DELANGE'S CHARACTERIZATION OF THE SINE FUNCTION

by CHIN-HUNG CHING and CHARLES K. CHUI

(Received 10 November, 1972)

1. Introduction and results. In [2], H. Delange gives the following characterization of the sine function.

THEOREM A. *$f(x) = \sin x$ is the only infinitely differentiable real-valued function on the real line such that $f'(0) = 1$ and*

$$|f^{(n)}(x)| \leqq 1 \tag{1}$$

for all real x and $n = 0, 1, 2, \ldots$.

It is clear that, if f satisfies (1), then the analytic continuation of f is an entire function satisfying

$$|f(z)| \leqq \exp(|\mathrm{Im}\, z|)$$

for all z in the complex plane. Hence f is of at most order one and type one. In this note, we prove the following theorem.

THEOREM 1. *Let f be an entire function of at most order one and type one, such that $f(x)$ is real, $|f(x)| \leqq 1$ for all real x, and*

$$f'(x_0)^2 + f(x_0)^2 \geqq 1, \qquad f'(x_0) \neq 0 \tag{2}$$

for some real x_0. Then $f(z) = \sin(z+c)$ for some real constant c. In particular, if $f'(0) = 1$, then $f(z) = \sin z$ for all z.

The example $\sin z/z$ shows that the condition (2) above cannot be omitted. Also, it is easy to see that, in the above theorem, if f is of finite type $\sigma > 1$ and satisfies

$$\frac{1}{\sigma^2} f'(x_0)^2 + f(x_0)^2 \geqq 1, \qquad f'(x_0) \neq 0 \tag{2'}$$

instead, then, by considering $F(z) = f(z/\sigma)$, we can conclude that $f(z) = \sin(\sigma z + c)$. However, the function f defined by

$$f(z) = \frac{1}{ia} \int_{-\infty}^{\infty} t \exp[iatz/b] \exp[-|t| \log(1+t^2)]\, dt,$$

where

$$a = 2 \int_0^{\infty} t \exp[-|t| \log(1+t^2)]\, dt$$

and

$$b = 2 \int_0^{\infty} t^2 \exp[-|t| \log(1+t^2)]\, dt,$$

is an entire function of order one and maximal type, such that $f'(0) = 1$, $f(x)$ is real and $|f(x)| \leqq 1$ for all real x.

https://doi.org/10.1017/S0017089500002147 Published online by Cambridge University Press

2. Proof of Theorem 1. For $-1 < \xi < 1$, it is easy to show that

$$\exp(i\alpha\xi)\cos\alpha - i\xi\exp(i\alpha\xi)\sin\alpha = \sum_{k=-\infty}^{\infty} c_k \exp(ik\pi\xi), \tag{3}$$

where α is real and

$$c_k = \frac{(-1)^k \sin^2\alpha}{(\alpha - k\pi)^2}. \tag{3'}$$

Let $\hat{f}(\xi)$ be the Fourier transform of $f(x)$ (x real). Then, by the Paley–Wiener Theorem [cf. **1**, p. 103], the support of $\hat{f}$ lies in $[-1, 1]$. Hence, multiplying (3) by $\hat{f}(-(1+\varepsilon)\xi)$, we have

$$\hat{f}(-(1+\varepsilon)\xi)\cos\alpha - i\xi\hat{f}(-(1+\varepsilon)\xi)\sin\alpha = \sum_{k=-\infty}^{\infty} c_k \exp[i(k\pi-\alpha)\xi]\hat{f}(-(1+\varepsilon)\xi), \tag{4}$$

where $\varepsilon > 0$. Here, the series converges uniformly since the support of $\hat{f}(-(1+\varepsilon)\xi)$ lies in $(-1, 1)$. Applying the inverse Fourier transform to both sides of (4), we obtain

$$f\left(\frac{x}{1+\varepsilon}\right)\cos\alpha + \frac{1}{1+\varepsilon}f'\left(\frac{x}{1+\varepsilon}\right)\sin\alpha = \sum_{k=-\infty}^{\infty} c_k f\left(\frac{x}{1+\varepsilon} - k\pi + \alpha\right).$$

Now f is continuous and the series converges uniformly for any positive ε. By the boundedness of $f(x)$ for any $x > 0$, we can let ε go to zero and obtain, using (3′),

$$f(x)\cos\alpha + f'(x)\sin\alpha = \sum_{k=-\infty}^{\infty} \frac{(-1)^k \sin^2\alpha}{(\alpha - k\pi)^2} f(x - k\pi + \alpha). \tag{5}$$

If we take $f(x) = \cos x$, then (5) yields the well-known formula

$$\sum_{k=-\infty}^{\infty} \frac{\sin^2\alpha}{(\alpha - k\pi)^2} = 1. \tag{6}$$

Thus we have

$$1 - [f(x)\cos\alpha + f'(x)\sin\alpha] = \sum_{k=-\infty}^{\infty} \frac{[1-(-1)^k f(x-k\pi+\alpha)]}{(\alpha-k\pi)^2}\sin^2\alpha. \tag{7}$$

From the condition (2), we can assume that

$$f(x_0) = A\cos\alpha_0 \quad \text{and} \quad f'(x_0) = A\sin\alpha_0 \tag{8}$$

for some $A \geqq 1$ and some real α_0. Let $x = x_0$ and $\alpha = \alpha_0$ in (7); we have

$$\sum_{k=-\infty}^{\infty} \frac{[1-(-1)^k f(x_0-k\pi+\alpha_0)]}{(\alpha_0-k\pi)^2}\sin^2\alpha_0 = 1 - A \leqq 0.$$

Since $|f(x)| \leqq 1$, we can conclude that, for $k = 0, \pm 1, \ldots,$

$$1 = (-1)^k f(x_0 - k\pi + \alpha). \tag{9}$$

https://doi.org/10.1017/S0017089500002147 Published online by Cambridge University Press

As (5) holds for all entire functions of at most order one and type one, we also have

$$F(x)\cos\alpha+F'(x)\sin\alpha=\sum_{k=-\infty}^{\infty}\frac{(-1)^k\sin^2\alpha}{(\alpha-k\pi)^2}F(x+\alpha-k\pi), \tag{10}$$

with

$$F(x)=f(x_0+\alpha_0+x).$$

By letting $\alpha=-x$ in (10), we obtain, from (9) and (6),

$$F(x)\cos x-F'(x)\sin x=\sin^2 x\sum_{k=-\infty}^{\infty}\frac{1}{(x+k\pi)^2}=1. \tag{11}$$

Integrating (11) gives

$$F(x)=\cos x+b\sin x \tag{12}$$

for some real constant b. Since f is a translation of F, (12) implies that

$$f(x)=d\sin(x+c),$$

where d is positive and c is real. Since $f(x)$ is bounded by one and $f^2+f'^2$ is not less than one at some point x_0, d must be equal to one and the proof is completed.

3. Final remarks. We should like to point out the essential difference between our proof and Delange's. In Delange's paper [**2**], some rather complicated residues are computed and Liouville's Theorem is used to give Theorem A. Here, we use the Paley–Wiener Theorem and finally solve the equation (11) to prove Theorem 1. Of course, Theorem 1 can also be obtained by applying a Phragmén–Lindelöf theorem and Delange's method. It can also be proved by combining a result of Bernstein (cf. page 206 in [**1**]) and a result of Duffin and Schaeffer [**3**].

REFERENCES

1. R. P. Boas, *Entire Functions* (New York, 1954).
2. H. Delange, Caractérisations des fonctions circulaires, *Bull. Sc. Math.* **91** (1967), 65–73.
3. R. J. Duffin and A. C. Schaeffer, On the extension of a functional inequality of S. Bernstein to non-analytic functions, *Bull. Amer. Math. Soc.* **46** (1940), 356–363.

UNIVERSITY OF MELBOURNE
PARKVILLE, VICTORIA
AUSTRALIA 3052

TEXAS A AND M UNIVERSITY
COLLEGE STATION, TEXAS 77801