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A REMARK ON NONEXPANSIVE MAPPINGS

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Let X be a closed convex subset of a Banach space and let $T:X\to X$ be a nonexpansive mapping, i.e.

$$\|Tx-Ty\|\le\|x-y\| \quad \text{for } x, y\in X.$$

It is well known that the set Fix T of fixed points of T may be empty unless X has some "nice" geometrical and topological properties [see e.g. [1], [2], [3], [4]].

We are going to prove a fixed point theorem assuming nothing more about the regularity of X but putting some additional condition on the mapping T itself.

Let us call a nonexpansive mapping $T:X\to X$ "rotative" if there exists an integer $n\ge 2$ and a real number $a<n$ such that for any $x\in X$

$$\|x-T^n x\|\le a\,\|x-Tx\|, \tag{1}$$

and use the notation "k-rotative" if T is rotative for $n=k$.

We should note that if T is nonexpansive, then for each positive integer m, $\|x-T^m x\|\le m\,\|x-Tx\|$. One can observe that if T is n-rotative, it is also m-rotative for $m>n$.

The simplest examples of rotative nonexpansive mappings are all contractions, rotations of Euclidean space $\mathbb{R}^n$ or any periodic nonexpansive mappings in any Banach space.

THEOREM. *If a nonexpansive mapping $T:X\to X$ is rotative then Fix $T\ne\emptyset$.*

Proof. Since T is nonexpansive, $I-\alpha T$ is invertible for each $\alpha\in(0,1)$; thus let us define a mapping $F_\alpha:X\to X$ by $F_\alpha=(I-\alpha T)^{-1}(1-\alpha)I$. It is easy to see that Fix $T=$ Fix F_α and F_α is also nonexpansive. Actually we have

$$y=F_\alpha x \Leftrightarrow y=(1-\alpha)x+\alpha Ty,$$

implying

$$F_\alpha x=(1-\alpha)x+\alpha TF_\alpha x,\ F_\alpha^2 x=(1-\alpha)F_\alpha x+\alpha TF_\alpha^2 x,\ \dots,\ \text{etc.}$$

and

$$(1-\alpha)(x-F_\alpha x)=\alpha(F_\alpha x-TF_\alpha x).$$

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Thus we have

$$\begin{aligned}\|F_\alpha x - F_\alpha^2 x\| &= \|F_\alpha x - (1-\alpha)F_\alpha x - \alpha T F_\alpha^2 x\| = \alpha \|F_\alpha x - TF_\alpha^2 x\| \\ &\leq \alpha \|F_\alpha x - T^n F_\alpha x\| + \alpha \|T^n F_\alpha x - TF_\alpha^2 x\| \\ &\leq \alpha a \|F_\alpha x - TF_\alpha x\| + \alpha \|T^{n-1} F_\alpha x - F_\alpha^2 x\| \\ &= (1-\alpha)a \|x - F_\alpha x\| + \alpha \|T^{n-1} F_\alpha x - F_\alpha^2 x\|;\end{aligned}$$

i.e.,

$$\|F_\alpha x - F_\alpha^2 x\| \leq (1-\alpha)a \|x - F_\alpha x\| + \alpha \|T^{n-1} F_\alpha x - F_\alpha^2 x\|. \tag{2}$$

Using only the nonexpansive property of T, we proceed by induction to establish the following inequality which is needed:

$$\alpha \|T^{k-1} F_\alpha x - F_\alpha^2 x\| \leq [(k-1) - k\alpha + \alpha^k] \|x - F_\alpha x\| + \alpha^k \|F_\alpha x - F_\alpha^2 x\|. \tag{3}$$

For $k = 2$

$$\begin{aligned}\alpha \|TF_\alpha x - F_\alpha^2 x\| &= \alpha \|TF_\alpha x - (1-\alpha)F_\alpha x - \alpha TF_\alpha^2 x - \alpha TF_\alpha x + \alpha TF_\alpha x\| \\ &\leq \alpha(1-\alpha) \|TF_\alpha x - F_\alpha x\| + \alpha^2 \|TF_\alpha x - TF_\alpha^2 x\| \\ &\leq (1-\alpha)^2 \|x - F_\alpha x\| + \alpha^2 \|F_\alpha x - F_\alpha^2 x\|.\end{aligned}$$

For $k = n+1$, we have

$$\begin{aligned}\alpha \|T^n F_\alpha x - F_\alpha^2 x\| &= \alpha \|(1-\alpha)T^n F_\alpha x + \alpha T^n F_\alpha x - (1-\alpha)F_\alpha x - \alpha TF_\alpha^2 x\| \\ &\leq \alpha(1-\alpha) \|T^n F_\alpha x - F_\alpha x\| + \alpha^2 \|T^n F_\alpha x - TF_\alpha^2 x\| \\ &\leq n\alpha(1-\alpha) \|TF_\alpha x - F_\alpha x\| + \alpha^2 \|T^{n-1} F_\alpha x - F_\alpha^2 x\| \\ &= n(1-\alpha)^2 \|x - F_\alpha x\| + \alpha^2 \|T^{n-1} F_\alpha x - F_\alpha^2 x\|\end{aligned}$$

and by the induction hypothesis

$$\begin{aligned}\alpha \|T^n F_\alpha x - F_\alpha^2 x\| &\leq n(1-\alpha)^2 \|x - F_\alpha x\| + \alpha[(n-1) - n\alpha + \alpha^n] \|x - F_\alpha x\| \\ &\quad + \alpha^{n+1} \|F_\alpha x - F_\alpha^2 x\| \\ &= [n + (n+1)\alpha + \alpha^{n+1}] \|x - F_\alpha x\| + \alpha^{n+1} \|F_\alpha x - F_\alpha^2 x\|\end{aligned}$$

as desired.

From (2) and (3) we conclude that

$$\begin{aligned}\|F_\alpha x - F_\alpha^2 x\| &\leq \frac{(1-\alpha)a + (n-1) - n\alpha + \alpha^n}{1-\alpha^n} \|x - F_\alpha x\| \\ &= \left[(a+n)\left(\sum_{i=0}^{n-1} \alpha^i\right)^{-1} - 1\right] \|x - F_\alpha x\| \\ &= g(\alpha) \|x - F_\alpha x\|.\end{aligned}$$

Since g is continuous and decreasing on $(0, 1\rangle$ with $g(1) < 1$, then there exists $b \in (0, 1)$ such that $g(\alpha) < 1$ for each $\alpha \in (b, 1)$. This observation implies the

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convergence of iterates $\{F^n_\alpha x\}$ for any $x \in X$, thus the existence of fixed point for T.

One can notice that the set Fix T is a nonexpansive retract of X. Actually the mapping $Rx = \lim_{n\to\infty} F^n_\alpha x$ is the retraction of X onto Fix T.

We feel that rotativeness is quite natural metrical assumption. However we are aware of the fact that for concrete mapping T it may be difficult to check whether it is rotative or not.

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