associated with this category. While these constructions are more or less obvious generalizations of the constructions in Section 1 of [K 3] (for K_0 and K_1), the subsequent construction of symmetric power operations on relative Grothendieck groups (in the sense of [B]) is new. We furthermore show that these operations are compatible with the maps in the localization sequence. Finally, we present some cases in which the relative Grothendieck groups can be identified with Grothendieck groups of certain torsion modules.

By a (quasi-)coherent $\mathcal{O}_X G$-module we mean a (quasi-)coherent $\mathcal{O}_X$-module $\mathcal{P}$ together with an action of G on $\mathcal{P}$ by $\mathcal{O}_X$-homomorphisms. Homomorphisms and exact sequences of quasi-coherent $\mathcal{O}_X G$-modules are defined in the obvious way. We call a coherent $\mathcal{O}_X G$-module $\mathcal{P}$ *locally projective* iff the stalk $\mathcal{P}_x$ is a projective $\mathcal{O}_{X,x} G$-module for all $x \in X$. Let $K_0(\mathcal{O}_X G)$ denote the Grothendieck group of all locally projective $\mathcal{O}_X G$-modules.

Remark 1.1. If $X = \mathrm{Spec}(A)$ is affine, then $\mathcal{P}$ is locally projective if and only if $P := H^0(X, \mathcal{P})$ is a finitely generated projective AG-module. (Easy to prove.)

We are now going to construct the above-mentioned symmetric power operations. As in Section 1 of [K 3], it is convenient to introduce the following categories. For any $i \geqslant 1$, let $\mathcal{M}_i$ denote the smallest full subcategory of the Abelian category of all coherent $\mathcal{O}_X G$-modules which is closed under extensions and kernels of $\mathcal{O}_X G$-epimorphisms and which contains all the modules of the form $\mathrm{Sym}^{i_1}_{\mathcal{O}_X}(\mathcal{P}_1) \otimes_{\mathcal{O}_X} \ldots \otimes_{\mathcal{O}_X} \mathrm{Sym}^{i_r}_{\mathcal{O}_X}(\mathcal{P}_r)$ where $\mathcal{P}_1, \ldots, \mathcal{P}_r$ are locally projective coherent $\mathcal{O}_X G$-modules, $i_1, \ldots, i_r$ are natural numbers with $i_1 + \cdots + i_r = i$, and G acts diagonally. So, $\mathcal{M}_1$ is the category of all locally projective coherent $\mathcal{O}_X G$-modules. By Proposition 1.1 in [K 3], the category $\mathcal{M}_i$ is contained in $\mathcal{M}_1$ if $\gcd(i, \mathrm{ord}(G))$

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is invertible on X. It is easy to see that, for all $i, j \geqslant 1$, the functor

$$\mathcal{M}_i \times \mathcal{M}_j \to \mathcal{M}_{i+j}, \quad (\mathcal{P}, \mathcal{Q}) \mapsto \mathcal{P} \otimes_{\mathcal{O}_X} \mathcal{Q},$$

is well-defined and bi-exact (cf. Lemma 1.2 in [K 3]). In particular, we obtain products $K_0(\mathcal{M}_i) \times K_0(\mathcal{M}_j) \to K_0(\mathcal{M}_{i+j})$, $i, j \geqslant 1$, and the set $1 + \prod_{i \geqslant 1} K_0(\mathcal{M}_i)t^i$ consisting of all power series $1 + \sum_{i \geqslant 1} a_i t^i$ with $a_i \in K_0(\mathcal{M}_i)$ forms an Abelian group with respect to multiplication of power series. As usual, one shows that the association $[\mathcal{P}] \mapsto \sum_{i \geqslant 0} [\mathrm{Sym}^i_{\mathcal{O}_X}(\mathcal{P})] t^i$ can be extended to a well-defined homomorphism

$$\sigma : K_0(\mathcal{O}_X G) \to 1 + \prod_{i \geqslant 1} K_0(\mathcal{M}_i) t^i$$

(see Section 1 of Chapter V in [FL] and Lemma 1.3 in [K 3]). The ith component of this homomorphism is denoted by σ^i. We have for all $x, y \in K_0(\mathcal{O}_X G)$:

$$\begin{aligned}\sigma^i(x - y) &= \sum_{\substack{a \geqslant 0, b_1, \ldots, b_u \geqslant 1 \\ a + b_1 + \ldots + b_u = i}} (-1)^u \sigma^a(x) \sigma^{b_1}(y) \cdots \sigma^{b_u}(y) \\ &= \sum_{\substack{a, b_1, \ldots, b_u \geqslant 1 \\ a + b_1 + \ldots + b_u = i}} (-1)^u (\sigma^a(x) - \sigma^a(y)) \sigma^{b_1}(y) \cdots \sigma^{b_u}(y)\end{aligned}$$

in $K_0(\mathcal{M}_i)$ (cf. Section 2 in [G 2]). If $\gcd(i, \mathrm{ord}(G))$ is invertible on X, let σ^i also denote the composition

$$K_0(\mathcal{O}_X G) \xrightarrow{\sigma^i} K_0(\mathcal{M}_i) \xrightarrow{\mathrm{can}} K_0(\mathcal{O}_X G).$$

The map σ^i is called ith *symmetric power operation*.

Now, let $K_0(\mathbb{Z}, \mathcal{M}_i)$ denote the Grothendieck group of all pairs $(\mathcal{P}, \alpha)$ where $\mathcal{P}$ is an object of $\mathcal{M}_i$ and α is an $\mathcal{O}_X G$-automorphism of $\mathcal{P}$. We put $K_0(\mathbb{Z}, \mathcal{O}_X G) := K_0(\mathbb{Z}, \mathcal{M}_1)$. As above, the association $((\mathcal{P}, \alpha), (\mathcal{Q}, \beta)) \mapsto (\mathcal{P} \otimes_{\mathcal{O}_X} \mathcal{Q}, \alpha \otimes_{\mathcal{O}_X} \beta)$ induces a multiplication map

$$K_0(\mathbb{Z}, \mathcal{M}_i) \times K_0(\mathbb{Z}, \mathcal{M}_j) \to K_0(\mathbb{Z}, \mathcal{M}_{i+j})$$

(for all $i, j \geqslant 1$) and the association $(\mathcal{P}, \alpha) \mapsto \sum_{i \geqslant 0} (\mathrm{Sym}^i_{\mathcal{O}_X}(\mathcal{P}), \mathrm{Sym}^i_{\mathcal{O}_X}(\alpha)) t^i$ induces a homomorphism

$$\sigma \colon K_0(\mathbb{Z}, \mathcal{O}_X G) \to 1 + \prod_{i \geqslant 1} K_0(\mathbb{Z}, \mathcal{M}_i) t^i.$$

By restricting, we obtain symmetric power operations

$$\sigma^i \colon \tilde{K}_0(\mathbb{Z}, \mathcal{O}_X G) = \tilde{K}_0(\mathbb{Z}, \mathcal{M}_1) \to \tilde{K}_0(\mathbb{Z}, \mathcal{M}_i), \quad i \geqslant 1,$$

between the reduced Grothendieck groups

$$\tilde{K}_0(\mathbb{Z}, \mathcal{M}_i) := \ker(K_0(\mathbb{Z}, \mathcal{M}_i) \to K_0(\mathcal{M}_i), \ [\mathcal{P}, \alpha] \mapsto [\mathcal{P}]), \quad i \geqslant 1.$$

We denote the factor group of $K_0(\mathbb{Z}, \mathcal{M}_i)$ modulo the subgroup generated by the relations of the form $[\mathcal{P}, \alpha\beta] - [\mathcal{P}, \alpha] - [\mathcal{P}, \beta]$ by $K_1^{\det}(\mathcal{M}_i)$. (Note that in particular $[\mathcal{P}, \mathrm{id}]$ is in the group of relations.) If $X = \mathrm{Spec}(A)$ is affine, the group $K_1^{\det}(\mathcal{O}_X G) = K_1^{\det}(\mathcal{M}_1)$ coincides with the usual Bass-Whitehead group $K_1(AG)$ of the group ring AG (by Remark 1.1). In the sequel, we consider $K_1^{\det}(\mathcal{M}_i)$ as the factor group of $\tilde{K}_0(\mathbb{Z}, \mathcal{M}_i)$ modulo the subgroup I_i generated by the relations of the form $[\mathcal{P}, \alpha\beta] - [\mathcal{P}, \alpha] - [\mathcal{P}, \beta] + [\mathcal{P}, \mathrm{id}]$. Since

$$\begin{aligned}
&([\mathcal{P}, \alpha\beta] - [\mathcal{P}, \alpha] - [\mathcal{P}, \beta] + [\mathcal{P}, \mathrm{id}]) \cdot [\mathcal{Q}, \gamma] \\
&\quad = ([\mathcal{P}\otimes\mathcal{Q}, \alpha\beta\otimes\gamma] - [\mathcal{P}\otimes\mathcal{Q}, \alpha\otimes\gamma] - [\mathcal{P}\otimes\mathcal{Q}, \beta\otimes\mathrm{id}] + [\mathcal{P}\otimes\mathcal{Q}, \mathrm{id}\otimes\mathrm{id}]) - \\
&\qquad - ([\mathcal{P}\otimes\mathcal{Q}, \beta\otimes\gamma] - [\mathcal{P}\otimes\mathcal{Q}, \mathrm{id}\otimes\gamma] - [\mathcal{P}\otimes\mathcal{Q}, \beta\otimes\mathrm{id}] + [\mathcal{P}\otimes\mathcal{Q}, \mathrm{id}\otimes\mathrm{id}]),
\end{aligned}$$

the group $I_i K_0(\mathbb{Z}, \mathcal{M}_j)$ is contained in I_{i+j} and we obtain a multiplication map

$$K_1^{\det}(\mathcal{M}_i) \times K_1^{\det}(\mathcal{M}_j) = \tilde{K}_0(\mathbb{Z}, \mathcal{M}_i)/I_i \times \tilde{K}_0(\mathbb{Z}, \mathcal{M}_j)/I_j \to K_1^{\det}(\mathcal{M}_{i+j})$$

(for all $i, j \geqslant 1$) which is obviously trivial, i.e., the product of any two power series $\sum_{i\geqslant 0} x_i t^i$, $\sum_{i\geqslant 0} y_i t^i$ in $1 + \prod_{i\geqslant 1} K_1^{\det}(\mathcal{M}_i)t^i$ is $1 + \sum_{i\geqslant 1}(x_i + y_i)t_i$.

LEMMA 1.2. *The homomorphism* $\sigma : \tilde{K}_0(\mathbb{Z}, \mathcal{O}_X G) \to 1 + \prod_{i\geqslant 1} \tilde{K}_0(\mathbb{Z}, \mathcal{M}_i)t^i$ *induces a homomorphism* $\sigma : K_1^{\det}(\mathcal{O}_X G) \to 1 + \prod_{i\geqslant 1} K_1^{\det}(\mathcal{M}_i)t^i$. *Each component* $\sigma^i : K_1^{\det}(\mathcal{O}_X G) \to K_1^{\det}(\mathcal{M}_i)$ *of* σ *is a homomorphism.*

Proof. Let $\mathcal{P} \in \mathcal{M}_1$ and $\alpha, \beta \in \mathrm{Aut}_{\mathcal{O}_X G}(\mathcal{P})$. We write S for Sym. Then, for all $a \geqslant 1$, the element

$$\begin{aligned}
&[\mathrm{S}^a(\mathcal{P}\oplus\mathcal{P}), \mathrm{S}^a(\alpha\beta\oplus\mathrm{id})] - [\mathrm{S}^a(\mathcal{P}\oplus\mathcal{P}), \mathrm{S}^a(\alpha\oplus\beta)] \\
&\quad = \sum_{c=0}^{a}\Big([\mathrm{S}^c(\mathcal{P})\otimes\mathrm{S}^{a-c}(\mathcal{P}), \mathrm{S}^c(\alpha\beta)\otimes\mathrm{S}^{a-c}(\mathrm{id})] - [\mathrm{S}^c(\mathcal{P})\otimes\mathrm{S}^{a-c}(\mathcal{P}), \mathrm{S}^c(\alpha)\otimes\mathrm{id}] - \\
&\qquad - [\mathrm{S}^c(\mathcal{P})\otimes\mathrm{S}^{a-c}(\mathcal{P}), \mathrm{S}^c(\beta)\otimes\mathrm{id}] + [\mathrm{S}^c(\mathcal{P})\otimes\mathrm{S}^{a-c}(\mathcal{P}), \mathrm{id}\otimes\mathrm{id}]\Big) - \\
&\qquad - \sum_{c=0}^{a}\Big([\mathrm{S}^c(\mathcal{P})\otimes\mathrm{S}^{a-c}(\mathcal{P}), \mathrm{S}^c(\alpha)\otimes\mathrm{S}^{a-c}(\beta)] - [\mathrm{S}^c(\mathcal{P})\otimes\mathrm{S}^{a-c}(\mathcal{P}), \mathrm{S}^c(\alpha)\otimes\mathrm{id}] - \\
&\qquad - [\mathrm{S}^c(\mathcal{P})\otimes\mathrm{S}^{a-c}(\mathcal{P}), \mathrm{id}\otimes\mathrm{S}^{a-c}(\beta)] + [\mathrm{S}^c(\mathcal{P})\otimes\mathrm{S}^{a-c}(\mathcal{P}), \mathrm{id}\otimes\mathrm{id}]\Big)
\end{aligned}$$

is contained in I_a. Since

$$\sigma^i(x - y) = \sum_{\substack{a, b_1, \ldots, b_u \geqslant 1 \\ a + b_1 + \ldots + b_u = i}} (-1)^u(\sigma^a(x) - \sigma^a(y))\sigma^{b_1}(y)\cdots\sigma^{b_u}(y)$$

(for all $x, y \in K_0(\mathbb{Z}, \mathcal{M}_1)$), this implies that the element

$$\sigma^i([\mathcal{P}, \alpha\beta] - [\mathcal{P}, \alpha] - [\mathcal{P}, \beta] + [\mathcal{P}, \mathrm{id}]) = \sigma^i([\mathcal{P}\oplus\mathcal{P}, \alpha\beta\oplus\mathrm{id}] - [\mathcal{P}\oplus\mathcal{P}, \alpha\oplus\beta])$$

is contained in I_i, as was to be shown. For all $x, y \in K_1^{\det}(\mathcal{M}_1)$, we have

$$\sigma(x+y) = \sigma(x) \cdot \sigma(y) = 1 + \sum_{i \geqslant 1} (\sigma^i(x) + \sigma^i(y))t^i \quad \text{in} \quad 1 + \prod_{i \geqslant 1} K_1^{\det}(\mathcal{M}_i)t^i;$$

thus, σ^i is a homomorphism for all $i \geqslant 1$.

Now, let $j : U \to X$ be a morphism between Noetherian schemes. Similarly to Secion 5 of Chapter VII in [B], let $K_0(\mathrm{co}(j_i^*))$ denote the Grothendieck group of all triples $(\mathcal{P}, \alpha, \mathcal{Q})$ where $\mathcal{P}$ and $\mathcal{Q}$ are objects in $\mathcal{M}_i$ and $\alpha\colon j^*(\mathcal{P}) \to j^*(\mathcal{Q})$ is an $\mathcal{O}_U G$-isomorphism. As above, the association

$$((\mathcal{P}, \alpha, \mathcal{Q}), (\mathcal{P}', \alpha', \mathcal{Q}')) \mapsto (\mathcal{P} \otimes_{\mathcal{O}_X} \mathcal{P}', \alpha \otimes_{\mathcal{O}_U} \alpha', \mathcal{Q} \otimes_{\mathcal{O}_X} \mathcal{Q}')$$

induces, for all $i, i' \geqslant 1$, a multiplication map

$$K_0(\mathrm{co}(j_i^*)) \times K_0(\mathrm{co}(j_{i'}^*)) \to K_0(\mathrm{co}(j_{i+i'}^*))$$

and the association $(\mathcal{P}, \alpha, \mathcal{Q}) \mapsto \sum_{i \geqslant 0} (\mathrm{Sym}^i_{\mathcal{O}_X}(\mathcal{P}), \mathrm{Sym}^i_{\mathcal{O}_U}(\alpha), \mathrm{Sym}^i_{\mathcal{O}_X}(\mathcal{Q}))t^i$ induces a homomorphism

$$\sigma : K_0(\mathrm{co}(j_1^*)) \to 1 + \prod_{i \geqslant 1} K_0(\mathrm{co}(j_i^*))t^i.$$

By restricting, we obtain symmetric power operations

$$\sigma^i : \tilde{K}_0(\mathrm{co}(j_1^*)) \to \tilde{K}_0(\mathrm{co}(j_i^*)), \quad i \geqslant 1,$$

between the reduced Grothendieck groups

$$\tilde{K}_0(\mathrm{co}(j_i^*)) := \ker(K_0(\mathrm{co}(j_i^*)) \to K_0(\mathcal{M}_i),\ [\mathcal{P}, \alpha, \mathcal{Q}] \mapsto [\mathcal{P}]).$$

Let $K_0(j_i^*)$ denote the factor group of $K_0(\mathrm{co}(j_i^*))$ modulo the subgroup generated by the relations of the form $[\mathcal{P}, \beta\alpha, \mathcal{R}] - [\mathcal{P}, \alpha, \mathcal{Q}] - [\mathcal{Q}, \beta, \mathcal{R}]$ (see also Proposition (5.1) on p. 370 in [B]). In the sequel, we consider $K_0(j_i^*)$ as the factor group of $\tilde{K}_0(\mathrm{co}(j_i^*))$ modulo the subgroup I_i generated by the elements of the form $[\mathcal{P}, \beta\alpha, \mathcal{R}] - [\mathcal{P}, \alpha, \mathcal{Q}] - [\mathcal{Q}, \beta, \mathcal{R}] + [\mathcal{Q}, \mathrm{id}, \mathcal{Q}]$. As above, one easily sees that $I_i K_0(\mathrm{co}(j_{i'}^*))$ is contained in $I_{i+i'}$ and we obtain a multiplication map

$$K_0(j_i^*) \times K_0(j_{i'}^*) \to K_0(j_{i+i'}^*)$$

for all $i, i' \geqslant 1$ which however (in contrast to $K_1^{\det}$) seems not to be trivial in general.

LEMMA 1.3. *The homomorphism* $\sigma : \tilde{K}_0(\mathrm{co}(j_1^*)) \to 1 + \prod_{i \geqslant 1} \tilde{K}_0(\mathrm{co}(j_i^*))t^i$ *induces a homomorphism* $\sigma : K_0(j_1^*) \to 1 + \prod_{i \geqslant 1} K_0(j_i^*)t^i$.

Proof. Similarly to Lemma 1.2.

The association $[\mathcal{P}, \alpha, \mathcal{Q}] \mapsto [\mathcal{Q}] - [\mathcal{P}]$ obviously defines a homomorphism

$$\nu_i : K_0(j_i^*) \to K_0(\mathcal{M}_i)$$

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for all $i \geqslant 1$. (Side Remark: If we would have chosen the map $[\mathcal{P}, \alpha, \mathcal{Q}] \mapsto [\mathcal{Q}]$ in the definition of the reduced Grothendieck group, then we would have to replace v_i by $-v_i$ in the following lemma.)

LEMMA 1.4. *The multiplication maps are compatible with the homomorphisms* v_i, $i \geqslant 1$. *The same holds for the symmetric power operations* σ^i, $i \geqslant 1$; *i.e., the following diagram commutes for all* $i \geqslant 1$:

$$\begin{array}{ccc} K_0(j_1^*) & \xrightarrow{v_1} & K_0(\mathcal{M}_1) \\ \big\downarrow \sigma^i & & \big\downarrow \sigma^i \\ K_0(j_i^*) & \xrightarrow{v_i} & K_0(\mathcal{M}_i). \end{array}$$

Proof. We only prove the assertion for σ^i. Let $\mathcal{P}, \mathcal{Q}, \mathcal{R} \in \mathcal{M}_1$ and let $\alpha : j^*(\mathcal{P}) \xrightarrow{\sim} j^*(\mathcal{Q})$ and $\beta : j^*(\mathcal{Q}) \xrightarrow{\sim} j^*(\mathcal{R})$ be $\mathcal{O}_U G$-isomorphisms. We again write S for Sym. Then we have in $K_0(\mathcal{M}_i)$:

$$\begin{aligned} v_i\sigma^i(\mathcal{P}, \alpha, \mathcal{Q}) &= v_i\sigma^i((\mathcal{P}, \alpha, \mathcal{Q}) - (\mathcal{P}, \mathrm{id}, \mathcal{P})) \\ &= v_i\Bigg(\sum_{\substack{a \geqslant 0, b_1, \ldots, b_u \geqslant 1 \\ a+b_1+\ldots+b_u=i}} (-1)^u \Big(\mathrm{S}^a(\mathcal{P}) \otimes \mathrm{S}^{b_1}(\mathcal{P}) \otimes \ldots \otimes \mathrm{S}^{b_u}(\mathcal{P}), \mathrm{S}^a(\alpha) \otimes \mathrm{id} \otimes \ldots \otimes \mathrm{id}, \\ &\qquad \mathrm{S}^a(\mathcal{Q}) \otimes \mathrm{S}^{b_1}(\mathcal{P}) \otimes \ldots \otimes \mathrm{S}^{b_u}(\mathcal{P})\Big)\Bigg) \\ &= \sum_{\substack{a, b_1, \ldots, b_u \geqslant 1 \\ a+b_1+\ldots+b_u=i}} (-1)^u([\mathrm{S}^a(\mathcal{Q})] - [\mathrm{S}^a(\mathcal{P})]) \cdot [\mathrm{S}^{b_1}(\mathcal{P}) \otimes \ldots \otimes \mathrm{S}^{b_u}(\mathcal{P})] \\ &= \sigma^i([\mathcal{Q}] - [\mathcal{P}]) = \sigma^i v_1(\mathcal{P}, \alpha, \mathcal{Q}). \end{aligned}$$

We now assume that $U = \mathrm{Spec}(F)$ is affine. Then, by Proposition (2.1) on p. 393 in [B], the association $(\overset{m}{\oplus} FG, \alpha) \mapsto (\overset{m}{\oplus} \mathcal{O}_X G, \alpha, \overset{m}{\oplus} \mathcal{O}_X G)$ induces a *connecting homomorphism*

$$\partial : K_1(FG) \to K_0(j_1^*)$$

with $v_1 \circ \partial = 0$.

LEMMA 1.5. *Let* $\gcd(i, \mathrm{ord}(G))$ *be invertible on* X. *Then we have:*

$$\sigma^i \circ \partial = \partial \circ \sigma^i \quad \textit{in} \quad \mathrm{Hom}(K_1(FG), K_0(j_1^*)).$$

The multiplication maps are compatible with ∂ *(in the obvious sense), too. In particular, the multiplication on* Image(∂) *is trivial and the operation* σ^i *is a homomorphism on* Image(∂).

Proof. Easy.

PROPOSITION 1.6. *The following sequence is exact:*

$$K_1(FG) \xrightarrow{\partial} K_0(j_1^*) \xrightarrow{\nu_1} K_0(\mathcal{O}_X G) \xrightarrow{j^*} K_0(FG).$$

Proof. Apply Theorem (2.2)(b) on p. 396 in [B].

Now, let $\mathcal{H}$ denote the category of all coherent $\mathcal{O}_X G$-modules $\mathcal{V}$ which allow a resolution of length $\leqslant 1$ by locally projective coherent $\mathcal{O}_X G$-modules and for which $j^*(\mathcal{V}) = 0$ holds. Furthermore, let $K_0 T(\mathcal{O}_X G)$ denote the Grothendieck group of $\mathcal{H}$. By mapping the class $[\mathcal{V}]$ of a coherent $\mathcal{O}_X G$-module $\mathcal{V}$ with the resolution $0 \to \mathcal{P} \xrightarrow{\alpha} \mathcal{Q} \to \mathcal{V} \to 0$ and with $j^*(\mathcal{V}) = 0$ to the element $(\mathcal{P}, j^*(\alpha), \mathcal{Q})$ in $K_0(j_1^*)$, we obviously obtain a homomorphism

$$\psi : K_0 T(\mathcal{O}_X G) \to K_0(j_1^*).$$

PROPOSITION 1.7. *The homomorphism ψ is bijective in the following cases:*

(a) $X = \mathrm{Spec}(A)$ *is affine, F is the localization A_S of A by a multiplicative set S of non-zero-divisors in A, and $j\colon U = \mathrm{Spec}(F) \to X = \mathrm{Spec}(A)$ is the canonical morphism.*

(b) *The morphism $j : U = \mathrm{Spec}(F) \to X$ is an open immersion and the ideal $\mathcal{I}$ of the complement $Y := X \backslash U$ is locally generated by a non-zero-divisor.*

(c) *X is a Dedekind scheme (i.e., Noetherian, regular, irreducible, and* $\dim(X) = 1$*), F is the function field of X and $j\colon U = \mathrm{Spec}(F) \to X$ is the canonical morphism.*

Proof. The assertion (a) follows from (the proof of) Theorem (5.8) on p. 429 in [B]. In the case (b), we construct an inverse map as follows: Let $(\mathcal{P}, \alpha, \mathcal{Q})$ be a generator of $K_0(j_1^*)$. Then, the image of the composition

$$\tilde{\alpha} : \mathcal{P} \xrightarrow{\mathrm{can}} j_* j^*(\mathcal{P}) \xrightarrow{j_*(\alpha)} j_* j^*(\mathcal{Q}) = \cup_{n \geqslant 0} \mathcal{I}^{-n} \mathcal{Q}$$

(see Lemma 2 on p. 231 in [G 1] for the last equality) is contained in $\mathcal{I}^{-n}\mathcal{Q}$ for some $n \geqslant 0$. We put

$$\phi(\mathcal{P}, \alpha, \mathcal{Q}) := [\mathrm{coker}(\mathcal{P} \overset{\tilde{\alpha}}{\hookrightarrow} \mathcal{I}^{-n}\mathcal{Q})] - [\mathrm{coker}(\mathcal{Q} \overset{\mathrm{can}}{\hookrightarrow} \mathcal{I}^{-n}\mathcal{Q})] \in K_0 T(\mathcal{O}_X G).$$

As in loc. cit., one easily checks that the association $(\mathcal{P}, \alpha, \mathcal{Q}) \mapsto \phi(\mathcal{P}, \alpha, \mathcal{Q})$ induces a well-defined map $\phi : K_0(j_1^*) \to K_0 T(\mathcal{O}_X G)$ which is an inverse of ψ. In the case (c), we construct an inverse map as follows. Let $(\mathcal{P}, \alpha, \mathcal{Q})$ be a generator of $K_0(j_1^*)$. The isomorphism $\alpha\colon j^*(\mathcal{P}) \xrightarrow{\sim} j^*(\mathcal{Q})$ can be extended to an isomorphism $\mathcal{P}|_U \xrightarrow{\sim} \mathcal{Q}|_U$ where U is an open subset of X. The ideal $\mathcal{I}$ of the complement $Y := X \backslash U$ is then locally generated by a non-zero-divisor. We now define $\phi(\mathcal{P}, \alpha, \mathcal{Q})$ as in the case (b). As in loc. cit., one again easily checks that the association $(\mathcal{P}, \alpha, \mathcal{Q}) \mapsto \phi(\mathcal{P}, \alpha, \mathcal{Q})$ induces a well-defined map $\phi : K_0(j_1^*) \to K_0 T(\mathcal{O}_X G)$ which is an inverse of ψ.

Remark 1.8. We assume that one of the conditions (a), (b), (c) of Proposition 1.7 holds.

(a) The K-theory space of the exact category $\mathcal{H}$ is homotopy equivalent to the homotopy fibre of the canonical continuous map from the K-theory space of $\mathcal{M}_1$ to the K-theory space of the exact category consisting of all finitely generated projective FG-modules (see [G 1] and [AB]). Hence, we have a long exact (localization) sequence

$$\ldots \to K_1(FG) \to K_0T(\mathcal{O}_XG) \to K_0(\mathcal{O}_XG) \to K_0(FG).$$

The end of this sequence can be identified with the exact sequence in Proposition 1.6 by virtue of Proposition 1.7.

(b) If $\gcd(i, \mathrm{ord}(G))$ is invertible on X, we obtain a symmetric power operation $\sigma^i : K_0T(\mathcal{O}_XG) \to K_0T(\mathcal{O}_XG)$ by virtue of the isomorphism ψ. It maps the class $[\mathcal{V}]$ of a coherent $\mathcal{O}_XG$-module $\mathcal{V}$ in $\mathcal{H}$ with the resolution $0 \to \mathcal{P} \stackrel{\alpha}{\to} \mathcal{Q} \to \mathcal{V} \to 0$ to the element

$$\sum_{\substack{a, b_1, \ldots, b_u \geqslant 1 \\ a + b_1 + \ldots + b_u = i}} (-1)^u \Big[\mathrm{coker}\Big(\mathrm{Sym}^a(\mathcal{P}) \otimes \mathrm{Sym}^{b_1}(\mathcal{P}) \otimes \ldots \otimes \mathrm{Sym}^{b_u}(\mathcal{P}) \xrightarrow{\mathrm{Sym}^a(\alpha)\otimes \mathrm{id} \otimes \ldots \otimes \mathrm{id}} \mathrm{Sym}^a(\mathcal{Q}) \otimes \mathrm{Sym}^{b_1}(\mathcal{P}) \otimes \ldots \otimes \mathrm{Sym}^{b_u}(\mathcal{P})\Big)\Big].$$

(Note that the contribution of $a = 0$ would be 0 to this sum.)
Alternatively, the operation σ^i on $K_0T(\mathcal{O}_XG)$ can also be constructed as follows. Let $\mathcal{E}$ denote the exact category of all short exact sequences $0 \to \mathcal{P} \to \mathcal{Q} \to \mathcal{V} \to 0$ with $\mathcal{P}, \mathcal{Q} \in \mathcal{M}_1$ and $\mathcal{V} \in \mathcal{H}$. Then, we have a canonical isomorphism

$$K_0T(\mathcal{O}_XG) = K_0(\mathcal{H}) \cong \ker(K_0(\mathcal{E}) \to K_0(\mathcal{O}_XG), \quad [0 \to \mathcal{P} \to \mathcal{Q} \to \mathcal{V} \to 0] \mapsto [\mathcal{P}]).$$

The association

$$[0 \to \mathcal{P} \stackrel{\alpha}{\to} \mathcal{Q} \to \mathcal{V} \to 0] \mapsto [0 \to \mathrm{Sym}^i(\mathcal{P}) \stackrel{\mathrm{Sym}^i(\alpha)}{\longrightarrow} \mathrm{Sym}^i(\mathcal{Q}) \to \mathrm{coker}(\mathrm{Sym}^i(\alpha)) \to 0]$$

induces an operation σ^i on $K_0(\mathcal{E})$ as usual. It is then easy to check that its restriction to $K_0T(\mathcal{O}_XG)$ coincides with the operation σ^i constructed above. Moreover, the latter construction can be extended to all higher K-groups $K_q(\mathcal{H})$, $q \geqslant 0$, by using the methods of [G 2]. On the other hand, we have a symmetric power operation σ^i on the K-theory space of $\mathcal{M}_1$ and on the K-theory space of the category consisting of all finitely generated projective modules (see Section 1 in [K 3]), hence also on the homotopy fibre mentioned in (a) and finally on $K_q(\mathcal{H})$, $q \geqslant 0$. It seems to be plausible that these two constructions of σ^i on $K_q(\mathcal{H})$, $q \geqslant 0$, coincide. I hope to say more on this in a future paper.

2. Symmetric Power Operations on Locally Free Classgroups of Dedekind Schemes

Let X be a Dedekind scheme (i.e. Noetherian, regular, irreducible and $\dim(X) \leqslant 1$) with function field F, and let G be a finite group.

First, we recall the definition of the locally free classgroup $\mathrm{Cl}(\mathcal{O}_X G)$ (see [AB] or [BC]). Using the tools developed in Section 1 and a theorem of Swan, we then show that the locally free classgroup coincides with the analogously defined locally projective classgroup and that the operations σ^i, $i \geqslant 1$, constructed in Section 1 are homomorphisms on $\mathrm{Cl}(\mathcal{O}_X G)$. Furthermore, we prove the following concrete interpretations of the operations σ^i, $i \geqslant 1$, on $\mathrm{Cl}(\mathcal{O}_X G)$. Firstly, if G is Abelian and $\gcd(i, \mathrm{ord}(G)) = 1$, then pulling back the action of G on locally free $\mathcal{O}_X G$-modules along the automorphism $G \to G$, $g \mapsto g^i$, induces the operation σ^i on $\mathrm{Cl}(\mathcal{O}_X G)$. Secondly, if X is a smooth curve over an (algebraically closed or) finite field L such that the characteristic of L does not divide the order of G, then the identification of the locally free with the locally projective classgroup allows us a simple module theoretic description of the isomorphism between $\mathrm{Cl}(\mathcal{O}_X G)$ and $\mathrm{Hom}_{\mathrm{Galois}}(K_0(\bar{L}G), \mathrm{Cl}(\bar{X}))$ (developed in [AB]), and the operation σ^i on $\mathrm{Cl}(\mathcal{O}_X G)$ is dual to the adjoint Adams operation $\hat{\psi}^i$ on $K_0(\bar{L}G)$ with respect to this isomorphism. The proof of the latter result presented here can also be applied in the number field case and then simplifies the proof of Theorem 3.7 in [K 3].

A coherent $\mathcal{O}_X G$-module $\mathcal{P}$ is called *locally free over* $\mathcal{O}_X G$ iff the stalk $\mathcal{P}_x$ is a free $\mathcal{O}_{X,x}G$-module for all $x \in X$. By Proposition (30.17) on p. 627 in [CR], this is equivalent to the condition that $\mathcal{P}_x \otimes_{\mathcal{O}_{X,x}} \hat{\mathcal{O}}_{X,x}$ is a free $\hat{\mathcal{O}}_{X,x}G$-module for all closed points $x \in X$. (Here, $\hat{\mathcal{O}}_{X,x}$ denotes the $\mathfrak{m}_x$-adic completion of $\mathcal{O}_{X,x}$ and $\mathfrak{m}_x$ the maximal ideal in $\mathcal{O}_{X,x}$.) Let $K_0^{\mathrm{lf}}(\mathcal{O}_X G)$ denote the Grothendieck group of all coherent $\mathcal{O}_X G$-modules which are locally free over $\mathcal{O}_X G$.

Remark 2.1. Let $X = \mathrm{Spec}(A)$ be affine. Then we also write $K_0^{\mathrm{lf}}(AG)$ for $K_0^{\mathrm{lf}}(\mathcal{O}_X G)$. This is the Grothendieck group considered for instance in [F 1]. If A is a local Dedekind domain, then the rank (over AG) induces an isomorphism $K_0^{\mathrm{lf}}(AG) \tilde{\to} \mathbb{Z}$. If $\mathrm{char}(A) = 0$ and and no prime divisor of $\mathrm{ord}(G)$ is a unit in A, then any finitely generated projective AG-module is already locally free by Swan's theorem (see Theorem (32.11) on p. 676 in [CR]). The same holds if $p = \mathrm{char}(A) > 0$ and G is a p-group since then the group rings $\mathcal{O}_{X,x}G$, $x \in X$, are local rings. We will prove in Proposition 2.4 that the locally free classgroup defined below always coincides with the analogously defined locally projective classgroup.

DEFINITION 2.2. The group

$$\mathrm{Cl}(\mathcal{O}_X G) := \ker(K_0^{\mathrm{lf}}(\mathcal{O}_X G) \xrightarrow{\mathrm{can}} K_0^{\mathrm{lf}}(FG) \cong \mathbb{Z})$$

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is called the *locally free classgroup associated with* X *and* G.

Let $K_0T(\mathcal{O}_X G)$ (resp., $K_0^{\mathrm{lf}}T(\mathcal{O}_X G)$) denote the Grothendieck group of all coherent $\mathcal{O}_X G$-modules which are $\mathcal{O}_X$-torsion modules and which allow a resolution of length $\leqslant 1$ by locally projective (resp., locally free) $\mathcal{O}_X G$-modules. The notation $K_0T(\mathcal{O}_X G)$ obviously agrees with the notation introduced in Section 1 (if $j\colon U = \mathrm{Spec}(F) \to X$ is the canonical morphism).

LEMMA 2.3. *The canonical homomorphisms*

$$K_0T(\mathcal{O}_X G) \to \bigoplus_{x\in X \text{ closed}} K_0T(\mathcal{O}_{X,x}G)$$

and

$$K_0^{\mathrm{lf}}T(\mathcal{O}_X G) \to \bigoplus_{x\in X \text{ closed}} K_0^{\mathrm{lf}}T(\mathcal{O}_{X,x}G)$$

are bijective.

Proof. Let x be a closed point of X and V a finitely generated $\mathcal{O}_{X,x}G$-module which is $\mathcal{O}_{X,x}$-torsion and which allows an $\mathcal{O}_{X,x}G$-projective (resp., $\mathcal{O}_{X,x}G$-free) resolution $0 \to P \to Q \overset{\epsilon}{\to} V \to 0$. Let $i : \mathrm{Spec}(\mathcal{O}_{X,x}) \hookrightarrow X$ denote the inclusion. It suffices to show that $i_*(V)$ has a (global) locally projective (resp., locally free) resolution of length $\leqslant 1$. If P and Q are $\mathcal{O}_{X,x}G$-free, i.e. if they are isomorphic to $\overset{m}{\oplus}\mathcal{O}_{X,x}G$ for some $m \geqslant 0$, then the composition $\tilde{\epsilon} : \overset{m}{\oplus}\mathcal{O}_X G \overset{\mathrm{can}}{\longrightarrow} i_*(\overset{m}{\oplus}(\mathcal{O}_{X,x}G)) \overset{i_*(\epsilon)}{\longrightarrow} i_*(V)$ is surjective and $\ker(\tilde{\epsilon})$ is a locally free $\mathcal{O}_X G$-module, i.e., $i_*(V)$ has a locally free resolution of length 1. If P and Q are only projective over $\mathcal{O}_{X,x}G$, we choose a (non-equivariant) surjective homomorphism $\mathcal{E} \to i_*(V)$ with a locally free $\mathcal{O}_X$-module $\mathcal{E}$. Then, the induced homomorphism $\tilde{\epsilon} : \mathcal{O}_X G \otimes_{\mathcal{O}_X} \mathcal{E} \to i_*(V)$ is an equivariant surjection and the coherent $\mathcal{O}_X G$-module $\ker(\epsilon)$ is locally projective by Schanuel's Lemma, i.e., $i_*(V)$ has a locally projective resolution of length 1.

PROPOSITION 2.4. *The canonical homomorphism* $K_0^{\mathrm{lf}}(\mathcal{O}_X G) \to K_0(\mathcal{O}_X G)$ *induces an isomorphism*

$$\mathrm{Cl}(\mathcal{O}_X G) \tilde{\to} \ker(K_0(\mathcal{O}_X G) \overset{\mathrm{can}}{\longrightarrow} K_0(FG)).$$

Proof. We have a natural commutative diagram of groups

$$\begin{array}{ccccccc} K_1(FG) & \longrightarrow & K_0^{\mathrm{lf}}T(\mathcal{O}_X G) & \longrightarrow & K_0^{\mathrm{lf}}(\mathcal{O}_X G) & \longrightarrow & K_0^{\mathrm{lf}}(FG) \\ \| & & \downarrow & & \downarrow & & \downarrow \\ K_1(FG) & \longrightarrow & K_0T(\mathcal{O}_X G) & \longrightarrow & K_0(\mathcal{O}_X G) & \longrightarrow & K_0(FG); \end{array}$$

here, the lower row is the exact localization sequence constructed in Proposition 1.6 and Proposition 1.7; the maps in the upper row are defined as in the lower row;

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one can prove as in Section 1 or as in Theorem 1(ii) on p. 3 in [F 2] that also the upper sequence is exact. Thus, it suffices to prove that the map $K_0^{\mathrm{lf}} T(\mathcal{O}_X G) \to K_0 T(\mathcal{O}_X G)$ is bijective. By Lemma 2.3, it furthermore suffices to prove that the map $K_0^{\mathrm{lf}} T(\mathcal{O}_{X,x} G) \to K_0 T(\mathcal{O}_{X,x} G)$ is bijective for all closed points $x \in X$. We have a natural commutative diagram of groups

$$\begin{array}{ccccccccc}
K_1(\mathcal{O}_{X,x}G) & \longrightarrow & K_1(FG) & \longrightarrow & K_0^{\mathrm{lf}} T(\mathcal{O}_{X,x}G) & \longrightarrow & 0 & & \\
\| & & \| & & \downarrow & & & & \\
K_1(\mathcal{O}_{X,x}G) & \longrightarrow & K_1(FG) & \longrightarrow & K_0 T(\mathcal{O}_{X,x}G) & \longrightarrow & K_0(\mathcal{O}_{X,x}G) & \longrightarrow & K_0(FG)
\end{array}$$

with exact rows (e.g., see Theorem 1(ii) on p. 3 in [F 2]). Furthermore, the map $K_0(\mathcal{O}_{X,x}G) \to K_0(FG)$ is injective by a theorem of Swan (see Theorem (32.1) on p. 671 in [CR]). This proves Proposition 2.4.

Let $K_0(G, X)$ denote the Grothendieck group of all coherent $\mathcal{O}_X G$-modules which are locally free as $\mathcal{O}_X$-modules.

COROLLARY 2.5. *If* $\mathrm{ord}(G)$ *is invertible on* X, *the Cartan homomorphism* $K_0^{\mathrm{lf}}(\mathcal{O}_X G) \to K_0(G, \mathcal{O}_X)$ *induces an isomorphism*

$$\mathrm{Cl}(\mathcal{O}_X G) \tilde{\to} \ker\Big(K_0(G, X) \xrightarrow{\mathrm{can}} K_0(G, F) \cong K_0(FG)\Big).$$

Proof. This immediately follows from Proposition 2.4 and the fact that a finitely generated $\mathcal{O}_{X,x}G$-module is projective over $\mathcal{O}_{X,x}G$ if and only if it is projective over $\mathcal{O}_{X,x}$.

Now, we fix $i \in \mathbb{N}$ such that $\gcd(i, \mathrm{ord}(G))$ is invertible on X. By Section 1, we have a symmetric power operation $\sigma^i : K_0(\mathcal{O}_X G) \to K_0(\mathcal{O}_X G)$. By restricting, we obtain an operation σ^i on $\ker(K_0(\mathcal{O}_X G) \to K_0(FG)) \cong \mathrm{Cl}(\mathcal{O}_X G)$. In the same way, we obtain a multiplication map on $\mathrm{Cl}(\mathcal{O}_X G)$.

PROPOSITION 2.6. *The multiplication on* $\mathrm{Cl}(\mathcal{O}_X G)$ *is trivial and the operation* σ^i *on* $\mathrm{Cl}(\mathcal{O}_X G)$ *is a homomorphism.*

Proof. Since the canonical homomorphism $K_0 T(\mathcal{O}_X G) \to \mathrm{Cl}(\mathcal{O}_X G)$ is surjective, it suffices to show the corresponding assertions for $K_0 T(\mathcal{O}_X G)$ (by Lemma 1.4). By Lemma 2.3, we may furthermore assume that $X = \mathrm{Spec}(A)$ where A is a local Dedekind domain. Then, the connecting homomorphism $\partial : K_1(FG) \to K_0 T(\mathcal{O}_X G)$ is surjective (see the proof of Proposition 2.4), and Proposition 2.6 follows from Lemma 1.5.

THEOREM 2.7. *Let G be Abelian and* $\gcd(i, \mathrm{ord}(G)) = 1$. *We fix* $i' \in \mathbb{N}$ *such that* $ii' \equiv 1 \bmod e(G)$ *where* $e(G)$ *denotes the exponent of G. Let* $\phi_{i'}$ *denote both the*

$\mathcal{O}_X$-algebra automorphism $\mathcal{O}_X G \to \mathcal{O}_X G$ given by $[g] \mapsto [g^{i'}]$ and the automorphism of $K_0(\mathcal{O}_X G)$ or $\mathrm{Cl}(\mathcal{O}_X G)$ induced by the association $[\mathcal{P}] \mapsto [\mathcal{O}_X G \otimes_{\mathcal{O}_X G} \mathcal{P}]$ (where $\mathcal{O}_X G$ is considered as an $\mathcal{O}_X G$-algebra via $\phi_{i'}$). Then we have $\sigma^i = \phi_{i'}$ on $\mathrm{Cl}(\mathcal{O}_X G)$.

Proof. As in Proposition 2.6, it suffices to show the corresponding assertion for $K_1(FG)$ where $\phi_{i'}$ on $K_1(FG)$ is defined analogously. Since FG is semilocal and commutative, the canonical homomorphism $(FG)^\times \to K_1(FG)$ is bijective (see Corollary (9.2) on p. 267 in [B]). Under this isomorphism, the automorphism $\phi_{i'}$ corresponds to the restriction of the (analogously defined) automorphism $\phi_{i'}$ of FG. Thus it suffices to show that the following diagram commutes:

$$\begin{array}{ccc} (FG)^\times & \xrightarrow{\sim} & K_1(FG) \\ \big\downarrow{\scriptstyle \phi_{i'}} & & \big\downarrow{\scriptstyle \sigma^i} \\ (FG)^\times & \xrightarrow{\sim} & K_1(FG). \end{array}$$

Now, let W be a local domain of characteristic 0 whose residue class field is isomorphic to F. (If $\mathrm{char}(F) = 0$, we may choose F itself for W. If $p = \mathrm{char}(F) > 0$, the ring of infinite Witt vectors over F associated with the prime p is such a ring.) Since the group ring WG is semilocal and commutative, the canonical map $(WG)^\times \to K_1(FG)$ is bijective (see loc. cit.) and the canonical homomorphism $(WG)^\times \to (FG)^\times$ is surjective. Thus it suffices to show that the following diagram commutes:

$$\begin{array}{ccc} (WG)^\times & \xrightarrow{\sim} & K_1(WG) \\ \big\downarrow{\scriptstyle \phi_{i'}} & & \big\downarrow{\scriptstyle \sigma^i} \\ (WG)^\times & \xrightarrow{\sim} & K_1(WG). \end{array}$$

In a similar way, we conclude that it suffices to show that the corresponding diagram commutes if W is replaced by the quotient field Q of W and finally by the algebraic closure $\bar{Q}$ of Q. In the latter case, the commutativity follows from Theorem 1.6(d) in [K 3], Theorem 3.3 in [K 1], and Lemma 3.6(b) in [K 3]. This ends the proof of Theorem 2.7.

Remark 2.8. Let $\gcd(i, \mathrm{ord}(G)) = 1$. Theorem 2.7 implies in particular that $\sigma^{i+e(G)} = \sigma^i$ on $\mathrm{Cl}(\mathcal{O}_X G)$ if G is Abelian. This also holds if $X = \mathrm{Spec}(\mathcal{O}_F)$ where $\mathcal{O}_F$ is the ring of integers in a number field F (see Corollary 3.8 in [K 3]) or if X is a smooth curve over a finite field (this follows from Theorem 2.10). It is not clear to me whether this is true in general.

Now, let L be an algebraically closed field such that $\mathrm{char}(L)$ does not divide $\mathrm{ord}(G)$, and let $p : X \to \mathrm{Spec}(L)$ be an irreducible smooth curve over L. Then, for any finitely generated LG-module V, the pull-back $p^*(V)$ is a locally projective coherent $\mathcal{O}_X G$-module. Furthermore, for any locally projective coherent

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$\mathcal{O}_X G$-module $\mathcal{P}$, $\mathcal{P}'$, the $\mathcal{O}_X$-module $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{P}, \mathcal{P}') \cong \mathcal{P}^\vee \otimes_{\mathcal{O}_X} \mathcal{P}$ is again a locally projective $\mathcal{O}_X G$-module. Finally, for any locally projective $\mathcal{O}_X G$-module $\mathcal{P}$, the $\mathcal{O}_X$-module $\mathcal{P}^G$ of G-fixed elements is locally free since $\mathrm{ord}(G)$ is invertible on X. Thus, we obtain a well-defined homomorphism

$$\begin{aligned} K_0(\mathcal{O}_X G) &\longrightarrow \mathrm{Hom}(K_0(LG), K_0(X)) \\ [\mathcal{P}] &\longmapsto ([V] \mapsto [\mathrm{Hom}_{\mathcal{O}_X G}(p^*(V), \mathcal{P})]). \end{aligned}$$

This homomorphism is bijective (see the proof of Proposition (2.2) on p. 133 in [S]) and induces an isomorphism

$$\mathrm{Cl}(\mathcal{O}_X G) \xrightarrow{\sim} \mathrm{Hom}(K_0(LG), \mathrm{Cl}(X)) \tag{2}$$

by Proposition 2.4.

Let ψ^i denote the ith Adams operation on $K_0(LG)$. In the sequel, we will identify $K_0(LG)$ with the ring of virtual characters of G. Then ψ^i maps a character χ to the character $G \to L$, $g \mapsto \chi(g^i)$. Let $\hat{\psi}^i$ denote the adjoint operation (with respect to the usual character pairing). Note that the assumption $\mathrm{char}(L) \nmid \mathrm{ord}(G)$ implies that $\gcd(i, \mathrm{ord}(G))$ is invertible on X for all $i \in \mathbb{N}$. To avoid further definitions we will use the somewhat complicated notation $\mathrm{Hom}(f, B)$ (and similar notations) in the usual functorial way (for any Abelian group B and any homomorphism $f: A \to A'$).

THEOREM 2.9. *Under the isomorphism (2), the operation σ^i on $\mathrm{Cl}(\mathcal{O}_X G)$ corresponds to the endomorphism $\mathrm{Hom}(\hat{\psi}^i, \mathrm{Cl}(X))$ of $\mathrm{Hom}(K_0(LG), \mathrm{Cl}(X))$.*

Proof. By Theorem 3.3 on p. 145 in [K 1] and Theorem 1.6(d)(ii) in [K 3], the operation σ^i on $K_1(FG)$ (constructed, e.g. in Section 1) corresponds to the endomorphism $\mathrm{Hom}(\hat{\psi}^i, K_1(F))$ of $\mathrm{Hom}(K_0(LG), K_1(F))$ under the isomorphism

$$\begin{aligned} K_1(FG) &\xrightarrow{\sim} \mathrm{Hom}(K_0(LG), K_1(F)) \\ (P, \alpha) &\longmapsto \big([V] \mapsto (\mathrm{Hom}_{FG}(F \otimes_L V, P), \mathrm{Hom}_{FG}(F \otimes_L V, \alpha))\big). \end{aligned}$$

For any closed point $x \in X$, the association

$$[M] \mapsto ([V] \mapsto [\mathrm{Hom}_{\mathcal{O}_{X,x} G}(\mathcal{O}_{X,x} \otimes_L V, M)])$$

induces an isomorphism $K_0 T(\mathcal{O}_{X,x} G) \xrightarrow{\sim} \mathrm{Hom}(K_0(LG), K_0 T(\mathcal{O}_{X,x}))$ (both sides are isomorphic to $K_0(LG)$!) such that the following diagram commutes:

$$\begin{array}{ccc} K_1(FG) & \xrightarrow{\quad \partial \quad} & K_0 T(\mathcal{O}_{X,x} G) \\ \downarrow \wr & & \downarrow \wr \\ \mathrm{Hom}(K_0(LG), K_1(F)) & \xrightarrow{\mathrm{Hom}(K_0(LG), \partial)} & \mathrm{Hom}(K_0(LG), K_0 T(\mathcal{O}_{X,x})). \end{array}$$

Hence, by Lemma 1.5, the operation σ^i on $K_0 T(\mathcal{O}_{X,x} G)$ corresponds to the endomorphism $\mathrm{Hom}(\hat{\psi}^i, K_0 T(\mathcal{O}_{X,x}))$ of $\mathrm{Hom}(K_0(LG), K_0 T(\mathcal{O}_{X,x}))$. Under the

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isomorphism of Lemma 2.3, the operation σ^i on $K_0T(\mathcal{O}_XG)$ obviously corresponds to the endomorphism $\bigoplus_{x\in X \text{ closed}} \sigma^i$ of $\bigoplus_{x\in X \text{ closed}} K_0T(\mathcal{O}_{X,x}G)$. Thus, under the isomorphism

$$K_0T(\mathcal{O}_XG) \cong \mathrm{Hom}(K_0(LG), K_0T(\mathcal{O}_X)),\ [\mathcal{M}] \mapsto ([V] \mapsto [\mathrm{Hom}_{\mathcal{O}_XG}(p^*(V), \mathcal{M})]),$$

the operation σ^i on $K_0T(\mathcal{O}_XG)$ corresponds to the endomorphism $\mathrm{Hom}(\hat{\psi}^i, K_0T(\mathcal{O}_X))$ of $\mathrm{Hom}(K_0(LG), K_0T(\mathcal{O}_X))$. Furthermore, the following diagram obviously commutes:

$$\begin{array}{ccc} K_0T(\mathcal{O}_XG) & \xrightarrow{\text{can}} & K_0(\mathcal{O}_XG) \\ \downarrow\wr & & \downarrow\wr \\ \mathrm{Hom}(K_0(LG), K_0T(\mathcal{O}_X)) & \xrightarrow{\text{can}} & \mathrm{Hom}(K_0(LG), K_0(X)). \end{array}$$

Now, Theorem 2.9 follows from Lemma 1.4 and Proposition 1.6.

Now, let L be a finite field with $\mathrm{char}(L) \nmid \mathrm{ord}(G)$ and $p: X \to \mathrm{Spec}(L)$ an irreducible smooth curve over L. Let $\bar{L}$ denote an algebraic closure of L and $\bar{p}: \bar{X} := X \times_L \bar{L} \to \mathrm{Spec}(\bar{L})$ the corresponding curve over $\bar{L}$. Then, the composition of the canonical map $K_0(\mathcal{O}_XG) \to K_0(\mathcal{O}_{\bar{X}}G)$ with the isomorphism $K_0(\mathcal{O}_{\bar{X}}G) \cong \mathrm{Hom}(K_0(\bar{L}G), K_0(\bar{X}))$ constructed above obviously induces a homomorphism

$$K_0(\mathcal{O}_XG) \to \mathrm{Hom}_{\mathrm{Gal}(\bar{L}/L)}(K_0(\bar{L}G), K_0(\bar{X})).$$

THEOREM 2.10. *This homomorphism is bijective. In particular, we obtain an isomorphism*

$$\mathrm{Cl}(\mathcal{O}_XG) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Gal}(\bar{L}/L)}(K_0(\bar{L}G), \mathrm{Cl}(\bar{X})).$$

Under this isomorphism, the operation σ^i on $\mathrm{Cl}(\mathcal{O}_XG)$ corresponds to the endomorphism $\mathrm{Hom}_{\mathrm{Gal}(\bar{L}/L)}(\hat{\psi}^i, \mathrm{Cl}(\bar{X}))$ of $\mathrm{Hom}_{\mathrm{Gal}(\bar{L}/L)}(K_0(\bar{L}G), \mathrm{Cl}(\bar{X}))$.

Proof. The bijectivity can be shown as in Section 6 of [AB] using Morita equivalence and the Galois descent property $K_0(X \times_L L') \cong K_0(\bar{X})^{\mathrm{Gal}(\bar{L}/L')}$ (for any finite extension $L \subseteq L' \subset \bar{L}$ of L). Proposition 2.4 then yields the Hom-description of the classgroup. The last assertion immediately follows from Theorem 2.9.

3. Equivariant Riemann–Roch Type Formulas for Tame Extensions of Dedekind Schemes

The aim of this section is to prove Theorem A and Theorem B presented in the introduction.

Let Y be a Dedekind scheme and G a finite group of order n. Let $\mathrm{Ind}_1^G: \mathrm{Cl}(\mathcal{O}_Y) \to \mathrm{Cl}(\mathcal{O}_YG)$ and $\mathrm{Ind}_1^G: K_0T(\mathcal{O}_Y) \to K_0^{\mathrm{lf}}T(\mathcal{O}_YG)$ denote the induction maps. The following lemma generalizes Lemma 2.6 on p. 933 in [BC].

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LEMMA 3.1. *The image of the natural multiplication maps*

$$K_0T(\mathcal{O}_Y)\times K_0^{\mathrm{lf}}(\mathcal{O}_YG)\to K_0^{\mathrm{lf}}T(\mathcal{O}_YG)\quad\text{and}\quad \mathrm{Cl}(\mathcal{O}_Y)\times K_0^{\mathrm{lf}}(\mathcal{O}_YG)\to \mathrm{Cl}(\mathcal{O}_YG)$$

is contained in $\mathrm{Ind}_1^G K_0T(\mathcal{O}_Y)$ *resp.* $\mathrm{Ind}_1^G\mathrm{Cl}(\mathcal{O}_Y)$.

Proof. The assertion for the first map is clear. The assertion for the second map follows from this since the natural map $K_0T(\mathcal{O}_Y)\to\mathrm{Cl}(\mathcal{O}_Y)$ is surjective.

Now, let F/E be a finite Galois extension of the function field E of Y with Galois group G. Let X denote the normalization of Y in F. Then X is a Dedekind scheme endowed with a natural G-action and the corresponding G-morphism $f: X\to Y$ is finite (see the proof of Theorem (8.1) on p. 47 in [N]). We assume that f is tamely ramified. As in Lemma 5.5 in [K 3], one easily shows that then, for any locally free coherent $\mathcal{O}_X$-module $\mathcal{E}$ with (semilinear) G-action, the direct image $f_*(\mathcal{E})$ is a locally free coherent $\mathcal{O}_YG$-module in the sense of Section 2. Let $K_0(G,X)$ denote the Grothendieck group of all such modules $\mathcal{E}$. Thus, we have a homomorphism

$$f_*: K_0(G,X)\to K_0^{\mathrm{lf}}(\mathcal{O}_YG),\quad [\mathcal{E}]\mapsto[f_*(\mathcal{E})].$$

The different $\mathcal{D}:=\mathcal{D}_{X/Y}:=\mathcal{A}\mathrm{nn}_{\mathcal{O}_X}(\Omega^1_{X/Y})$ is a G-stable ideal in $\mathcal{O}_X$, hence a module $\mathcal{E}$ as above. The following proposition generalizes formula (2.8) on p. 933 in [BC].

PROPOSITION 3.2. *For all* $x\in K_0(G,X)$ *we have:*

$$f_*\left(x\cdot\sum_{i=0}^{n-1}[\mathcal{D}^{-i}]\right)=0\quad\text{in}\quad K_0^{\mathrm{lf}}(\mathcal{O}_YG)/(\mathrm{Ind}_1^G\mathrm{Cl}(\mathcal{O}_Y)\oplus n\mathbb{Z}[\mathcal{O}_YG]).$$

Proof. We may assume that $x=[\mathcal{E}]$ where $\mathcal{E}$ is a module as above. Let $r:=\mathrm{rank}_{\mathcal{O}_X}(\mathcal{E})$. Then we have:

$$\begin{aligned}&\sum_{i=0}^{n-1}\Big([f_*(\mathcal{E}\otimes\mathcal{D}^{-i})]-r[\mathcal{O}_YG]\Big)\\&=n\Big([f_*(\mathcal{E})]-r[\mathcal{O}_YG]\Big)+\sum_{i=1}^{n-1}\Big([f_*(\mathcal{E}\otimes\mathcal{D}^{-i})]-[f_*(\mathcal{E})]\Big)\quad\text{in}\quad \mathrm{Cl}(\mathcal{O}_YG).\end{aligned}$$

In the sequel, let $\mathcal{M}\mapsto\mathcal{M}^t$ denote the forgetful functor from the category of $\mathcal{O}_YG$-modules to the category of $\mathcal{O}_Y$-modules. (We will consider $\mathcal{M}^t$ also as an $\mathcal{O}_YG$-module with trivial G-action.) Then, the elements $[f_*(\mathcal{O}_X)^t]-n[\mathcal{O}_Y]$ and $[f_*(\mathcal{E})^t]-nr[\mathcal{O}_Y]$ are contained in $\mathrm{Cl}(\mathcal{O}_Y)$. Hence, we have by Lemma 3.1:

$$\begin{aligned}&n([f_*(\mathcal{E})]-r[\mathcal{O}_YG])\\&\quad=[f_*(\mathcal{O}_X)^t\otimes f_*(\mathcal{E})]-[f_*(\mathcal{E})^t\otimes\mathcal{O}_YG]\quad\text{in}\quad \mathrm{Cl}(\mathcal{O}_YG)/\mathrm{Ind}_1^G\mathrm{Cl}(\mathcal{O}_Y).\end{aligned}$$

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The homomorphism

$$f_*(\mathcal{O}_X)^t \otimes f_*(\mathcal{E}) \to f_*(\mathcal{E})^t \otimes \mathcal{O}_Y G, \quad a \otimes b \mapsto \sum_{g \in G} ag(b) \otimes [g^{-1}],$$

of $\mathcal{O}_Y G$-modules is generically bijective since F/E is a Galois extension and any finitely generated module over the twisted group ring $F\#G$ is isomorphic to $\overset{m}{\oplus} F$ for some $m \geqslant 0$. In particular, this map is a monomorphism and the cokernel $\mathcal{R}_{X/Y}(\mathcal{E})$ is an $\mathcal{O}_Y G$-torsion module. Hence, it suffices to show that we have:

$$[\mathcal{R}_{X/Y}(\mathcal{E})] = \sum_{i=1}^{n-1} [f_*(\mathcal{E} \otimes \mathcal{D}^{-i}/\mathcal{O}_X)] \quad \text{in} \quad K_0^{\mathrm{lf}} T(\mathcal{O}_Y G)/\mathrm{Ind}_1^G K_0 T(\mathcal{O}_Y).$$

(The notation $\mathcal{E} \otimes \mathcal{D}^{-i}/\mathcal{O}_X$ means $\mathcal{E} \otimes (\mathcal{D}^{-i}/\mathcal{O}_X)$, of course. Similar simplifying notations will be used also below.) By Lemma 2.3, it furthermore suffices to show that we have

$$[\mathcal{R}_{X/Y}(\mathcal{E})_y] = \sum_{i=1}^{n-1} [f_*(\mathcal{E} \otimes \mathcal{D}^{-i}/\mathcal{O}_X)_y] \quad \text{in} \quad K_0^{\mathrm{lf}} T(\mathcal{O}_{Y,y} G)/\mathrm{Ind}_1^G K_0 T(\mathcal{O}_{Y,y})$$

for all closed points $y \in Y$.

We now fix $y \in Y$ and $x \in X$ with $f(x) = y$. Let $G_x := \{g \in G : xg = x\}$ denote the decomposition group of x. Furthermore, let $f' : X' := \mathrm{Spec}(\hat{\mathcal{O}}_{X,x}) \to \mathrm{Spec}(\hat{\mathcal{O}}_{Y,y}) =: Y'$ denote the induced G_x-morphism where $\hat{}$ denotes completion. We identify the category of coherent torsion modules on Y' with the category of coherent torsion modules on Y supported in y. An easy generalization of Corollary 3.11(b) on p. 239 in [C] shows that $\mathcal{R}_{X/Y}(\mathcal{E})_y$ is isomorphic to the direct sum of $[G : G_x]$ copies of $\mathrm{Ind}_{G_x}^G \mathcal{R}_{X'/Y'}(\hat{\mathcal{E}}_x)$. Furthermore, it is clear that $f_*(\mathcal{E} \otimes \mathcal{D}_{X/Y}^{-i}/\mathcal{O}_X)_y$ is isomorphic to $\mathrm{Ind}_{G_x}^G f'_*(\hat{\mathcal{E}}_x \otimes \mathcal{D}_{X'/Y'}^{-i}/\mathcal{O}_{X'})$ for all $i \geqslant 0$. For $i \equiv j \bmod \mathrm{ord}(G_x)$, we finally have

$$\begin{aligned}&[f'_*(\hat{\mathcal{E}}_x \otimes \mathcal{D}_{X'/Y'}^{-i}/\mathcal{O}_{X'})] \\ &\quad = [f'_*(\hat{\mathcal{E}}_x \otimes \mathcal{D}_{X'/Y'}^{-j}/\mathcal{O}_{X'})] \quad \text{in} \quad K_0^{\mathrm{lf}} T(\mathcal{O}_{Y'} G_x)/\mathrm{Ind}_1^{G_x} K_0 T(\mathcal{O}_{Y'})\end{aligned}$$

since the ideal $\mathcal{D}_{X'/Y'}^{\mathrm{ord}(G_x)}$ of $\mathcal{O}_{X'}$ can be written as $(f')^*(\mathfrak{a})$ with some ideal $\mathfrak{a}$ in $\mathcal{O}_{Y'}$ and since, for any locally free coherent $\mathcal{O}_{Y'}G$-module $\mathcal{P}$, we have

$$[\mathcal{P}/\mathfrak{a}\mathcal{P}] = [\mathcal{O}/\mathfrak{a} \otimes \mathcal{P}] = 0 \quad \text{in} \quad K_0^{\mathrm{lf}}(\mathcal{O}_{Y'} G_x)/\mathrm{Ind}_1^{G_x} K_0 T(\mathcal{O}_{Y'})$$

by Lemma 3.1. Thus it suffices to prove that

$$[\mathcal{R}_{X'/Y'}(\hat{\mathcal{E}}_x)] = \sum_{i=1}^{\mathrm{ord}(G_x)-1} [f'_*(\hat{\mathcal{E}}_x \otimes \mathcal{D}_{X'/Y'}^{-i}/\mathcal{O}_{X'})] \quad \text{in} \quad K_0 T(\mathcal{O}_{Y'} G_x)/\mathrm{Ind}_1^{G_x} K_0 T(\mathcal{O}_{Y'}).$$

We now write G for G_x, X for X', $\mathcal{E}$ for $\hat{\mathcal{E}}_x$, and so on. Let $\Delta \subseteq G$ denote the inertia group, e the order of Δ, $\mathfrak{P}$ the ideal in $\mathcal{O}_X$ which corresponds to the closed point in X,

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and χ the Δ-module $\mathfrak{P}/\mathfrak{P}^2$. We decompose $f: X \to Y$ into $X \xrightarrow{g} Z \xrightarrow{h} Y$ where $Z := \mathrm{Spec}(\Gamma(X, \mathcal{O}_X)^{\Delta})$; i.e. the function field of Z is the inertia field of F/E. Since $K_0(G, X)$ is generated by the classes of fractional G-stable ideals in $\mathcal{O}_X$ (see Lemma 5.5(c) in [K 3]), we may assume that $\mathcal{E} = \mathfrak{P}^j$ for some $j \in \mathbb{Z}$. An easy generalization of Corollary 3.8 on p. 236 and Theorem 2.8 on p. 222 in [C] shows that we have the following isomorphisms:

$$\begin{aligned}\mathcal{R}_{X/Y}(\mathfrak{P}^j) &\cong \mathrm{Ind}_{\Delta}^{G} h_*(\mathcal{R}_{X/Z}(\mathfrak{P}^j))\\ &\cong \mathrm{Ind}_{\Delta}^{G} h_*\left(\bigoplus_{i=1}^{e-1} g_*\left((\mathfrak{P}^j/\mathfrak{P}^{j+i})^t \otimes \chi^{j+i}\right)\right)\\ &\cong \mathrm{Ind}_{\Delta}^{G} f_*\left(\bigoplus_{i=1}^{e-1}(\mathfrak{P}^j/\mathfrak{P}^{j+i})^t \otimes \chi^{j+i}\right).\end{aligned}$$

Thus we have

$$[\mathcal{R}_{X/Y}(\mathfrak{P}^j)] = \sum_{i=1}^{e-1} i[\mathrm{Ind}_{\Delta}^{G} f_*(\chi^{j+i})] \quad \text{in} \quad K_0T(\mathcal{O}_Y G).$$

Since $\mathcal{D} = \mathfrak{P}^{e-1}$ and $\mathfrak{P}^e = f^*(\mathfrak{p})$ (where $\mathfrak{p}$ is the ideal in $\mathcal{O}_Y$ which corresponds to the closed point in Y), we can conclude as above using Lemma 3.1:

$$\begin{aligned}&\sum_{i=1}^{n-1}[f_*(\mathfrak{P}^j \otimes \mathcal{D}^{-i}/\mathcal{O}_X)]\\ &\quad = \frac{n}{e}\sum_{i=1}^{e-1}[f_*(\mathfrak{P}^j \otimes \mathcal{D}^{-i}/\mathcal{O}_X)] = \frac{n}{e}\sum_{i=1}^{e-1}[f_*(\mathfrak{P}^{j+i}/\mathfrak{P}^{j+e})]\\ &\quad = \frac{n}{e}\sum_{i=1}^{e-1} i[f_*(\mathfrak{P}^{j+i}/\mathfrak{P}^{j+i+1})] \quad \text{in} \quad K_0T(\mathcal{O}_Y G)/\mathrm{Ind}_1^G K_0T(\mathcal{O}_Y).\end{aligned}$$

(For the second equality note that $\mathcal{D}^{-i}/\mathcal{O}_X = \mathfrak{P}^{i(1-e)}/\mathcal{O}_X$ is isomorphic to $\mathfrak{P}^i/\mathfrak{P}^{ie}$ which has a filtration with quotients $\mathfrak{P}^i/\mathfrak{P}^e$ and $\mathfrak{P}^e/\mathfrak{P}^{ie}$.) Thus it suffices to prove that the $\mathcal{O}_Y G$-modules $\mathrm{Ind}_{\Delta}^{G} f_*(\chi^i)$ and $\overset{n/e}{\oplus} f_*(\mathfrak{P}^i/\mathfrak{P}^{i+1})$ are isomorphic for all $i \in \mathbb{Z}$. For this, we consider the $\mathcal{O}_Y G$-homomorphism

$$\begin{aligned} h_*(\mathcal{O}_Z)^t \otimes f_*(\mathfrak{P}^i/\mathfrak{P}^{i+1}) &\longrightarrow \mathrm{Maps}_{\Delta}(G, f_*(\mathfrak{P}^i/\mathfrak{P}^{i+1}))\\ a \otimes b &\longmapsto (g \mapsto ag(b)).\end{aligned}$$

This homomorphism is bijective since h is unramified (e.g. see pp. 214-215 in [C]). Furthermore, the left hand side is obviously isomorphic to $\overset{n/e}{\oplus} f_*(\mathfrak{P}^i/\mathfrak{P}^{i+1})$ and the right hand side is isomorphic to $\mathrm{Ind}_{\Delta}^{G} f_*(\chi^i)$. So, Proposition 3.2 is proved.

Now, let $k \in \mathbb{N}$ with $\gcd(k, n) = 1$ and $k' \in \mathbb{N}$ with $kk' \equiv 1 \bmod n$. Let σ^k denote the kth symmetric power operation on $K_0(G, Y)$ and ψ^k the kth Adams operation on $K_0(G, Y)$ or $K_0(G, X)$ (e.g., see Section 1 in [K 3]). The composition of the

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map $f_* : K_0(G, X) \to K_0^{\mathrm{lf}}(\mathcal{O}_Y G)$ with the Cartan homomorphism $K_0^{\mathrm{lf}}(\mathcal{O}_Y G) \to K_0(G, Y)$ is denoted by f_* again. Finally, let $\hat{K}_0(G, Y)[k^{-1}]$ denote the J-adic completion of $K_0(G, Y)[k^{-1}]$ where $J := \ker(K_0(G, Y) \xrightarrow{\mathrm{rank}} \mathbb{Z})[k^{-1}]$ is the augmentation ideal in $K_0(G, Y)[k^{-1}]$.

THEOREM 3.3. *For all* $x \in K_0(G, X)$ *we have*

$$\sigma^k(f_*(x) - \mathrm{rank}(x) \cdot [\mathcal{O}_Y G]) = f_*\left(\sum_{i=0}^{k'-1} [\mathcal{D}^{-ik}] \cdot \psi^k(x)\right)$$

in $\hat{K}_0(G, Y)[k^{-1}]/(\mathrm{Ind}_1^G K_0(Y))\hat{K}_0(G, Y)[k^{-1}]$.

Proof. Let

$$\hat{f}_* : \hat{K}_0(G, X)[k^{-1}] := K_0(G, X) \otimes_{K_0(G,Y)} \hat{K}_0(G, Y)[k^{-1}] \to \hat{K}_0(G, Y)[k^{-1}]$$

denote the homomorphism which is induced by $f_* : K_0(G, X) \to K_0(G, Y)$, and let $\theta^k(\mathcal{D}^{-1}) := 1 + [\mathcal{D}^{-1}] + \cdots + [\mathcal{D}^{-(k-1)}] \in K_0(G, X)$ denote the Bott element. As in Theorem 5.4 in [K 3], one easily deduces the following assertion from the equivariant Adams–Riemann–Roch theorem (see Theorem (4.5) in [K 2]): The element $\theta^k(\mathcal{D}^{-1})$ is invertible in $\hat{K}_0(G, X)[k^{-1}]$ and we have

$$\psi^k(f_*(x)) = \hat{f}_*(k \cdot \theta^k(\mathcal{D}^{-1})^{-1} \cdot \psi^k(x)) \quad \text{in} \quad \hat{K}_0(G, Y)[k^{-1}]$$

for all $x \in K_0(G, X)$. Furthermore, we have

$$\theta^k(\mathcal{D}^{-1}) \cdot \left(\sum_{i=0}^{k'-1} [\mathcal{D}^{-ik}]\right) = \sum_{j=0}^{k-1} \sum_{i=0}^{k'-1} [\mathcal{D}^{-(j+ik)}] = \sum_{i=0}^{kk'-1} [\mathcal{D}^{-i}] = [\mathcal{O}_X] + \sum_{i=1}^{kk'-1} [\mathcal{D}^{-i}]$$

in $K_0(G, X)$. Thus, we have

$$\theta^k(\mathcal{D}^{-1})^{-1} = \sum_{i=0}^{k'-1} [\mathcal{D}^{-ik}] - \theta^k(\mathcal{D}^{-1})^{-1} \sum_{i=1}^{kk'-1} [\mathcal{D}^{-i}] \quad \text{in} \quad \hat{K}_0(G, X)[k^{-1}].$$

Hence, we obtain the equality

$$\begin{aligned} \psi^k(f_*(x)) &= k \cdot \hat{f}_*\left(\left(\sum_{i=0}^{k'-1} [\mathcal{D}^{-ik}] - \theta^k(\mathcal{D}^{-1})^{-1} \cdot \sum_{i=1}^{kk'-1} [\mathcal{D}^{-i}]\right) \cdot \psi^k(x)\right) \\ &= k \cdot f_*\left(\sum_{i=0}^{k'-1} [\mathcal{D}^{-ik}] \cdot \psi^k(x)\right) \quad \text{in} \quad \hat{K}_0(G, Y)[k^{-1}]/(\mathrm{Ind}_1^G K_0(Y))\hat{K}_0(G, Y)[k^{-1}] \end{aligned}$$

by Proposition 3.2. Since we have $\psi^k = k \cdot \sigma^k$ on $\mathrm{Cl}(\mathcal{O}_Y G)$ (by Proposition 2.6) and $\psi^k([\mathcal{O}_Y G]) = [\mathcal{O}_Y G]$ (by Theorem 1.6(e) in [K 3]), this implies Theorem 3.3.

Note that the formula of Theorem 3.3 lives within the somewhat complicated group $\hat{K}_0(G, Y)[k^{-1}]/(\mathrm{Ind}_1^G K_0(Y))\hat{K}_0(G, Y)[k^{-1}]$. The next proposition computes this group in a special case.

PROPOSITION 3.4. *Let L be an algebraically closed field, Y a projective smooth irreducible curve over L, and $n = \mathrm{ord}(G)$ a power of a prime $l \neq \mathrm{char}(L)$. Let I denote the augmentation ideal in $K_0(LG)$. Then we have:*

$$\hat{K}_0(G, Y)[k^{-1}] \cong K_0(Y)[k^{-1}] \oplus I \otimes \mathbb{Z}_l \oplus I \otimes \mathbb{Z}_l;$$

under this isomorphism, the extended ideal $(\mathrm{Ind}_1^G K_0(Y))\hat{K}_0(G, Y)[k^{-1}]$ corresponds to the subgroup $\{(ny, ([\mathbb{Z}G] - n) \otimes \mathrm{rank}(y), ([\mathbb{Z}G] - n) \otimes \deg\det(y)) : y \in K_0(Y)[k^{-1}]\}$ of $K_0(Y)[k^{-1}] \oplus I \otimes \mathbb{Z}_l \oplus I \otimes \mathbb{Z}_l$.

Proof. The canonical map $K_0(LG) \otimes K_0(Y) \to K_0(G, Y)$ is an isomorphism by Proposition (2.2) on p. 133 in [S]. Since the augmentation ideal of $K_0(Y)$ is nilpotent (e.g., by Proposition 2.6) and the I-adic topology on I coincides with the l-adic topology (see Proposition 1.1 on p. 277 in [AT]), the completion $\hat{K}_0(G, Y)[k^{-1}]$ is isomorphic to the direct sum of $K_0(Y)[k^{-1}]$ and the l-adic completion of $I \otimes K_0(Y)[k^{-1}]$. Furthermore, we have $K_0(Y) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathrm{Pic}^0(Y)$ where $\mathrm{Pic}^0(Y)$ denotes the group of line bundles on Y of degree 0. Since $\mathrm{Pic}^0(Y)$ is an l-divisible group (see item (iv) on p. 42 in [M]), the l-adic completion of $I \otimes K_0(Y)[k^{-1}]$ is isomorphic to $I \otimes \mathbb{Z}_l \oplus I \otimes \mathbb{Z}_l$. Thus, we have

$$\hat{K}_0(G, Y)[k^{-1}] \cong K_0(Y)[k^{-1}] \oplus I \otimes \mathbb{Z}_l \oplus I \otimes \mathbb{Z}_l.$$

Under the isomorphism $K_0(G, Y) \cong K_0(LG) \otimes K_0(Y)$, the ideal $\mathrm{Ind}_1^G K_0(Y)$ of the ring $K_0(G, Y)$ corresponds to the ideal $\mathrm{Ind}_1^G K_0(L) \otimes K_0(Y)$ $(\cong K_0(Y))$ of $K_0(LG) \otimes K_0(Y)$ which is generated by the element $[\mathbb{Z}G] \otimes 1 = n \otimes 1 + ([\mathbb{Z}G] - n) \otimes 1$. One easily deduces the second assertion of Proposition 3.4 from this. (Note that $[\mathbb{Z}G] \cdot x = 0$ for all $x \in I$.)

Now, let $f_* : K_0(G, X) \to \mathrm{Cl}(\mathcal{O}_Y G)$ denote the composition of $f_* : K_0(G, X) \to K_0^{\mathrm{lf}}(\mathcal{O}_Y G)$ with the canonical projection $K_0^{\mathrm{lf}}(\mathcal{O}_Y G) \cong \mathrm{Cl}(\mathcal{O}_Y G) \oplus \mathbb{Z}[\mathcal{O}_Y G] \to \mathrm{Cl}(\mathcal{O}_Y G)$.

THEOREM 3.5. *Suppose that one of the following conditions holds:*

(a) *$Y = \mathrm{Spec}(\mathcal{O}_E)$ where $\mathcal{O}_E$ is the ring of integers in a number field E.*
(b) *Y is an irreducible projective smooth curve over a finite field L and* $\gcd(\mathrm{char}(L), n) = 1$.
(c) *The group G is Abelian and $f : X \to Y$ is unramified.*
(d) $k = 1$.

Then for all $x \in K_0(G, X)$ we have,

$$\sigma^k(f_*(x)) = f_*\left(\sum_{i=0}^{k'-1} [\mathcal{D}^{-ik}] \cdot \psi^k(x)\right) \quad \textit{in} \quad \mathrm{Cl}(\mathcal{O}_Y G)/\mathrm{Ind}_1^G \mathrm{Cl}(\mathcal{O}_Y).$$

Proof. In the case (a), Theorem 3.5 can be deduced from Corollary 2.7 on p. 933 in [BC] using Theorem 3.7 and Lemma 5.5 in [K 3] (see also the proof of Theorem 5.6 in

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[K 3]). The same can be done in the case (b) by using Lemma 3.6(a) in [K 3] and Theorem 2.9 (in place of Theorem 3.7 in [K 3]) and an obvious generalization of Lemma 5.5(c) in [K 3]. (For completeness sake, we mention that it is easy to check that the additional assumptions in Theorem 2.1 on p. 932 in [BC] about the absolute discriminant or the characteristic of E are not necessary for Corollary 2.7 on p. 933 in [BC].) We now prove Theorem 3.5 in the case (c), i.e., we want to show the formula

$$\sigma^k(f_*(x)) = k' \cdot f_*(\psi^k(x)) \quad \text{in} \quad \mathrm{Cl}(\mathcal{O}_Y G)/\mathrm{Ind}_1^G \mathrm{Cl}(\mathcal{O}_Y) \tag{3}$$

for all $x \in K_0(G, X)$. First, we show that it suffices to prove the formula (3) for $x = 1 = [\mathcal{O}_X]$. Indeed, for an arbitrary $x \in K_0(G, X)$, there is a $y \in K_0(Y) \subseteq K_0(G, Y)$ such that $x = f^*(y)$ (e.g. see Theorem 1(B) on p. 112 in [M]). Furthermore, we have $\sigma^k(\mathrm{Ind}_1^G \mathrm{Cl}(\mathcal{O}_Y)) \subseteq \mathrm{Ind}_1^G \mathrm{Cl}(\mathcal{O}_Y)$. This follows from Proposition 1.1 in [K 3] as there is a polynomial $Q_k \in \mathbb{Z}[X_1, \ldots, X_k; Y_1, \ldots, Y_k]$ which is homogeneous of weight k in both sets of variables such that

$$\sigma^k(z \cdot [\mathcal{O}_Y G]) = Q_k\big(\sigma^1(z), \ldots, \sigma^k(z); [\mathrm{Sym}^1(\mathcal{O}_Y G)], \ldots, [\mathrm{Sym}^k(\mathcal{O}_Y G)]\big)$$

for all $z \in \mathrm{Cl}(\mathcal{O}_Y)$ (by Theorem 2.2 in [K 3]). Thus we have in $\mathrm{Cl}(\mathcal{O}_Y G)/\mathrm{Ind}_1^G \mathrm{Cl}(\mathcal{O}_Y)$:

$$\begin{aligned}
\sigma^k(f_*(x)) &= \sigma^k(f_*(f^*(y))) = \sigma^k(y \cdot f_*(1)) \quad \text{(Projection formula)}\\
&= \sigma^k(\mathrm{rank}(y) \cdot f_*(1)) \quad \text{(Lemma 3.1)}\\
&= \mathrm{rank}(y) \cdot \sigma^k(f_*(1)) \quad \text{(Proposition 2.6)}\\
&= \mathrm{rank}(y) \cdot k' \cdot f_*(1) \quad \text{(by assumption)}\\
&= k' \cdot \psi^k(y) \cdot f_*(1) \quad \text{(Lemma 3.1)}\\
&= k' \cdot f_*(\psi^k(f^*(y))) = k' \cdot f_*(\psi^k(x)) \quad \text{(Projection formula).}
\end{aligned}$$

We now prove formula (3) for $x = 1$. Since f is unramified, the scheme X is a principal G-bundle over Y (see Proposition 2.6 on p. 115 in [SGA 1]). There is a well-known natural bijection between the set of all principal G-bundles over Y and the cohomology group $H^1(Y, G)$. We write $[X]$ for the corresponding element in $H^1(Y, G)$. We define a new principal G-bundle $X_{k'}$ over Y as follows: $X_{k'} = X$ as Y-schemes and the new action $*$ of G on $X_{k'}$ is given by $x * g := xg^k$ for "$x \in X$" and $g \in G$. Then, it is easy to check that the association $X \mapsto X_{k'}$ corresponds to the multiplication with k' on $H^1(Y, G)$. Let $\mathrm{cl} : H^1(Y, G) \to \mathrm{Cl}(\mathcal{O}_Y G)$ denote the map which maps a principal G-bundle $f : X \to Y$ to the class $[f_*(\mathcal{O}_X)] - [\mathcal{O}_Y G]$. This map is a homomorphism by Theorem 5 and the subsequent remarks on p. 189 in [W] and by Proposition 3.9 in [AB]. Thus we have:

$$\begin{aligned}
\sigma^k(f_*([\mathcal{O}_X])) &= \phi_{k'}(\mathrm{cl}([X])) \quad \text{(Theorem 2.7)}\\
&= \mathrm{cl}([X_{k'}]) = \mathrm{cl}(k' \cdot [X]) = k' \cdot \mathrm{cl}([X]) = k' \cdot f_*([\mathcal{O}_X]).
\end{aligned}$$

in $\mathrm{Cl}(\mathcal{O}_Y G)$, as was to be shown. In the case (d), Theorem 3.5 immediately follows from Proposition 3.2.

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Remark 3.6. If one of the conditions (a), (b), (c), (d) of Theorem 3.5 is satisfied, then Theorem 3.3 follows from Theorem 3.5 by passing from $\mathrm{Cl}(\mathcal{O}_Y G) \subset K_0(\mathcal{O}_Y G)$ to $K_0(G, Y)$ via Cartan homomorphism and finally by passing from $K_0(G, Y)$ to the completion $\hat{K}_0(G, Y)[k^{-1}]$ of $K_0(G, Y)[k^{-1}]$. In particular, in the case (a), the formula of Theorem 3.3 is substantially weaker than the formula in Theorem 3.5, as already the passage from $K_0(\mathcal{O}_Y G)$ to $K_0(G, Y)$ loses much information. On the other hand, in the case (b), the formula of Theorem 3.5 modulo torsion follows from the formula in Theorem 3.3 if n is a power of a prime. This can be proved as follows. The Cartan homomorphism $K_0(\mathcal{O}_Y G) \to K_0(G, Y)$ is bijective since n is invertible on Y. Furthermore, the canonical map $\mathrm{Cl}(\mathcal{O}_Y G)/\mathrm{Ind}_1^G \mathrm{Cl}(\mathcal{O}_Y) \subseteq K_0(\mathcal{O}_Y G)/\mathrm{Ind}_1^G K_0(Y) \to K_0(\mathcal{O}_{\bar{Y}} G)/\mathrm{Ind}_1^G K_0(\bar{Y})$ is injective by Theorem 2.10. (Here, $\bar{Y}$ denotes the curve $Y \times_L \bar{L}$ over the algebraic closure $\bar{L}$ of L.) Hence, it suffices to prove the formula

$$\sigma^k(\bar{f}_*(x)) = \bar{f}_*\left(\sum_{i=0}^{k'-1} [\mathcal{D}_{\bar{X}/\bar{Y}}^{-ik}] \cdot \psi^k(x)\right) \quad \text{in} \quad K_0(G, \bar{Y})_{\mathbb{Q}}/\mathrm{Ind}_1^G K_0(\bar{Y})_{\mathbb{Q}} \tag{4}$$

for all $x \in K_0(G, \bar{X})$. Furthermore, we have $K_0(G, \bar{Y})_{\mathbb{Q}} \cong K_0(\bar{L}G)_{\mathbb{Q}} \otimes K_0(\bar{Y})_{\mathbb{Q}}$ and $K_0(G, \bar{Y})_{\mathbb{Q}}/\mathrm{Ind}_1^G K_0(\bar{Y})_{\mathbb{Q}} \cong I \otimes K_0(\bar{Y})_{\mathbb{Q}} \cong I_{\mathbb{Q}} \oplus I_{\mathbb{Q}}$ (see the proof of Proposition 3.4). On the other hand, $\left(\hat{K}_0(G, \bar{Y})[k^{-1}]/(\mathrm{Ind}_1^G K_0(\bar{Y}))\hat{K}_0(G, \bar{Y})[k^{-1}]\right)_{\mathbb{Q}}$ is isomorphic to $I \otimes \mathbb{Q}_l \oplus I \otimes \mathbb{Q}_l$ by Proposition 3.4. Hence, the canonical map

$$K_0(G, \bar{Y})_{\mathbb{Q}}/\mathrm{Ind}_1^G K_0(\bar{Y})_{\mathbb{Q}} \to \left(\hat{K}_0(G, \bar{Y})[k^{-1}]/(\mathrm{Ind}_1^G K_0(\bar{Y}))\hat{K}_0(G, \bar{Y})[k^{-1}]\right)_{\mathbb{Q}}$$

is injective, and formula (4) follows from Theorem 3.3.

Remark 3.7.

(a) If one of the conditions (a), (b), or (c) holds, Theorem 3.5 can be slightly strengthened: It suffices to assume that k' is an inverse modulo the exponent of G (see [BC] and Theorem 2.7, respectively). It is not clear to me whether this is true also in the case (d).

(b) Let Y be an irreducible smooth projective curve over a finite field L. Then, the case (c) is particularly interesting as complementary case of the semisimple case which is assumed in the case (b). Indeed, if G is an (Abelian) char(L)-group, then the tameness condition already implies that f is unramified.

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