Part 1.1. Theoretical Aspects

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On the Hydrodynamics of Stellar Pulsation and Stability

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To the memory of Paul Ledoux

Abstract. Much of our knowledge of the hydrodynamics of stellar oscillations is summarized in the now-classical article by Ledoux & Walraven in volume 51 of the Handbuch der Physik. It is from that article that I, and many others, first learned the subject. Most of what we have learned since, aside, perhaps, from what we have learned from nonlinear studies, derives from that work. Even today, what is written in that work is hardly out of date. It is not possible to do justice to even the most outstanding contents of Ledoux & Walraven's article in a single talk, so I highlight some aspects of further developments that have been of more recent interest, trying to set them into a context of current understanding.

1. Introduction

It is with very much pleasure that I report, albeit briefly, on some fluid-dynamical issues pertinent to the theory of stellar pulsation and stability, with some emphasis on issues that occupied the mind of Paul Ledoux, a man for whom I had enormous admiration and respect.

I was a student when I first thought deeply (i.e., as deeply as I was able) about stellar pulsation. I was a summer 'Fellow' (an excessively grand title for a student) in 1964 at the Woods Hole Oceanographical Institution. I was studying a class of approaches to a theory of nonlinear convection in pulsating stars, and naturally I became in need of a tutorial on stellar pulsation. The library of the Oceanographic Institution was of no use for satisfying my need. But next door is the Marine Biological Laboratory, whose library is much more extensive. Not surprisingly, the holdings in astrophysics were not very large there too, but the library did (and I hope it still does) have a complete set of the Handbuch der Physik, and therefore a copy of volume 51. And so it was that whereas many young students of the field simply go to bed with copies of volume 51 under their pillows, I was moved, in the absence of any alternative, actually to read it while I was awake. And what an education the experience was. Even today I can think of no better introduction to the subject. And if many of you readers of this report have in your minds solely the 252-page article on variable stars by

Ledoux & Walraven, let me urge you to read also the subsequent 84 pages, an excellent article by Paul Ledoux on stellar stability.

2. Some basic fluid mechanics

The fundamental equations of motion describing the fluid flow in a star with respect to an Eulerian coordinate system are:

$$\rho \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} \equiv \rho \left(\frac{\partial \boldsymbol{u}}{\partial \mathrm{t}} + \boldsymbol{u}.\nabla \boldsymbol{u} \right) = -\nabla p + \boldsymbol{\mathcal{F}} \ , \tag{2-1}$$

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\, \mathrm{div}\boldsymbol{u} = \frac{\partial \rho}{\partial t} + \mathrm{div}\rho\boldsymbol{u} = 0 \ , \tag{2-2}$$

which express the conservation of momentum and mass respectively; $\boldsymbol{u}$ is the fluid velocity, with Cartesian components u_i, ρ is the density, p the pressure, $\boldsymbol{\mathcal{F}}$ the body force per unit volume and t is time. The operator $\mathrm{D}/\mathrm{D}t$ is the material, or Lagrangian, time derivative, and represents the time derivative following the fluid. The only body force that I shall consider explicitly here is gravity, so $\boldsymbol{\mathcal{F}} = -\rho\nabla\Phi$, where Φ is the gravitational potential which satisfies

$$\nabla^2 \Phi = 4\pi G \rho \ , \tag{2-3}$$

in which G is the gravitational constant. To these must be added an energy equation, expressing the first law of thermodynamics. It may be written

$$\rho \left(\frac{\mathrm{D}\epsilon}{\mathrm{D}t} + p\frac{\mathrm{D}v}{\mathrm{D}t} \right) = \rho T \frac{\mathrm{D}s}{\mathrm{D}t} = \rho\varepsilon - \mathrm{div}\boldsymbol{F} \ , \tag{2-4}$$

where ϵ is the specific internal energy (i.e., internal energy per unit mass), $v = \rho^{-1}$ is the specific volume, T the temperature and s the specific entropy; ε is the rate of generation of heat per unit mass in the fluid (by, e.g., nuclear reactions) and $\boldsymbol{F}$ is the (microscopic) heat flux, which is dominated in most stars by either radiative transport (in nondegenerate stars) or conduction (in white dwarfs and neutron stars). The system of equations is completed by an equation of state: $p = p(\rho, T; \boldsymbol{X})$, where $\boldsymbol{X}$ represents the chemical composition, together with equations for determining ε and $\boldsymbol{F}$. Note that viscous stresses have been omitted from the momentum and energy equations, because they have little direct effect on all but the smallest scales of flow. They do have an indirect effect, however, by influencing the smaller-scale (perhaps turbulent) flow $\boldsymbol{u}_\mathrm{t}$ which must inevitably be present. To be more specific, it is often convenient to regard the variables in equations (2-1)–(2-4) to be 'coarse-grained' quantities obtained by averaging over some scale ℓ. Then the governing equations for those quantities are obtained by averaging the exact equations (which are essentially the same as equations (2-1)–(2-4), save the caveat concerning viscous stresses), which yield equations (2-1)–(2-4) with additional terms to account for motion on scales smaller than ℓ. The effect of the small-scale flow can be represented by an energy flux $\boldsymbol{F}_\mathrm{c}$, which must be added to $\boldsymbol{F}$, and the stress tensor $R_{ij} = \overline{\rho u_{\mathrm{t}i} u_{\mathrm{t}j}}$, where the overbar represents an appropriate coarse-grained ensemble average, whose divergence $\partial R_{ij}/\partial x_j$ must be subtracted from the i component $\mathcal{F}_i$ of $\boldsymbol{\mathcal{F}}$;

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and it must be taken into account also in the energy equation, for the turbulent cascade to smaller and smaller scales of motion is eventually dissipated into heat by thermal and viscous diffusion. Ledoux & Walraven (1958) discuss these terms in some detail.

Except typically in the surface layers of the star, the thermal (and viscous) diffusion timescales are very much greater than the characteristic dynamical timescales, and the motion is nearly adiabatic. Temperature, therefore, plays only a minor role in the dynamics, and it is prudent to express the energy equation directly in terms of pressure and density. This can be accomplished by applying the thermodynamic relation

$$(\gamma_3 - 1)\mathrm{d}\epsilon = v\mathrm{d}p + (1 + \gamma_1 - \gamma_3)p\mathrm{d}v \ , \tag{2-5}$$

in which the adiabatic exponents γ_1 and γ_3 are defined by

$$\gamma_1 := \left(\frac{\partial \mathrm{ln} p}{\partial \mathrm{ln} \rho}\right)_s \quad \text{and} \quad \gamma_3 := 1 + \left(\frac{\partial \mathrm{ln} T}{\partial \mathrm{ln} \rho}\right)_s \ , \tag{2-6}$$

yielding

$$\frac{\mathrm{D}\mathrm{ln}p}{\mathrm{D}t} - \gamma_1 \frac{\mathrm{D}\mathrm{ln}\rho}{\mathrm{D}t} = (\gamma_3 - 1)\frac{\rho T}{p}\frac{\mathrm{D}s}{\mathrm{D}t} \ . \tag{2-7}$$

For precisely isentropic variation the right-hand side of equation (2-7) vanishes, and temperature disappears completely from the system of dynamical equations (2-1)–(2-3) and (2-7). These equations, together with the constitutive equation expressing γ_1 in terms of p and ρ (and $\boldsymbol{X}$) which is derived from the equation of state, now express how $\boldsymbol{u}$ varies under the action of $\boldsymbol{\mathcal{F}}$ in terms of p and ρ alone.

In what follows it will be useful to express the isentropic variation of specific internal energy ϵ in relation to changes in specific volume. The first partial derivative $(\partial\epsilon/\partial v)_s$ follows immediately from the first law of thermodynamics, expressed in equation (2-4), and the second derivative is obtained directly by differentiation:

$$\left(\frac{\partial \epsilon}{\partial v}\right)_s = -p \ , \quad \left(\frac{\partial^2 \epsilon}{\partial v^2}\right)_s = \frac{\gamma_1 p}{v} \ . \tag{2-8}$$

The turbulent fluxes R_{ij} and F_{ci} of momentum and energy are important in convection zones and in shear layers that are unstable to small-scale perturbations. Their evaluation requires a theory of turbulence, which I shall not discuss here. However, I shall make a few remarks about what quantities should be calculated, because there has been some confusion in the literature. In particular, I derive an expression for the convective heat flux, by casting the equations of motion into conservative form.

An equation for the kinetic energy density can be obtained by taking the scalar product of equation (2-1) with $\boldsymbol{u}$. It can then be cast almost into conservative form by adding to it the product of $\frac{1}{2}\boldsymbol{u}.\boldsymbol{u} =: \frac{1}{2}u^2$ and equation (2-2), yielding

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2\right) + \frac{\partial}{\partial x_i}\left(\frac{1}{2}\rho u^2 u_i\right) + u_i\frac{\partial p}{\partial x_i} + \rho u_i \frac{\partial \Phi}{\partial x_i} = 0 \ , \tag{2-9}$$

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in which I have written the body force in terms of the gravitational potential. The thermal energy equation (2-4) may be treated similarly by adding to it the product of ϵ and equation (2-2):

$$\frac{\partial}{\partial t}(\rho\epsilon) + \frac{\partial}{\partial x_i}(\rho\epsilon u_i + F_i) + p\frac{\partial u_i}{\partial x_i} = \rho\varepsilon \, . \tag{2-10}$$

Both of these equations are derived and discussed by Ledoux & Walraven (1958). Equation (2-9) expresses the rate of change of kinetic energy per unit volume as the rate of working of the forces, both internal (via the pressure) and locally external (via the gravitational potential – although in an isolated system the gravitational forces are globally internal), less the divergence of the flux of kinetic energy. Equation (2-10) expresses the rate of change of internal energy $\rho\epsilon$ per unit volume in terms of the rate of generation of heat $\rho\varepsilon$ per unit volume and the rate of working of the internal (pressure) forces, less the divergence of the sum of the heat flux $\boldsymbol{F}$ and the flux $\rho\epsilon\boldsymbol{u}$ of internal energy. Because the internal pressure forces mediate the exchange of internal energy with macroscopic kinetic energy, *the macroscopic heat flux is not the flux of internal energy*, but is modified by the internal working of the fluid to be the flux of what meteorologists have called 'available energy', h. This is evident from the equation governing the total energy density, which is obtained by adding equations (2-9) and (2-10):

$$\frac{\partial}{\partial t}(\rho\epsilon + \tfrac{1}{2}\rho u^2 + \rho\Phi) + \frac{\partial}{\partial x_i}(\rho h u_i + \tfrac{1}{2}\rho u^2 u_i + \rho\Phi u_i + F_i) = \rho\left(\varepsilon + \frac{\partial\Phi}{\partial t}\right) . \tag{2-11}$$

The interpretation of the terms is obvious; in particular, the macroscopic heat flux is the flux of $\rho h = \rho\epsilon + p$, namely enthalpy. Note that equation (2-11) is valid for motion on all scales large enough for viscous stresses to be unimportant. If the motion is turbulent in three dimensions, cascade to very small scales is inevitable, and a viscous flux term must be incorporated.

If equation (2-11) is to represent the large-scale motion (excluding, for example, turbulent convection that may be present), then the mean convective energy flux $\boldsymbol{F}_{\rm c} = \overline{\rho h \boldsymbol{u}_{\rm t}} + \tfrac{1}{2}\overline{\rho u_{\rm t}^2 \boldsymbol{u}_{\rm t}}$ must be included in $\boldsymbol{F}$. In most stellar calculations, some form of mixing-length theory is usually employed to estimate $\boldsymbol{F}_{\rm c}$, and usually the contribution to the kinetic-energy flux is ignored because it depends critically on subtle fluctuation correlations which are difficult to estimate. It is more straightforward to estimate the heat flux because perturbations in h and $\boldsymbol{u}_{\rm t}$ are strongly correlated in the convection zone, where motion is driven by buoyancy. It is usual to separate the thermodynamical variables into mean and fluctuating parts — $\rho = \overline{\rho} + \rho'$ etc — and to assume that the fluctuations are small: $|\rho'| \ll \overline{\rho}$; moreover, one adopts a Lagrangian mean frame of reference in which the turbulent mass flux vanishes: $\overline{\rho\boldsymbol{u}_{\rm t}} = \boldsymbol{0}$. Then $\boldsymbol{F}_{\rm c} = \overline{\rho h' \boldsymbol{u}_{\rm t}} \simeq \overline{\rho}\,\overline{h'\boldsymbol{u}_{\rm t}}$, the second relation being obtained by retaining only the term that is of lowest order in fluctuation quantities. Then one can use (in a chemically homogeneous region) the thermodynamic relation $h' = c_p T' + (1-\delta)\rho^{-1}p'$, where $c_p := T(\partial s/\partial T)_p$ is the specific heat capacity at constant pressure and $\delta := -(\partial \ln\rho/\partial \ln T)_p$ is the dimensionless coefficient of thermal expansion. Note that, for a perfect gas, $\delta = 1$, and the second term in the expression for h' vanishes. In mixing-length theory it is normal to ignore that term even in zones of ionization of abundant elements, where δ deviates significantly from unity, either because (i) the theory is often local (valid if the mixing length ℓ associated with the largest eddies is much smaller than the characteristic scale height H of the background state, in which case Boussinesq scaling should apply (Spiegel and Veronis, 1960), although subsequent calibration of ℓ yields values comparable with H), or because (ii) fluctuations p' tend be out of phase with fluctuations T' and the vertical component $w_{\rm t}$ of $\boldsymbol{u}_{\rm t}$, so that $\overline{c_p}\,\overline{T'w_{\rm t}} \gg \overline{p'w_{\rm t}}$ (although numerical simulations show that $\overline{p'w_{\rm t}}$ is not always negligible). Consequently, one sets $\boldsymbol{F}_{\rm c} \simeq \overline{\rho}\,\overline{c_p}\,\overline{T'\boldsymbol{u}_{\rm t}}$. Note, however, that the appropriate coefficient c_p of specific heat is unambiguously at constant pressure and not, as has sometimes been posed, at constant volume.

Finally, it is instructive to record the equation of total energy of an isolated self-gravitating system, such as a star, which is obtained by integrating equation (2-11) over the volume $\mathcal{V}$ of the star. By integrating $\dot{\Phi}\nabla^2\Phi - \Phi\nabla^2\dot{\Phi}$ over $\mathcal{V}$, where $\dot{\Phi} := \partial\Phi/\partial t$, one can deduce immediately that $\int_{\mathcal{V}} \rho\dot{\Phi}\mathrm{d}V = \int_{\mathcal{V}} \dot{\rho}\Phi\mathrm{d}V$, whence

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{U} + \mathcal{T} + \Omega) = \mathcal{Q} - \mathcal{L}, \tag{2-12}$$

where $\mathcal{U} = \int_{\mathcal{V}} \rho\epsilon\mathrm{d}V$, $\mathcal{T} = \int_{\mathcal{V}} \frac{1}{2}\rho u^2\mathrm{d}V$ and $\Omega = \frac{1}{2}\int_{\mathcal{V}} \rho\Phi\mathrm{d}V$ are respectively the total internal, kinetic and gravitational energies of the star, $\mathcal{Q} = \int_{\mathcal{V}} \rho\varepsilon\mathrm{d}V$ is the rate of generation of heat and $\mathcal{L} = \int_{\mathcal{S}} \boldsymbol{F}.\mathrm{d}\boldsymbol{S}$ is the luminosity, the surface $\mathcal{S}$ which bounds $\mathcal{V}$ being sufficiently distant that all other surface integrals vanish.

3. Linearized perturbation equations

Consider a static equilibrium state given by $p_0(\boldsymbol{x}), \rho_0(\boldsymbol{x}), ...$ and $\boldsymbol{u} = \boldsymbol{0}$, satisfying equations (2-1), (2-3) and (2-4) with $\mathrm{D}/\mathrm{D}t = 0$. Equations governing small motion in the neighbourhood of that state can be obtained in the usual way in terms of the displacement $\boldsymbol{\xi}(\boldsymbol{x}, t)$ of any fluid element from its equilibrium position, together with either the Eulerian perturbations $p'(\boldsymbol{x}, t) = p(\boldsymbol{x}, t) - p_0(\boldsymbol{x})$ etc or Lagrangian perturbations $\delta p(\boldsymbol{x}, t) = p(\boldsymbol{x} + \boldsymbol{\xi}, t) - p_0(\boldsymbol{x}) = p' + \boldsymbol{\xi}.\nabla p_0 + \mathrm{O}(|\,\boldsymbol{\xi}\,|^2)$, etc from the equilibrium state, by substituting the decomposed expressions $p = p_0 + p'$ or $p = p_0 + \delta p$ etc into the governing equations and linearizing in perturbed quantities. Here I assume adiabatic perturbations, and therefore use the energy equation in the form (2-7) with the right-hand side set to zero. Note that, in view of the linearization and the fact that $\boldsymbol{u} = \boldsymbol{0}$ in the equilibrium state, it is immaterial whether $\boldsymbol{\xi}$ is considered to be Eulerian or Lagrangian, because the difference is quadratic in $\boldsymbol{\xi}$. The resulting equations are

$$\rho\frac{\partial^2\boldsymbol{\xi}}{\partial t^2} = -\nabla p' - \rho'\nabla\Phi - \rho\nabla\Phi'\,, \tag{3-1}$$

$$\delta\rho + \rho\,\mathrm{div}\boldsymbol{\xi} = \rho' + \mathrm{div}\rho\boldsymbol{\xi} = 0\,, \tag{3-2}$$

$$\nabla^2\Phi' = 4\pi G\rho'\,, \tag{3-3}$$

$$\delta p = c^2\delta\rho\,, \tag{3-4}$$

and $\boldsymbol{u} = \partial\boldsymbol{\xi}/\partial t$, where $c^2 = \gamma_1 p/\rho$ is the square of the adiabatic sound speed, and for clarity I have, without ambiguity, dropped the subscripts from the equilibrium quantities. Because the equilibrium quantities are independent of time, I may seek solutions that vary exponentially with t, and adopt the complex representation

$$p'(\boldsymbol{x}, t) = p'(\boldsymbol{x})e^{-\mathrm{i}\omega t}\,, \quad \text{etc}, \tag{3-5}$$

trusting that my use of the same notation for the perturbation on the left-hand side and its amplitude on the right-hand side will cause no confusion. The equations for the amplitudes are then simply (3-1)–(3-4), with the left-hand side of equation (3-1) replaced by $-\omega^2\rho\boldsymbol{\xi}$. They are to be solved subject to appropriate boundary conditions, which are such as to yield an eigenvalue problem for ω^2. I shall have in mind that on the surface $\mathcal{S}$ of the star on which the dynamical boundary condition is applied the density ρ is negligible, and that the region outside it exerts no force on the star, so that $\delta p = 0$ (and $\boldsymbol{\xi}$ is finite) on $\mathcal{S}$. Moreover, $\Phi' \to 0$ as $|\boldsymbol{x}| \to \infty$, which in practice is achieved by matching Φ' and its gradient interior to $\mathcal{S}$ continuously onto a vacuum potential exterior to $\mathcal{S}$. These conditions permit no energy flux through $\mathcal{S}$, and, as will be evident in the next section, imply that ω^2 is real. In some more realistic cases, however, energy can propagate through surfaces that one might conveniently adopt for being the boundary of the star, such as when ω is high enough for acoustic propagation through the photosphere to be possible. There are many discussions of how one might proceed under those circumstances (e.g. Ledoux & Walraven, 1958; Unno et al., 1989; Gough, 1993).

If one's principal interest in the evolution of the perturbations is dynamical, as it must be if those perturbations are presumed to be adiabatic, then generally the hydrostatic versions of only equations (2-1) and (2-3) need be satisfied by the background state; any thermal evolution is typically so slow that it is safe to ignore it over the characteristic timescale $|\omega|^{-1}$. Consequently, ∇p_0 and $\nabla\rho_0$ are effectively parallel (or antiparallel) to the unit vertical vector $\boldsymbol{n} := |\nabla\Phi_0|^{-1}\nabla\Phi_0$.

It should be noted that the assumption $\boldsymbol{u} = \boldsymbol{0}$ was made purely for convenience, and was actually not necessary. The procedure can be generalized to study the development of perturbations from any background state with nonzero $\boldsymbol{u}$ for which there exists a frame of reference in which that state is independent of time (Lynden-Bell & Ostriker, 1967), although in that case more care must be exercised in the interpretation of $\boldsymbol{\xi}$ (cf §7).

4. A variational principle and some of its consequences

Ledoux & Walraven (1958) make extensive use of integral relations. Here I illustrate some of the principles, although I go beyond what was known when the article was written. Let us assume that the background state is sufficiently regular for the solutions of the eigenvalue problem posed by equations (3-1)–(3-5) and the conditions on $\mathcal{S}$ to be discrete. Then they can be labelled with an integer subscript such as i or j (not to be confused with the index denoting the component of a vector). Let us now take the scalar product of the equation (3-1) satisfied by $(\boldsymbol{\xi}_i, \omega_i)$ with the complex conjugate $\boldsymbol{\xi}_j^*$ of $\boldsymbol{\xi}_j$, use equations (3-2)–(3-4) to eliminate p_i', ρ_i' and Φ_i', and integrate over the volume $\mathcal{V}$ enclosed by $\mathcal{S}$.

It is possible to transform the integrals using the divergence theorem to render the volume integrals symmetric in $\boldsymbol{\xi}_i$ and $\boldsymbol{\xi}_j^*$, and with the idealized boundary conditions that I have adopted the surface integrals vanish. The outcome is

$$\mathcal{I}\omega_i^2 = \mathcal{K} \ , \tag{4-1}$$

where

$$\begin{aligned}\mathcal{K}(\boldsymbol{\xi}_i, \boldsymbol{\xi}_j^*) := \int_{\mathcal{V}} \Big[& \gamma_1 p \, \mathrm{div}\boldsymbol{\xi}_i \mathrm{div}\boldsymbol{\xi}_j^* + \nabla p.(\boldsymbol{\xi}_i \mathrm{div}\boldsymbol{\xi}_j^* + \boldsymbol{\xi}_j^* \mathrm{div}\boldsymbol{\xi}_i) \\ & + \nabla p.\nabla \ln\rho \, \boldsymbol{n}.\boldsymbol{\xi}_i \boldsymbol{n}.\boldsymbol{\xi}_j^* \Big] \mathrm{d}V \\ & - G \iint_{\mathcal{V}} \frac{\mathrm{div}[\rho(\boldsymbol{x})\boldsymbol{\xi}_i(\boldsymbol{x})] \, \mathrm{div}'[\rho(\boldsymbol{x}')\boldsymbol{\xi}_j^*(\boldsymbol{x}')]}{|\boldsymbol{x} - \boldsymbol{x}'|} \mathrm{d}V \mathrm{d}V'\end{aligned} \tag{4-2}$$

and

$$\mathcal{I}(\boldsymbol{\xi}_i, \boldsymbol{\xi}_j^*) := \int_{\mathcal{V}} \rho \, \boldsymbol{\xi}_i . \boldsymbol{\xi}_j^* \, \mathrm{d}V. \tag{4-3}$$

It is evident from the symmetry of $\mathcal{I}$ and $\mathcal{K}$ that the problem is self-adjoint, and that equation (4-1) is satisfied also by ω_j^{*2}. Consequently $(\omega_i^2 - \omega_j^{*2})\mathcal{I} = 0$, from which it follows, by setting $j = i$, that ω_i^2 is real, and so also are the eigenfunctions (aside from an arbitrary multiplicative constant). It follows also that $\mathcal{I} = 0$ if $\omega_i \neq \omega_j$. This orthogonality of the displacement eigenfunctions is convenient for carrying out perturbation theory to determine the oscillations of, say, a slowly rotating star (cf §7) in which there is perhaps a weak magnetic field, by expanding the eigenfunctions as a linear combination of the eigenfunctions of a corresponding field-free nonrotating star, which are much easier than the true eigenfunctions to calculate. The faithfulness of the expansion, provided it is taken far enough (formally, provided that all terms are included) is assured because the eigenfunctions are complete (Eisenfeldt, 1969).

It follows also from the symmetry of $\mathcal{I}$ and $\mathcal{K}$ that the equation

$$\omega^2 = \frac{K(\boldsymbol{\xi})}{I(\boldsymbol{\xi})} =: \Sigma^2(\boldsymbol{\xi}) \ , \tag{4-4}$$

where $I(\boldsymbol{\xi}) := \mathcal{I}(\boldsymbol{\xi}, \boldsymbol{\xi}^*)$ and $K(\boldsymbol{\xi}) := \mathcal{K}(\boldsymbol{\xi}, \boldsymbol{\xi}^*)$, provides a variational principle (e.g. Chandrasekhar, 1964); the stationary values of Σ^2 amongst all differentiable vector functions $\boldsymbol{\xi}$ that satisfy the boundary conditions on $\mathcal{S}$ are the eigenvalues ω_i^2 of equations (3-1)–(3-4), and occur at the eigenfunctions $\boldsymbol{\xi}_i$. The integral K measures the rate of conversion of internal and potential energy into kinetic energy; the integral I is called the inertia of the mode (Ledoux & Walraven call it the moment of inertia).

The variational principle (4-4) has been used for estimating eigenfrequencies by substituting approximations $\boldsymbol{\xi}_\mathrm{e}$ to $\boldsymbol{\xi}$ into Σ^2; if one introduces a parameter ε (not to be confused with the energy generation rate in equation (2-4)) as a measure of the error $|\boldsymbol{\xi} - \boldsymbol{\xi}_\mathrm{e}|$, which is presumed to be small, then the error in the estimate $\Sigma^2(\boldsymbol{\xi}_\mathrm{e})$ of ω^2 is $\mathrm{O}(\varepsilon^2)$ and is therefore even smaller. The integral

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relation (4-4) can also be used to set bounds on ω^2, and thereby to provide conditions for instability; I shall present examples in the following two sections. In the early days, Ledoux (1945) also used a different relation derived from a virial theorem to estimate the effect of rotation on the almost radial pulsations of stars; the method was developed further by Chandrasekhar and used extensively by him in a series of papers to estimate the frequencies of oscillation of a variety of self-gravitating configurations. But equations (4-1)–(4-3) are generally to be preferred because of their variational property. The variational property enabled Ledoux & Pekeris (1941) to use the radial version (5-1) of (4-1) to estimate eigenfrequencies of radial modes of a star. It is used in asteroseismology today for calculating the effect on ω^2 of small perturbations to the background state, for example, which can be expressed as integrals of those perturbations weighted by kernels that depend on the eigenfunctions of the unperturbed state, but not on their perturbations (e.g. Gough, 1985; Gough & Thompson, 1991).

5. Radial perturbations to a spherically symmetrical star

On setting $\boldsymbol{\xi} = (r\hat{\xi}(r), 0, 0)$ with respect to spherical polar coordinates (r, θ, ϕ), equations (4-2)–(4-4) may be written, after some manipulation, in the form

$$\omega^2 = \Sigma^2 \equiv \frac{\displaystyle\int_0^R \left\{\gamma_1 p r^4 \left(\frac{\mathrm{d}\hat{\xi}}{\mathrm{d}r}\right)^2 - r^3 \frac{\mathrm{d}}{\mathrm{d}r}[3\gamma_1 - 4)p]\hat{\xi}^2\right\}\mathrm{d}r}{\displaystyle\int_0^R \rho r^4 \hat{\xi}^2 \mathrm{d}r}\,, \tag{5-1}$$

where R is the radius of the star. The Euler-Lagrange equation which must be satisfied at the stationary values of Σ^2 is

$$\frac{1}{r^4}\frac{\mathrm{d}}{\mathrm{d}r}\left(\gamma_1 p r^4 \frac{\mathrm{d}\hat{\xi}}{\mathrm{d}r}\right) + \left\{\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}[(3\gamma_1 - 4)p] + \omega^2 \rho\right\}\hat{\xi} = 0\,. \tag{5-2}$$

This equation can also be derived directly by substituting $(r\hat{\xi}, 0, 0)$ for $\boldsymbol{\xi}$ into equations (3-1)–(3-5) and eliminating p', ρ' and Φ' in favour of $\hat{\xi}$. The equation must be solved subject to appropriate boundary conditions, which here I take to be $\delta \ln p = 0$ ($\mathrm{d}r^3\hat{\xi}/\mathrm{d}r = 0$) at $r = R$ and a regularity condition at the coordinate singularity $r = 0$ (which requires $\xi := r\hat{\xi} = 0$ at $r = 0$).

Equation (5-1) has been used for bounding the frequency of the fundamental radial mode (e.g. Ledoux & Pekeris, 1941). Since Σ^2 is evidently bounded below, the lowest value of ω^2 must be a minimum. If the star is known (or conjectured) to be dynamically stable, evaluating Σ^2 for any function $\hat{\xi}$ that satisfies the boundary conditions must (or is conjectured to) bound the fundamental frequency above; if the function $\hat{\xi}$ contains free parameters, the bound is tightest when Σ^2 is minimized with respect to those parameters, and may be a good estimate of ω^2. Estimates of higher-order frequencies could be obtained by the Rayleigh-Ritz method, although nowadays it is more prudent simply to search amongst solutions of equation (5-2) that have been obtained numerically.

What is the condition for the star to be dynamically stable? If γ_1 were constant, equation (5-1) could be simplified, and ω^2 could be bounded below by ignoring the first, positive, term in the integrand in the numerator:

$$\omega^2 \geq (3\gamma_1 - 4)\frac{\int_0^R r^{-1} g\rho r^4 \hat{\xi}^2 \mathrm{d}r}{\int_0^R \rho r^4 \hat{\xi}^2 \mathrm{d}r} , \tag{5-3}$$

where $g = Gm/r^2$ is the acceleration due to gravity, $m(r) = 4\pi \int \rho r^2 \mathrm{d}r$ being the mass enclosed by the sphere of constant r. In obtaining this equation I have used the equation of hydrostatic support: $\mathrm{d}p/\mathrm{d}r = -g\rho$. If, furthermore, the mean density $\langle\rho\rangle := m/(\frac{4}{3}\pi r^3)$ to radius r is everywhere greater than its value at the surface (which is satisfied by most stellar models, and certainly by all convectively stable stars), then $r^{-1}g$ may be replaced by its surface value without violating the inequality, yielding

$$\omega^2 > (3\gamma_1 - 4)\frac{GM}{R^3}. \tag{5-4}$$

It follows that ω^2 can be negative only if $\gamma_1 < 4/3$, which provides a necessary condition for dynamical instability.

The more realistic case in which γ_1 varies is much harder to study. Evidently one can from equation (5-1) obtain stability criteria in terms of average values of γ_1, weighted by functions that depend on the a-priori-unknown eigenfunctions of the problem. Indeed, Ledoux (1946) addressed the matter in that way, and made estimates of stability for certain explicit examples. But there are some general results that do not rely on such averages of γ_1. They were first obtained by Freeman Dyson (unpublished) in 1960, using an energy principle. The following argument is based on Dyson's discussion, and illustrates the power of the energy principle to derive rigorous, albeit limited, conclusions. It also exhibits a connexion between the energy principle and the variational principle (5-1) governing the eigenvalue equation (5-2).

Imagine an equilibrium state to be perturbed adiabatically to a new state (here, by spherically symmetrical radial displacement $\xi = r\hat{\xi}$), and compute the total energy difference ΔE between the new and the original states. If $\Delta E > 0$ for all possible displacements, then energy would have to be injected into the system from outside to attain any change in state; the original state is at an energy minimum, and is therefore intrinsically stable. If, on the other hand, there exists a displacement for which $\Delta E < 0$, then there is energy available to drive a flow away from the original state of equilibrium, and one might conjecture that the equilibrium is unstable; one cannot know that it is unstable, however, because evolution from equilibrium occurs only by displacement that satisfies the equations of motion, and, unless one demonstrates that ΔE can be negative for at least one such admissible displacement, instability is not established.

The energy of a spherically symmetrical stellar configuration is given by the following integral over the mass M of the star:

$$E = \int_0^M (\epsilon - Gm/r)\mathrm{d}m . \tag{5-5}$$

In terms of the volume $V = 4\pi r^3/3$ of the sphere of radius r, the specific volume is $v = \mathrm{d}V/\mathrm{d}m$; moreover, the equation of hydrostatic support may be written

$$\frac{\mathrm{d}p}{\mathrm{d}m} = -\frac{Gm}{4\pi r^4} . \tag{5-6}$$

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One now considers the change ΔE resulting from a radial displacement ξ, associated with which is a Lagrangian perturbation δv to v, evaluated to second order in $\hat{\xi}$:

$$\Delta E = \int_0^M \left\{ \left(\frac{\partial \epsilon}{\partial v}\right)_s \delta v + \frac{1}{2}\left(\frac{\partial^2 \epsilon}{\partial v^2}\right)_s (\delta v)^2 + ... + \frac{Gm}{r}\left(\hat{\xi} - \hat{\xi}^2 + ...\right)\right\} \mathrm{d}m, \tag{5-7}$$

in which

$$\delta v = \frac{\mathrm{d}\delta V}{\mathrm{d}m} = 4\pi \frac{\mathrm{d}}{\mathrm{d}m}[r^3(\hat{\xi} + \hat{\xi}^2 + ...)] \,. \tag{5-8}$$

With the help of the first of equations of (2-8) one can show that the first-order contribution $\Delta_1 E$ to ΔE vanishes, as must be the case for any equilibrium:

$$\Delta_1 E = \int_0^M \left[-4\pi p \frac{\mathrm{d}}{\mathrm{d}m}(r^3\hat{\xi}) + \frac{Gm}{r}\hat{\xi}\right] \mathrm{d}m \,, \tag{5-9}$$

which, after integrating the first term by parts, becomes

$$\Delta_1 E = \int_0^M \left(4\pi r^3 \frac{\mathrm{d}p}{\mathrm{d}m} + \frac{Gm}{r}\right)\hat{\xi}\, \mathrm{d}m \,, \tag{5-10}$$

which vanishes in view of the equilibrium equation (5-6). The second-order term is

$$\Delta_2 E = \int_0^M \left\{-p\frac{\mathrm{d}}{\mathrm{d}m}(4\pi r^3\hat{\xi}^2) + \frac{\gamma_1 p}{2v}\left[\frac{\mathrm{d}}{\mathrm{d}m}(4\pi r^3\hat{\xi})\right]^2 - \frac{Gm}{r}\hat{\xi}^2\right\} \mathrm{d}m \,. \tag{5-11}$$

On integrating the first term by parts and combining it with the last using equation (5-6), one obtains

$$\Delta_2 E = \frac{1}{2}\int_0^{\mathcal{V}} \left(\gamma_1 p \chi^2 + \frac{4}{3V}\frac{\mathrm{d}p}{\mathrm{d}V}\psi^2\right) \mathrm{d}V \,, \tag{5-12}$$

where $\psi := 4\pi r^3\hat{\xi}$, $\chi := \mathrm{d}\psi/\mathrm{d}V = r^{-2}\mathrm{d}(r^3\hat{\xi})/\mathrm{d}r = \mathrm{div}\boldsymbol{\xi}$ and $\mathcal{V} = V(r) = 4\pi R^3/3$ is the volume of the star.

It is useful to transfer part of the second term in the integrand to the first in such a manner as to render the second term positive definite, as has been a common practice in studying magnetohydrodynamic stability (e.g. Bernstein et al., 1958), and was carried out, for example, by Gough & Tayler (1966) to derive sufficient conditions for convective stability in the presence of an imposed magnetic field. Dyson achieved it by integrating the second term by parts and expressing the outcome in terms of the square of the derivative of the relative displacement $\hat{\xi}$ and the square of χ:

$$\Delta_2 E = \tfrac{1}{2}\int_0^{\mathcal{V}} \left[\left(\gamma_1 - \tfrac{4}{3}\right)\chi^2 + \tfrac{4}{3}\left(r\tfrac{\mathrm{d}\hat{\xi}}{\mathrm{d}r}\right)^2\right] p\, \mathrm{d}V. \tag{5-13}$$

For perturbations of infinitesimal amplitude about a state of equilibrium, the sign of ΔE is the sign of $\Delta_2 E$.

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It follows immediately from equation (5-13) that if $\gamma_1 > 4/3$ everywhere, then $\Delta_2 E > 0$ for all displacements ξ from the equilibrium state. Consequently, a necessary condition for dynamical instability is that $\gamma_1 < 4/3$ somewhere.

Can one also find a condition to ensure that $\Delta_2 E > 0$? If one considers an homologous perturbation, for example, which satisfies $\hat{\xi} = \hat{\xi}_0 =$ constant, one obtains

$$\Delta_2 E = \tfrac{9}{2}\hat{\xi}_0^2 \int_0^{\mathcal{V}} \left(\gamma_1 - \tfrac{4}{3}\right) p \, \mathrm{d}V \ , \tag{5-14}$$

which can be positive only if $\gamma_1 > 4/3$ somewhere. However, as I pointed out previously, one cannot deduce from this argument that this is a necessary condition for stability, because the specific function $\hat{\xi}$ that was chosen does not in general satisfy the equations of motion, and may therefore not be a dynamically allowable perturbation. Nevertheless, the condition on γ_1 is suggestive.

To proceed further one can attempt to minimize $\Delta_2 E$, and find the condition for that minimum to be positive. Because the value of $\Delta_2 E$ depends on the amplitude of the relative displacement function $\hat{\xi}$, the minimization must be accomplished subject to some normalization condition to control the amplitude. Any normalization may be adopted; here I choose $I \equiv 4\pi \int \rho r^4 \hat{\xi}^2 \mathrm{d}r = 1$. Then the Euler-Lagrange equation for the stationary values of $\Delta_2 E$ is simply equation (5-2) with the place of ω^2 taken by a Lagrange multiplier whose value at the stationary points is $2\Delta_2 E$. Indeed, the expression (5-13) for $\Delta_2 E$ is simply 2π times the numerator in equation (5-1). (It is now evident why I chose my particular normalization condition.) The condition that $\Delta_2 E > 0$ for all functions $\hat{\xi}$ that satisfy the boundary conditions is therefore equivalent to demanding that all the eigenvalues ω^2 of equation (5-2) be positive.

With the power of the variational principle one can now proceed further. In view of the completeness of the eigenfunctions, the lowest eigenvalue bounds the functional Σ^2, or $\Delta_2 E$, below, and the stationary point corresponding to that eigenvalue must therefore be a minimum. It follows that the expression (5-14) cannot be less than that minimum, and must therefore be positive if the lowest permissible value of Σ^2 or $\Delta_2 E$ is positive. Evidently, it can be positive only if $\gamma_1 > 4/3$ somewhere, which now establishes a necessary condition for the dynamical stability of the star to spherically symmetrical adiabatic perturbations.

So far as I am aware, no further progress in this study has been made in general terms. Moreover, in most cases of interest the strict general results are not of great use. That is because the two necessary conditions on γ_1 for stability and instability are usually so far apart in practice that they can rarely be useful for determining the stability of a given system. What is needed is to take account of the structure of the eigenfunctions, as did Ledoux (1946), and this is difficult unless one resorts to specific examples. The reason for the difficulty is that the criterion is genuinely nonlocal, because it depends critically on gravitational forces, and therefore stability must depend on the overall structure of the equilibrium configuration, particularly of those regions that contain most of the mass of the star. In the next section I discuss a local instability, in which the perturbation to the gravitational potential plays no significant role for the neutral disturbance, and which, in contrast, can be analysed completely.

6. Nonradial motion: convective instability

The dynamical instability discussed in the previous section arises when the perturbed gravitational forces tending to augment a local compression dominate the pressure gradients that oppose it. Since the latter increase in magnitude as the local wavenumber of the perturbation increases at fixed amplitude of compression, whereas the former do not, it is the perturbation of greatest spatial scale that is the most unstable; the basic balance of forces is just as it is in the Jeans instability (Jeans, 1928). Thus it is the gravest mode of a star that is most unstable, and consequently, on the whole, nonradial modes (i.e. modes that are not spherically symmetrical – whose motion is not purely radial) are of lesser interest when considering gravitationally driven dynamical instability.

There is a nonradial dynamical instability associated with convection, however, which is not intimately concerned with gravitational perturbations. It occurs when a vertically displaced fluid element experiences a buoyancy force in such a direction as to reinforce the original displacement. It is because the condition under which adiabatic disturbances are marginally unstable does not depend on gravitational perturbations (nor diffusive processes, because viscosity – and thermal diffusion – are ignored), that it can be local. And because it actually is local, a complete analysis of the linearized instability problem is possible. I discuss that analysis here because the correct criterion, when written in terms of the temperature gradient, as is common practice in the literature on stellar structure, was elucidated by Ledoux (1947) and now bears his name. Recall that for adiabatic motion temperature is not a dynamically relevant variable, and the criterion is most lucidly written in terms of ρ, p and γ_1.

There are early discussions in the meteorological literature (e.g. Reye, 1872) of the condition for convective instability . An element of fluid was considered to be displaced vertically, slowly enough that it could be presumed to remain in pressure equilibrium with its surroundings yet fast enough for thermodynamic change to be adiabatic, and the sign of the resulting buoyancy force was then determined: the fluid element would continue on its path if the density gradient $\mathrm{d}\rho/\mathrm{d}p$ in the ambient medium is less than the corresponding derivative at constant entropy. This is essentially the argument most commonly used in the astrophysical literature today. An isothermal atmosphere is stable, and, as is also the case with liquids, convective instability can result from heating the layer from beneath, in order to produce a temperature gradient that reduces $\mathrm{d}\rho/\mathrm{d}p$ sufficiently to make it lower than $(\partial\rho/\partial p)_s$. Thus, when Schwarzschild (1906) introduced the argument to astrophysics, the criterion was written in terms of temperature gradients, but only for homogeneous fluids. Unfortunately the criterion was subsequently misapplied by others to fluids in which the chemical composition varies, until Ledoux (1947) pointed out the error.

I start my discussion with an application of the energy principle, to determine a necessary condition for instability. The perturbation I consider is an interchange, which, strictly speaking, is not dynamically admissible. Consider two small regions, 1 and 2, separated vertically by distance ξ. Each region i has volume V_i, and is so small that the fluid in it may be regarded as being uniform, with density ρ_i, pressure p_i and first adiabatic exponent γ_{1i}. Moreover, in anticipation of the criterion being local I shall eventually take the limit $\xi \to 0$, and in so doing I shall imagine the regions i to shrink too, in such a way that their linear dimensions always remain small compared with ξ. The original state is one of hydrostatic equilibrium, satisfying

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -g\rho \,, \tag{6-1}$$

where, as before, $g(r)$ is the acceleration due to gravity in the equilibrium state. Imagine now a second state obtained by exchanging the contents of regions 1 and 2, expanding or contracting the material uniformly and adiabatically to such an extent that the regions are exactly filled with the exchanged fluid. Then there need be no associated displacement of the ambient fluid. As in the previous section one now computes the excess energy of the second state over the energy of the original state. To obtain a stability criterion one needs to consider interchanges only in the vicinity of neutral stability, for which ΔE is small; in that case the ratio of the volumes of the two regions is close to unity, and one may set $V_2 = (1+\chi)V_1$ with χ small. (I argue this way solely for simplicity; one can easily generalize the argument, under certain circumstances, to arbitrary volume ratios.) As in the discussion of the dynamical instability in the previous section, the internal energy is expanded in powers of $(V_2 - V_1)/V_1 = \chi$, and the total energy excess is computed to second order in small quantities, yielding

$$\begin{aligned} \Delta E &= (p_2 - p_1)\chi V_1 + \gamma_{11} p_1 \chi^2 V_1 + \ldots + [\rho_1 - (1+\chi)\rho_2] V_1 \int g \,\mathrm{d}\xi + \ldots \\ &= \left[\gamma_1 p \chi^2 + 2\frac{\mathrm{d}p}{\mathrm{d}r}\chi\xi + \frac{1}{\Gamma_1 p}\left(\frac{\mathrm{d}p}{\mathrm{d}r}\right)^2 \xi^2\right] V + \ldots \,, \end{aligned} \tag{6-2}$$

where

$$\Gamma_1 := \frac{\mathrm{d}\ln p}{\mathrm{d}\ln\rho} \tag{6-3}$$

and in which I have ignored the perturbation to the gravitational potential, anticipating that it is not of crucial importance. In obtaining the second line of equation (6-2), p_2 and ρ_2 were expanded as Taylor series to first order in ξ about p_1 and ρ_1, and equation (6-1) was used to combine two terms proportional to $\chi\xi$ into one. It was then possible without ambiguity to omit the subscripts that denote the region to which the variables pertain. The expression in the second line of equations (6-2) is a quadratic form in χ and ξ, and since $\gamma_1 pV > 0$, ΔE must be positive for all χ and ξ if $\Gamma_1^{-1} > \gamma_1^{-1}$. This is a sufficient condition for stability. Consequently its converse, which may be rewritten

$$\frac{1}{\Gamma_1} < \frac{1}{\gamma_1} \,, \tag{6-4}$$

is the required necessary condition for instability.

To obtain a sufficient condition for instability one can consider the variational principle (4-4) without the second integral in equation (4-2). As in the previous section, one invokes completeness to justify that if a vector function $\boldsymbol{\xi}$ can be found that renders K negative, there is at least one eigensolution with $\omega_i^2 < 0$; then the system is unstable. Therefore consider for each value of r a

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perturbation with $\boldsymbol{n}.\boldsymbol{\xi}$ ($=\xi$, say) and div$\boldsymbol{\xi}$ ($=\chi$, say) localized in an infinitesimal neighbourhood of r. The integral K has the same sign as its integrand at r, which is given by the quantity in square brackets in the second line of equations (6-2). That quantity can be negative only if condition (6-4) is violated. Therefore condition (6-4) is both necessary and sufficient for instability.

Of course, in neither this discussion nor the discussion of the spherically symmetric dynamical stability of the previous section was it actually necessary to use the energy principle; it is clear that in both cases one can argue solely from the variational principle. However, there are more complicated cases in which it is simpler to evaluate ΔE than it is to derive appropriate integral relations involving dynamically possible perturbations, so it is perhaps useful to exercise the method. Moreover, one can sometimes make further general progress with energy principles by using a variety of normalization conditions (e.g. Bernstein et al., 1958). Although the minima of ΔE may not then be related to growth rates in a simple way, the sign of ΔE does determine stability.

It is instructive to look at the ΔE-minimizing interchange. It satisfies $(V_2 - V_1)/V_1 = \chi = -(\gamma_1 p)^{-1}(\mathrm{d}p/\mathrm{d}r)\xi \simeq -\gamma_1(p_2 - p_1)/p_1$. In other words, the critical perturbation occurs with a volume ratio such that after putative interchange the fluids in both small regions are again in pressure equilibrium with their environment. This is the case of the K-minimizing perturbation too, and goes some way towards explaining why the somewhat cavalier early discussions such as Reye's and Schwarzschild's actually yield the correct criterion.

It is the case that at marginal stability, where $\Gamma_1 = \gamma_1$ (actually one requires one or more finite regions in which $\Gamma_1^{-1} = \gamma_1^{-1}$, and $\Gamma_1^{-1} > \gamma_1^{-1}$ elsewhere) the energy-minimizing perturbation also has its density equal to that of its immediate environment, and generates no buoyancy force. Therefore it could produce no perturbation to the gravitational field. One might therefore expect that if the perturbation to the gravitational potential were included in the analysis it would not influence the critical disturbance in the case of neutral stability. Consequently condition (6-4) would still hold. That this is actually so was demonstrated by Lebovitz (1965). At the time, the completeness of the eigenfunctions had not been established, and it was possible to establish only the sufficient condition for stability. Lebovitz (1966) demonstrated subsequently that the condition is also necessary, by applying a theorem by Laval, Mercier & Pellat (1965) on the growth of certain solutions of self-adjoint linear differential systems.

Condition (6-4) is satisfied either if $\Gamma_1 > \gamma_1$ or if $\Gamma_1 < 0$. Instability arising when $\Gamma_1 < 0$, namely when ρ increases upwards (p must decrease upwards), is usually called the Rayleigh-Taylor instability, because Rayleigh (1883) and Taylor (1950) discussed it for incompressible fluids (for which γ_1 is effectively infinite and can therefore never be exceeded by Γ_1), but evidently it is dynamically no different from convective instability.

As Ledoux (1947) pointed out, if one wishes to write the instability criterion in terms of temperature, then one must beware when there is a gradient of chemical composition in the equilibrium state. It must be accounted for when transforming Γ_1 into a temperature gradient, but not, of course, when transforming γ_1, for normally chemical composition remains constant following the flow. An exception to that can be when nuclear reactions are taking place, but if the timescales are such that chemical transmutations are dynamically important during the development of the perturbation it is likely that nonadiabatic effects are also not negligible, and the analysis is then rather more complicated (cf Gough, 1977). So let us assume that the Lagrangian perturbation to the abundance X_k of each chemical species k is zero. Then the condition (6-4) for convective instability may be rewritten

$$\nabla > \nabla_{\rm ad} + H_p \left(\frac{\partial \ln T}{\partial \ln \rho}\right)_{p,X_k} \left(\frac{\partial \ln \rho}{\partial X_k}\right)_{p,T} \frac{{\rm d}X_k}{{\rm d}r}\,, \tag{6-5}$$

in which I have used the summation convention. Here $\nabla := {\rm d}\ln T/{\rm d}\ln p$, $\nabla_{\rm ad} := (\partial \ln T/\partial \ln p)_s =: 1 - \gamma_2^{-1}$ and H_p is the pressure scale height. If the relative abundance of a single chemical species dominates the equation of state, then X_k can be reduced to a single variable, which for a perfect gas can be represented by the mean molecular mass μ. If the gas is not in a state of partial ionization, $(\partial \ln T/\partial \ln \rho)_{p,\mu} = -1$ and $(\partial \ln \rho/\partial \ln \mu)_{p,T} = +1$, and criterion (6-5) reduces to

$$\nabla > \nabla_{\rm ad} + \frac{{\rm d}\ln \mu}{{\rm d}\ln p}\,; \tag{6-6}$$

it shows that if μ decreases upwards, which is usually the case, the system is more stable than a naïve misapplication of the Schwarzschild criterion, $\nabla > \nabla_{\rm ad}$, might lead one to suspect. Condition (6-6) is essentially the Ledoux criterion.

Before concluding this discussion I make a short comment on convection in the presence of gradients of chemical composition, which in some circumstances is called semiconvection. In the case of ordinary convection, which would arise if the Ledoux condition were satisfied, the nonlinear development of the motion would typically be to homogenize the fluid, after which the Ledoux criterion reduces to the Schwarzschild criterion. The tendency of the instability is to move the mean stratification towards the state of neutral stability according to the joint criterion. But if $\nabla > \nabla_{\rm ad}$, yet condition (6-6) is not satisfied, motion can still arise, and modify the mean stratification to a state presumably somewhere between the two criteria. Astrophysicists were forced to address this issue when circumstances in massive stars were found in which, owing to a strong dependence of electron-scattering-dominated opacity on chemical composition, a consistent equilibrium configuration could not be found in which there is convection wherever condition (6-6) is satisfied and there is no convection where it is not (Schwarzschild & Härm, 1958). However, the potential for motion does not require opacity variations to force the issue in this way. The motion might be triggered by nonlinear perturbations of a kind different from convection, such as diffusively driven overstable oscillations, as can be the case of doubly diffusive convection (e.g. Turner, 1973). Understanding of this phenomenon has not advanced significantly in the last two decades or so, yet it is extremely important for modelling the cores of massive stars, and therefore for understanding the details of the late stages of stellar evolution.

7. Rotational splitting

In a spherically symmetrical, necessarily nonrotating and nonmagnetic star, the components of the displacement eigenfunction $\boldsymbol{\xi}$ may be written in complex separated form with respect to spherical polar coordinates (r, θ, ϕ) as

$$\boldsymbol{\xi}(\mathbf{r}, t) = \left(\xi(r) P_l^m,\ \frac{\eta(r)}{L}\frac{\mathrm{d}P_l^m}{\mathrm{d}\theta},\ \frac{\mathrm{i}m\eta(r)}{L\sin\theta}P_l^m\right) \mathrm{e}^{\mathrm{i}(m\phi-\omega t)}\ , \tag{7-1}$$

where $P_l^m(\cos\theta)$ is an appropriately normalized associated Legendre function of the first kind and $L = \sqrt{l(l+1)}$. The integers l and m are respectively the degree and order of the spherical harmonic $Y_l^m = P_l^m(\cos\theta)\mathrm{e}^{\mathrm{i}m\phi}$, with $l \geq 0$ and $|m| \leq l$, and are called the degree and azimuthal order of the mode of oscillation. (The azimuthal order should not be confused with the mass variable of §5.) For each (l, m) there is a discrete sequence of eigenfunctions $(\xi_{n,l}(r), \eta_{n,l}(r))$ which are labelled with the integer n, called the order of the mode (sometimes principal order or radial order), such that the associated eigenfrequency $\omega_{n,l}$ increases monotonically with n at fixed l. The eigenfrequencies ω and the eigenfunctions (ξ, η) – I drop the suffices where no ambiguity is likely to arise – are independent of m. That is obvious because m is an indicator of only the azimuthal variation of the motion, which depends on the axis of the coordinate system, and for a spherically symmetrical background state all axes are equivalent.

The degeneracy of the eigensolutions with respect to m is lifted by any symmetry-breaking agent, such as rotation. The effect on the eigenfrequencies was first studied by Cowling & Newing (1949) and subsequently by Ledoux (1951), in the case of uniform rotation. Separability of the eigenfunctions similar to that expressed by equation (7-1) is maintained if the axis of the coordinate system is chosen to correspond with the axis of rotation, although now the θ dependence is no longer precisely an associated Legendre function. The eigenfunctions continue to be complete (Dyson & Schutz, 1979).

It is easy to see how the eigenfrequencies are influenced when the rotation is uniform. Transform to a frame rotating with the star. Then the sole change to the governing equations is to replace the left-hand side of equation (3-1) by

$$\rho\left[\frac{\partial^2\boldsymbol{\xi}}{\partial t^2} + 2\boldsymbol{\Omega}_\wedge\frac{\partial\boldsymbol{\xi}}{\partial t} + \boldsymbol{\Omega}_\wedge(\boldsymbol{\Omega}_\wedge\boldsymbol{\xi})\right]\ ,$$

where $\boldsymbol{\Omega}$ is the angular velocity of the star relative to an inertial frame. Note that now $\boldsymbol{\xi}$ is the displacement eigenfunction relative to the rotating coordinate system. Of course the equilibrium variables ρ, c^2 and Φ are distorted from spherical symmetry by the centrifugal force, but they remain axisymmetric. Consequently the ϕ dependence remains sinusoidal.

In the case of slow rotation, satisfying $\Omega := |\boldsymbol{\Omega}| \ll \omega$, terms quadratic in Ω/ω can be ignored in the first instance, and the Coriolis term is the sole perturbation that remains. Following Ledoux (1951), one can then take the scalar product of the modified equation (3-1) with $\boldsymbol{\xi}^*$, as in §4, and obtain

$$(\omega^2 - 2\mathrm{i}\omega\mathcal{C})I = K\ , \tag{7-2}$$

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in which ω is the frequency in the rotating frame and the Coriolis factor $\mathcal{C}$ is given by

$$\mathcal{C} = \int_{\mathcal{V}} (\boldsymbol{\xi}_\wedge \boldsymbol{\xi}^*).\boldsymbol{\Omega}\rho \mathrm{d}V. \tag{7-3}$$

Because equation (7-2) without $\mathcal{C}$, namely equation (4-4), is known to constitute a variational principle, the distortion of the eigenfunctions $\boldsymbol{\xi}$ induced by rotation, which is $\mathrm{O}(\omega^{-1}\Omega)$, influences the quotient K/I by an amount which is only $\mathrm{O}(\omega^{-2}\Omega^2)$, and therefore for calculating the first-order perturbation to the eigenfrequencies it is adequate to substitute the undistorted eigenfunctions (7-1) of the corresponding nonrotating star into the integrals I, K and $\mathcal{C}$. Then the Coriolis term becomes

$$\mathcal{C} = \mathrm{i}mI^{-1}\int_0^R (2L^{-1}\xi\eta + L^{-2}\eta^2)\Omega\rho r^2 \mathrm{d}r =: \mathrm{i}mC\overline{\Omega}\,, \tag{7-4}$$

with $\overline{\Omega} = \Omega$, provided the normalization of P_l^m is chosen such that

$$I = \int_0^R (\xi^2 + \eta^2)\rho r^2 \mathrm{d}r. \tag{7-5}$$

Equation (7-2) can now be solved for the frequency ω in the rotating frame. It is consistent within this approximation to retain the Coriolis correction in the solution to equation (7-2) only to first order. Finally, one can perform a transformation into an inertial frame, yielding (using, I hope without confusion, the same notation ω for the frequency in the inertial frame),

$$\omega \simeq \pm\Sigma + (1 - C)m\overline{\Omega}. \tag{7-6}$$

Degeneracy splits the frequencies by an amount proportional to both $\overline{\Omega}$ and the dimensionless azimuthal wavenumber m. Thus, the angular phase propagation speed ω/m of any mode of a multiplet (i.e. the set of modes of like n and l) is independent of m, and consequently any disturbance comprising the modes of a single multiplet rotates with angular velocity $(1 - C)\overline{\Omega}$, relative to an inertial frame, without change of form.

It is the case, although it is not immediately obvious, that equations (7-2)–(7-3) hold also when Ω is a function of r and θ; and so does equation (7-4) when Ω is a function of r alone, provided that the equations are presumed now to apply in a frame rotating with angular velocity $\overline{\Omega} = I^{-1}\int(\xi^2 + \eta^2)\Omega\rho r^2 \mathrm{d}r$. In that case $\boldsymbol{\xi}$ is now to be interpreted as the position of an element of fluid undergoing pulsation relative to the position it would have had had the star not been pulsating; moreover, the formula for $\mathcal{C}$, which implicitly contains integrations over θ and ϕ weighted with the square of P_l^m, must be modified to take explicit account of the variation of Ω with θ. Moreover, even though the problem is not self-adjoint (for the integral $\mathcal{C}$ is antisymmetric, not symmetric, in $\boldsymbol{\xi}$ and $\boldsymbol{\xi}^*$), the entire generalization of equation (7-2) to the case when $\omega^{-1}\Omega$ is not small also constitutes a variational principle, provided that now $\boldsymbol{\xi}^*$ is interpreted as the adjoint displacement eigenfunction (which satisfies the complex conjugate of the differential equations satisfied by $(\xi(r),\ \eta(r))$ but with Ω replaced by $-\Omega$), as was shown by Lynden-Bell & Ostriker (1967) for the more general case of a background state in which there is an arbitrary stationary flow.

When Ω is a function of r alone, C is still proportional to m, and the azimuthal propagation of the modes remains nondispersive. But when Ω varies with θ, the integral corresponding to that in equation (7-4) now depends explicitly on m; propagation is no longer nondispersive, and the mode is sheared.

Art Cox once asked me why it is that C is always found to be positive, implying that any standing wave comprising the sum of two modes of equal amplitude with azimuthal degrees $\pm m$ always rotates more slowly than $\overline{\Omega}$. It is not evident from the form of the integral in equation (7-4) that that should be the case. For high-order g modes (having $n \gg l$), for example, $|\eta| \gg |\xi|$ almost everywhere, as Cowling (1941) first pointed out, and C is clearly positive. But for high-order p modes, $|\xi| \gg |\eta|$ almost everywhere, yet ξ and η oscillate approximately $\pi/2$ out of phase, so the sign of C appears to be indeterminate. Nevertheless, neither Cox nor I had ever seen a counterexample to Cox's postulate. So far as I am aware, Ledoux never commented on this point, and I spent some time trying to answer Cox's question without success. But at this meeting Wojtek Dziembowski told me that he has encountered (rare) instances in which $C < 0$, so the conjecture that C is always positive is not sustained.

8. Concluding Remarks

If instead of using the adiabatic equation (3-4) to obtain equation (4-4) one uses the nonadiabatic counterpart derived from equation (2-7), one obtains the generalization

$$(\omega^2 - 2\mathrm{i}\omega C)I - K = \mathrm{i}\omega^{-1}Q := \frac{\mathrm{i}}{\omega}\int_{\mathcal{V}}(\gamma_3 - 1)\frac{\delta\rho^*}{\rho}(\delta\varepsilon - \frac{1}{\rho}\delta\mathrm{div}\boldsymbol{F})\rho\mathrm{d}V. \qquad (8\text{-}1)$$

Since $\delta\varepsilon$ and $\delta\rho^*$ are typically in phase (because temperature usually increases on compression and ε is an increasing function of both ρ and T), $\delta\varepsilon$ is destabilizing; the effect of div$\boldsymbol{F}$ is generally stabilizing because δT, and consequently δp, are typically reduced in magnitude by radiative diffusion, and often by convection too, but can be destabilizing in the ionization zones of abundant elements where the increased opacity on compression can dam up the energy flowing through the star (Eddington, 1926). It can be shown, furthermore, that if $\delta\varepsilon$ and δdiv$\boldsymbol{F}$ are regarded as functions of $\boldsymbol{\xi}$ (which requires further equations describing the generation and transport of energy, which I have not presented here) and $\delta\rho^*$ is regarded as a function of $\boldsymbol{\xi}^*$, and if in addition $\boldsymbol{\xi}^*$ is considered to be adjoint to $\boldsymbol{\xi}$ rather than merely the complex conjugate, then equation (8-1) constitutes a variational principle (cf Roberts, 1960). However, since nonadiabatic processes are relatively small throughout most of the star, one might be tempted to assume that $\boldsymbol{\xi}$ and $\boldsymbol{\xi}^*$ are not very different from their adiabatic counterparts, and one might therefore replace them both by the adiabatic (nonrotating, if $|C| \ll \omega$) eigenfunctions. This has been called the quasiadiabatic approximation. The approximation is often not very good for p modes, which tend to have relatively large amplitudes in the surface regions where nonadiabatic effects are not everywhere small, but it should be good for g modes. Indeed, in 1972 Dilke and I used it to estimate the stability of the Sun to grave low-degree g modes using this approximation, and concluded that once sufficient time had elapsed on the main sequence for a substantial abundance of ^{3}He to have been built

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up near the edge of the core, overstability would ensue. This conclusion was subsequently reaffirmed by more careful calculations by Christensen-Dalsgaard et al. (1974), and by all those similar calculations published subsequently (such as that reported by Boury et al. (1975)), quasiadiabatic and nonadiabatic, in which the imbalance of the nuclear reactions was accounted for consistently.

The instability discussed by Dilke & Gough (1972) is driven by the thermal energy generated by the nuclear reactions in the core. It can arise with grave g modes partly because the damping of temperature fluctuations by thermal diffusion is relatively small (this requires the degree l to be small, but it is also helped by the evanescence of the modes in the convection zone which prevents the amplitudes from becoming excessively large in the surface layers, unlike the case for p modes of the Sun or grave low-degree g modes of earlier-type stars having shallower convection zones), and partly because the balance between the reactions of the p-p chain is upset by the g-mode perturbation, causing the perturbed energy generation rate to be rather more sensitive to temperature than it is when the reactions are in balance, thereby augmenting the destabilizing $\delta\varepsilon$ relative to what it would otherwise have been, a phenomenon which had been discussed previously by Ledoux & Sauvenier-Goffin (1950). However, having never computed nonradial stellar oscillations before, to evaluate the integrals in equation (8-1) Dilke and I used for our initial estimate the eigenfunctions of a polytrope of index 3 (which approximates the radiative interior of the Sun quite well) published by Cowling (1941), rather than computing eigenfunctions of a more realistic solar model.

The matter was taken up by Paul Ledoux in a presentation to IAU Symposium 59 in 1973. Ledoux also used the eigenfunctions of a polytrope of index 3, also in the quasiadiabatic approximation. And in many ways his discussion complements ours. However, in places Ledoux appears not entirely to agree with us, although he does not explicitly contradict what we say. When I encountered Paul a year or so later, I asked him why he had written his paper in so oblique a style, and it transpired that when he had repeated what he thought was the calculation that we had done he did not find overstability. Perturbed by the realization that we must have made an error, I embarked on a detailed questioning to ascertain what Paul had done. But it took a long time to establish a mutual understanding. The reason was that it had not crossed Paul's mind that we should have used an erroneous column in Cowling's table (just as it had not crossed our minds that Cowling would have made a mistake), yet we had. (I wonder, sometimes, whether I would have persuaded Fisher Dilke, and subsequently Jørgen Christensen-Dalsgaard, to work with me on the more realistic model had the original calculation been performed with the correct polytropic eigenfunctions.) 'But the error is pointed out in my article with Walraven', exclaimed Paul (which, of course, it is, in a brief footnote on p.523), 'so I presumed that you knew'. Evidently I had not digested the article sufficiently well. So I promptly read it again. And my advice to every reader is to do likewise.

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