

# NORMAL FUNCTIONS FOR ALGEBRAICALLY TRIVIAL CYCLES ARE ALGEBRAIC FOR ARITHMETIC REASONS

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## Abstract

For families of smooth complex projective varieties, we show that normal functions arising from algebraically trivial cycle classes are algebraic and defined over the field of definition of the family. In particular, the zero loci of those functions are algebraic and defined over such a field of definition. This proves a conjecture of Charles.

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## 1. Introduction

Let  $f : X \rightarrow B$  be a smooth surjective projective morphism of complex algebraic manifolds, let  $n$  be an integer, and let  $J^{2n+1}(X/B) \rightarrow B$  be the  $(2n + 1)$ th relative Griffiths intermediate Jacobian. If  $Z \in CH^{n+1}(X)$  is an algebraic cycle class such that for every  $\mathbf{b} \in B$  the Gysin fiber  $Z_{\mathbf{b}}$  is algebraically (respectively homologically) trivial, then there is an associated holomorphic function

$$\nu_Z : B \longrightarrow J^{2n+1}(X/B), \quad \nu_Z(\mathbf{b}) = \mathbf{A}J_{X_{\mathbf{b}}}(Z_{\mathbf{b}}),$$

where  $\mathbf{A}J_{X_{\mathbf{b}}} : CH^{n+1}(X_{\mathbf{b}})_{\text{hom}} \rightarrow J^{2n+1}(X_{\mathbf{b}})$  is the Abel–Jacobi map on homologically trivial cycles in the fiber  $X_{\mathbf{b}}$ . Such a function is called an algebraically

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motivated (respectively motivated) normal function motivated by the cycle class  $Z$ .

More generally, let  $B$  be a complex manifold and let  $\mathcal{H}$  be a variation of pure negative weight integral Hodge structures over  $B$ . In [Sai96], Saito defines the notion of an admissible normal function as a holomorphic section  $\nu : B \rightarrow J(\mathcal{H})$  of the associated family of generalized intermediate Jacobians  $J(\mathcal{H}) \rightarrow B$  that satisfies a version of Griffiths horizontality and has controlled asymptotic behavior near the boundary (see, for example, [BP13]). Despite the transcendental nature of the definition of admissible normal functions, there is the following conjecture due to Green and Griffiths (for example, [BP09, page 883], [Cha10, Conjecture 1], [Sch12, Conjecture 1.1], [BP13, page 1914]).

**CONJECTURE 1** (Green–Griffiths). *The zero locus of an admissible normal function on a complex algebraic manifold is algebraic.*

Proofs of this conjecture were given in a series of papers:  $\dim B = 1$  [Sai08, Corollary 1], [BP09, Theorem 4.5],  $\dim B \geq 1$  [Sch12, Theorem C], [BP13, Corollary 1.3] (see also [Cha14]). In this paper, we are interested in algebraic and arithmetic questions concerning motivated normal functions. First, for algebraically motivated (respectively motivated) normal functions, if  $X$ ,  $B$ ,  $f$ , and  $Z$  are all defined over a subfield  $F \subseteq \mathbb{C}$ , we say that the normal function  $\nu_Z$  is algebraically  $F$ -motivated (respectively  $F$ -motivated), and it is natural to ask whether the zero locus of  $\nu_Z$  in  $B$  is also defined over  $F$  [Cha10, page 2284] (see also [KP11, Conjecture 81]).

**CONJECTURE 2** (Charles). *Let  $F \subseteq \mathbb{C}$  be a subfield. The zero locus of an algebraically  $F$ -motivated (respectively  $F$ -motivated) normal function is algebraic and defined over  $F$ .*

Several partial results are known. Regarding the  $F$ -motivated case of Conjecture 2, a special case of a result of Saito [Sai16, Corollary 1] on admissible normal functions implies that if an irreducible component of the zero locus of an  $F$ -motivated normal function contains a point of  $B$  that is defined over  $F$ , then the entire component of the zero locus is defined over  $F$  (see also [Cha10, Theorem 3], [KP11, Theorem 89]). Regarding the algebraically  $F$ -motivated case of Conjecture 2, Kerr and Pearlstein have shown in [KP11, Conjecture 81,  $\widetilde{\mathfrak{Z}}\mathcal{L}(D, 1)_{alg}$ , Theorem 88] that the zero locus of an algebraically  $F$ -motivated normal function is an algebraic subset of  $B$  defined over a finite extension of  $F$ . All of the aforementioned results take as a starting point the validity of Conjecture 1.

In this paper, we directly prove Conjecture 2 in the algebraically  $F$ -motivated case. (In particular, we do not rely on the earlier work on Conjecture 1.) In fact, we prove a stronger result, namely that algebraically  $F$ -motivated normal functions are themselves algebraic and defined over  $F$ .

**THEOREM 1.** *Let  $f : X \rightarrow B$  be a smooth surjective projective morphism of complex algebraic manifolds (not necessarily connected), let  $n$  be a nonnegative integer, and let  $J^{2n+1}(X/B) \rightarrow B$  be the  $(2n + 1)$ th relative Griffiths intermediate Jacobian. There is a relative algebraic complex subtorus  $J_a^{2n+1}(X/B) \subseteq J^{2n+1}(X/B)$  over  $B$  such that for very general  $u \in B$ , the fiber  $J_a^{2n+1}(X/B)_u \subseteq J^{2n+1}(X_u)$  is the image  $J_a^{2n+1}(X_u)$  of the Abel–Jacobi map  $AJ_{X_u} : A^{n+1}(X_u) \rightarrow J^{2n+1}(X_u)$ , and for any algebraic cycle class  $Z \in CH^{n+1}(X)$  such that for every  $b \in B$  the Gysin fiber  $Z_b$  is algebraically trivial:*

- (1) *the normal function  $v_Z : B \rightarrow J^{2n+1}(X/B)$  has image contained in  $J_a^{2n+1}(X/B)$  and is an algebraic map;*
- (2) *if, moreover,  $X, B, f$ , and  $Z$  are all defined over a field  $F \subseteq \mathbb{C}$ , then so are  $J_a^{2n+1}(X/B)$  and the morphisms  $J_a^{2n+1}(X/B) \rightarrow B$  and  $v_Z$ .*

See Remark 5.2 for a caution about the notation  $J_a^{2n+1}(X/B)$ , and see the notation and conventions below for a reminder on very general points. Conjecture 1 in the algebraically motivated case follows immediately from Theorem 1(1), and in this way, we obtain a short proof of this case of the conjecture. Conjecture 2 in the algebraically  $F$ -motivated case follows immediately from Theorem 1(2). In summary, we have the following.

**COROLLARY 1.** *Let  $F \subseteq \mathbb{C}$  be a subfield. The zero locus of an algebraically  $F$ -motivated normal function is algebraic and defined over  $F$ .*

We review some special easy cases of Theorem 1, with an eye toward explaining why their generalization is not immediate. In the case of  $n + 1 = 1$ , that is, of  $\text{Pic}_{X/B}^0$ , and  $n + 1 = \dim_B X$ , that is, of  $\text{Alb}_{X/B}$ , it is well known that algebraically  $F$ -motivated normal functions are algebraic and defined over  $F$  (for example, [Gro62, Theorem VI.3.3], [Kle05, Definition 4.6 and Theorem 4.8]). In the case where  $B$  is quasiprojective,  $X = B \times Y$  for some smooth projective complex manifold  $Y$ , and  $f : X \rightarrow B$  is the first projection, part (1) of the theorem is elementary by embedding  $B$  in a smooth complex projective manifold  $\bar{B}$ , extending the cycle class  $Z$  to a cycle class  $\bar{Z}$  on  $\bar{B} \times Y$ , and obtaining a normal function  $v_{\bar{Z}} : \bar{B} \rightarrow \bar{B} \times J_a^{2n+1}(Y)$  that is a holomorphic map between complex projective manifolds. From our work in [ACMV18], one can then easily deduce

(2) of the theorem in this case, as well. The difficulty in using the same strategy to prove part (1) of the theorem in general is twofold. First, the family  $f : X \rightarrow \mathbf{B}$  may not extend to a smooth family over  $\overline{\mathbf{B}}$ , in which case it is difficult to know how to extend  $J_a^{2n+1}(X/\mathbf{B})$  and  $\nu_Z$  to the boundary. Second, even if one can extend  $f : X \rightarrow \mathbf{B}$  to a smooth family  $\tilde{f} : \overline{X} \rightarrow \overline{\mathbf{B}}$ , the geometric coniveau of the family can jump along a countable union of algebraic subsets of  $\overline{\mathbf{B}}$ , and so there is no obvious algebraic target  $J_a^{2n+1}(\overline{X}/\overline{\mathbf{B}})$  for an extended normal function.

One faces similar difficulties in trying to prove Conjecture 1, and the approach taken in [Sai08, BP09, Sch12] overcomes these complications by constructing Néron models for the relative intermediate Jacobians (see also [GGK10]) that provide manageable targets for extending admissible normal functions. In the special case where  $\dim \mathbf{B} = 1$ , Schnell and Kerr independently communicated to us arguments using these techniques to prove part (1) of the theorem, up to replacing the normal function  $\nu_Z$  with  $M \cdot \nu_Z$  for some integer  $M$ , depending on  $Z$ . It appears, however, that it would be difficult to extend these arguments to the case where  $\dim \mathbf{B} \geq 2$ . It also appears that it would be difficult to use these techniques to prove part (2) of the theorem regarding the field of definition, even in the case where  $\dim \mathbf{B} = 1$ .

The starting point of our proof consists in showing that, for a smooth projective variety  $X$  defined over a subfield  $K \subseteq \mathbb{C}$ , the kernel of the Abel–Jacobi map restricted to algebraically trivial cycles defined over  $K$  is independent of the choice of field embedding  $K \subseteq \mathbb{C}$ . This is embodied in Corollary 3.3; in fact, a stronger result is proved in Proposition 3.1 where it is shown that the distinguished model of [ACMV18] does not depend on a choice of field embedding. The proof uses in an essential way the fact proven in [ACMV19] that algebraically trivial cycles defined over  $K$  are parameterized by abelian varieties, and builds on our previous work [ACMV18]. Consequently, the relevant material of [ACMV18] is reviewed in Section 2. An important consequence of Proposition 3.1 is that an algebraically  $F$ -motivated normal function vanishes at a very general point if and only if it vanishes on a Zariski open subset and is therefore identically zero; see Example 3.6 and Remarks 3.7 and 3.8.

In fact, the initial step of our strategy is to consider a very general fiber  $X_u$  and, thanks to Proposition 3.1, to descend the image of the Abel–Jacobi map  $J_a^{2n+1}(X_u)$  for this fiber to an abelian variety over the generic point of  $\mathbf{B}$ , which admits a natural section related to the normal function. We then spread this abelian variety and section to a Zariski open subset of  $\mathbf{B}$ , all defined over  $F$ . In Theorem 4.1, we compare this abelian scheme together with the induced section to the analytic normal function. This is achieved through comparing the related variations of Hodge structures via an algebraic correspondence defined over  $K$ , provided by Theorem 2.1. There is a technical point here—the correspondence only identifies

the integral Hodge structures as well as our algebraic section and the analytic normal function, up to an integer multiple  $M$ ; in Theorem 4.1, we show that the image of the morphism of abelian varieties induced by the correspondence, inside the Griffiths intermediate Jacobian, is in fact the spread of our distinguished model and that the algebraic section and analytic normal function are identified. The final step is to extend this to all of  $\mathbf{B}$  (Section 5). We extend the relative algebraic torus over the generic points of codimension-1 boundary loci by using the good reduction of  $X$  and the Néron–Ogg–Shafarevich criterion and then extend over codimension-2 loci using the Faltings–Chai Extension Theorem. The normal function is handled separately at each step. In short, rather than having to worry about extending admissible normal functions to projective compactifications in order to obtain algebraicity as a consequence of Chow’s theorem, we extend algebraic maps defined over  $F$  on a Zariski open subset of  $\mathbf{B}$  to all of  $\mathbf{B}$ , and, in this way, we also manage to maintain control over the field of definition.

In forthcoming work [ACMV], we will study the notion of regular homomorphisms in the relative setting; Theorem 1 shows that the Abel–Jacobi map provides such a relative regular homomorphism.

Finally, although our results are algebraic in nature and only concern algebraically trivial cycle classes, there are important instances of families of varieties for which homological and algebraic equivalence of cycles in certain codimensions agree and for which the corresponding intermediate Jacobians are algebraic [BS83]. For example, a direct application of Theorem 1 concerns codimension-2 cycles on uniruled threefolds.

**COROLLARY 2.** *Let  $f : X \rightarrow \mathbf{B}$  be a smooth projective family of uniruled threefolds, defined over a field  $F \subseteq \mathbb{C}$ , and let  $Z \in \text{CH}^2(X)$  be a cycle class defined over  $F$  that is fiberwise homologically trivial. Then the analytic normal function  $v_Z$  is algebraic and defined over  $F$ . In particular, its zero locus is an algebraic subvariety of  $\mathbf{B}$  defined over  $F$ .*

In the opposite direction, as a specific example of a case where it is not clear how to apply our results to the  $F$ -motivated case of Conjecture 2, one can consider families of Calabi–Yau threefolds; specifically, for very general quintic threefolds, Voisin [Voi00, Theorem 2,3] has shown that the image of the Abel–Jacobi map on *algebraically* trivial codimension-2 cycle classes is trivial, while the image of the Abel–Jacobi map on *homologically* trivial codimension-2 cycle classes is a countable abelian group of infinite rank. In general, it seems to us that it would be interesting to construct a canonical arithmetic structure on the image of all homologically trivial cycles under the Abel–Jacobi map. Such a construction is a necessary first step for extending our methods to normal functions motivated by homologically trivial cycles.

**1.1. Notation and conventions.** Let  $X$  be a scheme of finite type over a field  $F \subseteq \mathbb{C}$ . We denote by  $X(F)$  the set of  $F$ -morphisms  $\text{Spec } F \rightarrow X$ . We denote by  $\mathbf{X} = X_{an}$  the associated complex analytic space. After identifying the sets

$$\begin{aligned} \mathbf{X} &= \{\mathfrak{p} \in |X_{\mathbb{C}}| : \mathfrak{p} \text{ is closed in the underlying topological space } |X_{\mathbb{C}}|\} \\ &= \{x \in X_{\mathbb{C}}(\mathbb{C}) : x(\text{Spec } \mathbb{C}) \text{ is closed in } |X_{\mathbb{C}}|\}, \end{aligned}$$

we say that  $x \in \mathbf{X}$  is *F-very general* if the corresponding morphism  $x : \text{Spec } \mathbb{C} \rightarrow X$  has image a generic point of  $|X|$ . If  $F$  is countable, then so is the collection of all closed algebraic subsets of  $X$  that are defined over  $F$  and not equal to  $X$ ; any  $F$ -very general point  $x$  is in the complement of the union of these closed algebraic subsets. If  $Y$  is merely a complex algebraic variety, then a very general point of  $Y$  is an  $F$ -very general point for some field of definition  $F$  of  $Y$  which is of finite type over  $\mathbb{Q}$ .

A *variety* over a field is a geometrically reduced separated scheme of finite type over that field. Given a smooth projective variety  $X$  over a field  $F \subseteq \mathbb{C}$ , we denote by  $\text{CH}^i(X)$  (respectively  $\text{CH}^i(\mathbf{X})$ ) the Chow group of codimension- $i$  algebraic cycle classes on  $X$  (respectively  $\mathbf{X}$ ), and by  $A^i(X)$  (respectively  $A^i(\mathbf{X})$ ) the subgroup of algebraically trivial algebraic cycle classes on  $X$  (respectively  $\mathbf{X}$ ).

For the remainder of the paper, the domain of the Abel–Jacobi map  $\text{AJ} : A^{n+1}(X) \rightarrow J^{2n+1}(X)$  is the group of algebraically trivial algebraic cycle classes. We denote by  $J_a^{2n+1}(X)$  the image of this Abel–Jacobi map, and by  $i_{a,X}^{2n+1} : J_a^{2n+1}(X) \rightarrow J^{2n+1}(X)$  the natural inclusion.

## 2. Distinguished models of intermediate Jacobians and distinguished normal functions

In this section, we recall some results from [ACMV17, ACMV18] regarding descending intermediate Jacobians to a field of definition.

**2.1. Distinguished models of intermediate Jacobians.** We start by recalling the main result of [ACMV18]. We note that while in Sections 2.1 and 2.2 we work over a field  $K \subseteq \mathbb{C}$ , starting from Section 3.2, we will implement these results in the case where the field  $K$  is the residue field of the generic point of the integral base  $B$  of a smooth projective family  $f : X \rightarrow B$ , all defined over a field  $F \subseteq \mathbb{C}$ .

**THEOREM 2.1** (Distinguished models [ACMV18, Theorem 1]). *Suppose  $X$  is a smooth projective variety over a field  $K \subseteq \mathbb{C}$ , with associated complex analytic space  $\mathbf{X}$ , and let  $n$  be a nonnegative integer. Then  $J_a^{2n+1}(\mathbf{X})$ , the algebraic complex torus that is the image of the Abel–Jacobi map  $\text{AJ} : A^{n+1}(X) \rightarrow J^{2n+1}(X)$ , admits*

a distinguished model  $J_{a,X/K}^{2n+1}$  over  $K$  such that the Abel–Jacobi map is  $\text{Aut}(\mathbb{C}/K)$ -equivariant. Moreover, there exist a positive integer  $M$  and a correspondence  $\Gamma \in \text{CH}^{\dim(J_{a,X/K}^{2n+1})+n}(J_{a,X/K}^{2n+1} \times_K X)$  such that the induced morphism  $\Gamma_* : J_a^{2n+1}(X) \rightarrow J_a^{2n+1}(X)$  is  $M \cdot i_{a,X}^{2n+1}$ ; that is,  $M$  times the natural inclusion.

REMARK 2.2 (Uniqueness of the distinguished model). By Chow’s rigidity theorem, an abelian variety  $A/\mathbb{C}$  descends to at most one model defined over  $\overline{K}$ . On the other hand, an abelian variety  $A/\overline{K}$  may descend to more than one model defined over  $K$ . Nevertheless, since  $\text{AJ} : A^{n+1}(X) \rightarrow J_a^{2n+1}(X)$  is surjective, the algebraic complex torus  $J_a^{2n+1}(X)$  admits at most one structure of a variety over  $K$  such that  $\text{AJ}$  is  $\text{Aut}(\mathbb{C}/K)$ -equivariant. More precisely, setting  $J_a^{2n+1}(X_{\mathbb{C}})$  to be the abelian variety associated with the algebraic complex torus  $J_a^{2n+1}(X)$ , there is an abelian variety  $J_{a,X/K}^{2n+1}$  which is unique up to unique isomorphism, such that there is an isomorphism  $(J_{a,X/K}^{2n+1})_{\mathbb{C}} \rightarrow J_a^{2n+1}(X_{\mathbb{C}})$  such that the induced action of  $\text{Aut}(\mathbb{C}/K)$  on  $J_a^{2n+1}(X)$  makes the Abel–Jacobi map  $\text{Aut}(\mathbb{C}/K)$ -equivariant. This is the sense in which  $J_a^{2n+1}(X)$  admits a distinguished model over  $K$ .

In [ACMV18, Theorem 1], we show that the correspondence  $\Gamma$  in Theorem 2.1 induces a morphism of complex tori  $\Gamma_* : J_a^{2n+1}(X) \rightarrow J_a^{2n+1}(X)$  with image  $J_a^{2n+1}(X)$ . In fact, this morphism respects  $K$ -structures.

LEMMA 2.3. *In the situation of Theorem 2.1, the morphism  $\Gamma_* : J_a^{2n+1}(X) \rightarrow J_a^{2n+1}(X)$  is induced by an isogeny  $\psi : J_{a,X/K}^{2n+1} \rightarrow J_{a,X/K}^{2n+1}$  over  $K$ .*

*Proof.* It suffices to show that  $\Gamma_*$  is  $\text{Aut}(\mathbb{C}/K)$ -equivariant on torsion. This is achieved (as in [ACMV18, (2.3)]) by identifying the map on  $N$ -torsion with the map  $\Gamma_* : H^1(J_a^{2n+1}(X_{\mathbb{C}}), \mu_N) \rightarrow H^1(J_a^{2n+1}(X_{\mathbb{C}}), \mu_N) \subseteq H^{2n+1}(X_{\mathbb{C}}, \mu_N^{\otimes(n+1)})$ . Let  $M$  be the exponent of  $\psi$  and let  $\tilde{\psi} : J_{a,X/K}^{2n+1} \rightarrow J_{a,X/K}^{2n+1}$  be such that  $\tilde{\psi} \circ \psi = M$ . With  $\Gamma_{\tilde{\psi}}$  the correspondence associated with the morphism  $\tilde{\psi}$ , let  $\Gamma' = \Gamma_{\tilde{\psi}} \circ \Gamma$ . It follows that the induced morphism  $\Gamma'_* : J_a^{2n+1}(X) \rightarrow J_a^{2n+1}(X)$  has image  $J_a^{2n+1}(X)$  and is given as  $M \cdot i_{a,X}^{2n+1}$ ; that is,  $M$  times the natural inclusion.  $\square$

REMARK 2.4 (Extensions  $K \subseteq L \subseteq \mathbb{C}$ ). In the notation of the theorem, suppose we have an intermediate field extension  $K \subseteq L \subseteq \mathbb{C}$ . Then the base change  $(J_{a,X/K}^{2n+1})_L$  is the distinguished model for  $X_L$ . Indeed, the distinguished model over  $L$  is determined uniquely by the fact that the Abel–Jacobi map for  $X$  is  $\text{Aut}(\mathbb{C}/L)$ -equivariant; but if the Abel–Jacobi map is  $\text{Aut}(\mathbb{C}/K)$ -equivariant with respect to the  $K$ -structure on  $J_{a,X/K}^{2n+1}$ , then it is  $\text{Aut}(\mathbb{C}/L)$ -equivariant for the  $L$ -structure on  $(J_{a,X/K}^{2n+1})_L$ .

**2.2. Distinguished normal functions.** In [ACMV17], we established some results regarding equivariant regular homomorphisms. In this section, we recall the consequences of that work in the context of normal functions and the distinguished model.

Let  $X$  be a smooth projective variety over  $K \subseteq \mathbb{C}$ . Given  $Z \in \text{CH}^{n+1}(X)$  with the base change  $Z_{\mathbb{C}}$  algebraically trivial, and  $\sigma \in \text{Aut}(\mathbb{C}/K)$ , we showed in [ACMV17] that the following diagram commutes:

$$\begin{CD} (\text{Spec } \mathbb{C})_{\mathbb{C}}(\mathbb{C}) @>w_{Z_{\mathbb{C}}}>> A^{n+1}(X_{\mathbb{C}}) @>AJ>> J_a^{2n+1}(X_{\mathbb{C}})(\mathbb{C}) \\ @| @VV\sigma^*V @VV\sigma(\mathbb{C})V \\ (\text{Spec } \mathbb{C})_{\mathbb{C}}(\mathbb{C}) @>w_{Z_{\mathbb{C}}}>> A^{n+1}(X_{\mathbb{C}}) @>AJ>> J_a^{2n+1}(X_{\mathbb{C}})(\mathbb{C}) \end{CD}$$

where  $(\text{Spec } \mathbb{C})_{\mathbb{C}}(\mathbb{C}) = \{\text{Id}_{\mathbb{C}}\}$  and  $w_{Z_{\mathbb{C}}}(\text{Id}_{\mathbb{C}}) = Z_{\mathbb{C}}$ . Indeed, the right-hand side is the precise meaning of the statement in Theorem 2.1 that the Abel–Jacobi map is  $\text{Aut}(\mathbb{C}/K)$ -equivariant, while the left-hand side is elementary (see [ACMV17, Remark 4.3] for more on this). This corresponds to the commutativity of the diagram of sets:

$$\begin{CD} (\text{Spec } \mathbb{C})_{an} @>w_Z>> A^{n+1}(X) @>AJ>> J_a^{2n+1}(X) \\ @| @VV\sigma^*V @VV\sigma V \\ (\text{Spec } \mathbb{C})_{an} @>w_Z>> A^{n+1}(X) @>AJ>> J_a^{2n+1}(X) \end{CD}$$

where  $w_Z((\text{Spec } \mathbb{C})_{an}) = Z$ , the complex analytic cycle class associated with  $Z$ , and  $\sigma : J_a^{2n+1}(X) \rightarrow J_a^{2n+1}(X)$  is the map of sets induced by  $\sigma(\mathbb{C})$  in the previous diagram. Note that  $AJ \circ w_Z = \nu_Z$ , the normal function associated with  $Z$ . As mentioned in [ACMV17, Remark 4.3], the commutativity of the diagrams above implies that  $AJ \circ w_{Z_{\mathbb{C}}}$ , and hence  $\nu_Z$ , descend to  $K$  to give a morphism

$$\delta_Z : \text{Spec } K \rightarrow J_{a,X/K}^{2n+1}. \tag{2.1}$$

We call this the *distinguished normal function* associated with  $Z$ .

**REMARK 2.5 (Uniqueness of the distinguished normal function).** The distinguished normal function is unique in the sense that given the distinguished model  $J_{a,X/K}^{2n+1}$  (unique up to unique isomorphism by Remark 2.2), there is a unique morphism  $\delta_Z : \text{Spec } K \rightarrow J_{a,X/K}^{2n+1}$  such that  $(\delta_Z)_{an} : (\text{Spec } \mathbb{C})_{an} \rightarrow J_a^{2n+1}(X)$  is equal to the analytic normal function  $\nu_Z$ .

**REMARK 2.6 (Extensions  $K \subseteq L \subseteq \mathbb{C}$ ).** In the notation of Theorem 2.1, suppose we have an intermediate field extension  $K \subseteq L \subseteq \mathbb{C}$ . In light of Remark 2.4,



we have a fibered product diagram

$$\begin{array}{ccc}
 J_{a, X_L/L}^{2n+1} & \longrightarrow & J_{a, X/K}^{2n+1} \\
 \delta_{Z_L} \uparrow \downarrow & & \downarrow \uparrow \delta_Z \\
 \text{Spec } L & \longrightarrow & \text{Spec } K
 \end{array}$$

**2.3. Review of the construction of the distinguished model.** Because it will be relevant later, we recall the construction of the distinguished model  $J_{a, X/K}^{2n+1}$  from [ACMV18]. The starting point is [ACMV18, Proposition 1.1], which provides a smooth projective geometrically integral curve  $C/K$  (admitting a  $K$ -point) and a correspondence  $\gamma \in \text{CH}^{n+1}(C \times_K X)$  such that the induced morphism of complex tori  $\gamma_* : J(C) \rightarrow J^{2n+1}(X)$  has image  $J_a^{2n+1}(X)$ . We thus obtain a short exact sequence of algebraic compact complex analytic groups  $0 \rightarrow P \rightarrow J(C) \rightarrow J_a^{2n+1}(X) \rightarrow 0$ , where  $P$  is defined to be the kernel.

The next step is to show that  $P$  descends to  $K$ . For this, it suffices to show that for every natural number  $N$ , the  $N$ -torsion  $P[N]$  is preserved by  $\text{Aut}(\mathbb{C}/K)$  (since torsion is dense in any subgroup scheme of an abelian variety; see, for example, [ACMV18, Lemma 2.3]). For this, one shows that  $P[N]$  is equal to the kernel of the morphism  $\gamma_* : H^1(C_{\mathbb{C}}, \mu_N) \rightarrow H^{2n+1}(X_{\mathbb{C}}, \mu_N^{\otimes(n+1)})$  (see [ACMV18, (2.3)]), which is equivariant as it is induced by a correspondence defined over  $K$ . Thus,  $P[N]$  is preserved by  $\text{Aut}(\mathbb{C}/K)$  so that  $P$  descends to a group scheme  $P/K$ , and, consequently,  $J_a^{2n+1}(X)$  descends to a model  $J_{a, X/K}^{2n+1}$  over  $K$ , as well. This is the distinguished model.

REMARK 2.7. We reiterate here that the distinguished model is unique up to unique isomorphism (see Remark 2.2) so that  $J_{a, X/K}^{2n+1}$  is, in fact, independent of the curve  $C$  and the correspondence  $\gamma$  used in the construction. In other words, given any smooth projective geometrically integral curve  $C'/K$  (admitting a  $K$ -point) and a correspondence  $\gamma' \in \text{CH}^{n+1}(C' \times_K X)$  such that the induced morphism of complex tori  $\gamma'_* : J(C') \rightarrow J^{2n+1}(X)$  defines a short exact sequence of algebraic compact complex analytic groups  $0 \rightarrow P' \rightarrow J(C') \rightarrow J_a^{2n+1}(X) \rightarrow 0$ , the descent datum on  $C'_{\mathbb{C}}$  defines  $J_{a, X/K}^{2n+1}$ .

### 3. Changing the embedding $K \subseteq \mathbb{C}$

In this section, we show that if  $K$  is a field of finite transcendence degree over  $\mathbb{Q}$ , then up to isomorphism, the distinguished model and distinguished normal function do not depend on the embedding  $K \subseteq \mathbb{C}$ . An important consequence is that the normal function associated with a fiberwise algebraically

trivial cycle defined over a field  $F$  of finite transcendence degree over  $\mathbb{Q}$  (that is, an algebraically  $F$ -motivated normal function in our terminology) vanishes at an  $F$ -very general point if and only if it vanishes at all  $F$ -very general points; see Example 3.6 and Remark 3.8.

**3.1. The distinguished normal function is independent of the field embedding.** The following proposition complements, in particular, Theorem 2.1 by showing that the distinguished model does not depend on the choice of an embedding  $K \subseteq \mathbb{C}$ .

PROPOSITION 3.1. *Let  $X$  be a smooth projective variety over a field  $K$  of finite transcendence degree over  $\mathbb{Q}$ , let  $n$  be a nonnegative integer, and let  $Z \in A^{n+1}(X)$  be an algebraically trivial cycle class. Let  $b_1, b_2 : K \hookrightarrow \mathbb{C}$  be two inclusions of fields and let  $L_i = b_i(K)$ , and denote by  $\sigma : \mathbb{C} \rightarrow \mathbb{C}$  an automorphism inducing a commutative diagram of field homomorphisms*

$$\begin{array}{ccccc}
 & & L_1 & \hookrightarrow & \mathbb{C} \\
 & b_1 \nearrow & \downarrow b_2 b_1^{-1} & & \downarrow \sigma \\
 K & & & & \\
 & b_2 \searrow & L_2 & \hookrightarrow & \mathbb{C}
 \end{array}$$

which exists due to the assumption that  $K$  is of finite transcendence degree over  $\mathbb{Q}$ .

For  $i = 1, 2$ , let  $X_{b_i}$  be the base change of  $X$  over  $b_i : \text{Spec } \mathbb{C} \rightarrow \text{Spec } K$ , with associated complex analytic space  $\mathcal{X}_{b_i}$ , let  $J_{a, X_{L_i}/L_i}^{2n+1}$  be the distinguished model of  $J_a^{2n+1}(\mathcal{X}_{b_i})$  over  $L_i$ , and let  $\delta_{Z_{L_i}} : \text{Spec } L_i \rightarrow J_{a, X_{L_i}/L_i}^{2n+1}$  be the distinguished normal function. Let  $J_a^{2n+1}(X_{b_i})$  be the complex abelian variety associated with  $J_a^{2n+1}(\mathcal{X}_{b_i})$ .

Then  $J_{a, X_{L_2}/L_2}^{2n+1}$  is the pullback of  $J_{a, X_{L_1}/L_1}^{2n+1}$  by  $b_1^{-1} b_2 : \text{Spec } L_2 \rightarrow \text{Spec } L_1$ , and there is a commutative fibered product diagram

$$\begin{array}{ccccc}
 J_{a, X/K}^{2n+1} & \longleftarrow & J_{a, X_{L_1}/L_1}^{2n+1} & \longleftarrow & J_a^{2n+1}(X_{b_1}) \\
 \delta_Z \updownarrow & & \updownarrow \delta_{Z_{L_1}} & & \downarrow \\
 J_{a, X_{L_2}/L_2}^{2n+1} & \longleftarrow & J_a^{2n+1}(X_{b_2}) & \longrightarrow & J_a^{2n+1}(X_{b_1}) \\
 \delta_{Z_{L_2}} \updownarrow & & \downarrow & & \downarrow \\
 \text{Spec } K & \longleftarrow & \text{Spec } L_1 & \longleftarrow & \text{Spec } \mathbb{C} \\
 & \swarrow b_2 & \downarrow b_1 & & \downarrow \sigma \\
 & & \text{Spec } L_2 & \longleftarrow & \text{Spec } \mathbb{C}
 \end{array}
 \tag{3.1}$$

where  $J_{a, X/K}^{2n+1}$  and  $\delta_Z : \text{Spec } K \rightarrow J_{a, X/K}^{2n+1}$  are defined from the rest of the diagram via fibered product.

REMARK 3.2. Proposition 3.1 allows one to define *the distinguished model*  $J_{a,X/K}^{2n+1}$  of the image of the Abel–Jacobi map, and *the distinguished normal function*  $\delta_Z$  associated with a cycle class  $Z \in \text{CH}^{n+1}(X)$ , without first needing to specify a particular inclusion  $K \hookrightarrow \mathbb{C}$ .

*Proof.* From the diagram (3.1), it is clear that it suffices to establish that there is a commutative fibered product diagram

$$\begin{array}{ccc}
 J_{a,X_{L_2}/L_2}^{2n+1} & \longrightarrow & J_{a,X_{L_1}/L_1}^{2n+1} \\
 \delta_{Z_{L_2}} \uparrow \downarrow & & \downarrow \uparrow \delta_{Z_{L_1}} \\
 \text{Spec } L_2 & \xrightarrow{b_1^{-1}b_2} & \text{Spec } L_1
 \end{array} \tag{3.2}$$

that is, it is enough to focus on the subdiagram that is the left-hand face of the cube in diagram (3.1). We break the proof into two parts. First, we establish the result for the distinguished models and, second, for the distinguished normal functions.

*Step 1. The distinguished models.* Let  $C$  be a geometrically integral curve over  $K$  (admitting a  $K$ -point) and let  $\gamma \in \text{CH}^{n+1}(C \times_K X)$  be a correspondence such that, for  $i = 1, 2$ , the induced morphisms of complex tori  $\gamma_{i*} : \mathbf{J}(C_i) \rightarrow \mathbf{J}^{2n+1}(X_i)$  have respective images equal to  $J_a^{2n+1}(X_i)$  [ACMV18, Proposition 1.1]. Here,  $X_i$  and  $C_i$  are the complex analytic spaces associated with the pullbacks of  $X$  and  $C$  to  $\text{Spec } \mathbb{C}$  via the given inclusions  $b_i : K \hookrightarrow \mathbb{C}$ . Thus, for  $i = 1, 2$ , we obtain short exact sequences

$$0 \longrightarrow P_i \longrightarrow \mathbf{J}(C_i) \longrightarrow \mathbf{J}_a^{2n+1}(X_i) \longrightarrow 0$$

where  $P_i$  is defined to be the kernel of the morphism of complex tori induced by  $\gamma$ . Moreover, we have seen in Remark 2.7 that  $P_i$  descends to an abelian scheme  $P_i$  over  $L_i$ . This gives short exact sequences

$$0 \longrightarrow P_i \longrightarrow J(C_{L_i}) \longrightarrow J_{a,X_{L_i}/L_i}^{2n+1} \longrightarrow 0$$

defining the distinguished models  $J_{a,X_{L_i}/L_i}^{2n+1}$ . We want to show that the distinguished models differ by base change over  $b_1^{-1}b_2 : \text{Spec } L_2 \rightarrow \text{Spec } L_1$ . We will do this by showing that  $P_1$  and  $P_2$  differ by base change over  $b_1^{-1}b_2$ .

Let  $P_{1,L_2} \subseteq J(C_{L_2})$  be the base change of  $P_1$  to  $L_2$ . To show that  $P_{1,L_2} = P_2$ , it suffices to show that for all natural numbers  $N$ , the  $N$ -torsion of  $P_{1,L_2}$  and  $P_2$  are equal; that is,  $P_{1,L_2}[N] = P_2[N]$ . But the  $N$ -torsion  $P_i[N]$  is equal to the kernel of the morphism  $\gamma_{i*} : H^1(C_{b_i}, \mu_N) \rightarrow H^{2n+1}(X_{b_i}, \mu_N^{\otimes(n+1)})$ ; see [ACMV18, (2.3)].

These are related by the diagram

$$\begin{CD}
 H^1(C_{b_1}, \mu_N) @>{(\gamma_1)_*}>> H^{2n+1}(X_{b_1}, \mu_N^{\otimes(n+1)}) \\
 @V{\simeq \downarrow \sigma^*}VV @VV{\simeq \downarrow \sigma^*}V \\
 H^1(C_{b_2}, \mu_N) @>{(\gamma_2)_*}>> H^{2n+1}(X_{b_2}, \mu_N^{\otimes(n+1)})
 \end{CD}$$

implying that  $P_{1,L_2}[N] = P_2[N]$ , completing the proof.

*Step 2. The distinguished normal functions.* We now show that the distinguished normal functions  $\delta_{Z_{L_1}}$  and  $\delta_{Z_{L_2}}$  fit into the fibered product diagram (3.2); that is, they agree under base change. To begin, recall that the distinguished normal function  $\delta_{Z_{L_i}}$  is characterized by the condition that  $(\delta_{Z_{L_i}})_{an} = \nu_{(Z_{b_i})_{an}}$ ; that is, the analytic map induced by  $\delta_{Z_{L_i}}$  agrees with the analytic normal function (Remark 2.5). Algebraically, this is the condition that  $(\delta_{Z_{L_i}})_{b_i} = AJ \circ w_{Z_{b_i}}$  (see Section 2.2). In other words, to complete the proof of the theorem, it suffices to show that  $(\delta_{Z_{L_1}})_{b_2} = (\delta_{Z_{L_2}})_{b_2}$ . Put differently, by virtue of the fact that  $J_a^{2n+1}(X_{b_1})$  and  $J_a^{2n+1}(X_{b_2})$  have been identified in Step 1 via base change over  $\sigma : \text{Spec } \mathbb{C} \rightarrow \text{Spec } \mathbb{C}$ , it suffices to show that the outer rectangle of the diagram

$$\begin{CD}
 (\text{Spec } \mathbb{C})_{\mathbb{C}}(\mathbb{C}) @>{w_{Z_{b_1}}}>> A^{n+1}(X_{b_1}) @>{AJ}>> J_a^{2n+1}(X_{b_1})(\mathbb{C}) \\
 @V{\downarrow \sigma}VV @VV{\downarrow \sigma^*}V @VV{\downarrow \sigma(\mathbb{C})}V \\
 (\text{Spec } \mathbb{C})_{\mathbb{C}}(\mathbb{C}) @>{w_{Z_{b_2}}}>> A^{n+1}(X_{b_2}) @>{AJ}>> J_a^{2n+1}(X_{b_2})(\mathbb{C})
 \end{CD} \tag{3.3}$$

is commutative. In other words, it suffices to show that  $\sigma AJ(Z_{b_1}) = AJ((Z_{b_2}))$ .

To do this, we first show the commutativity of the right-hand side of (3.3) on torsion (see also Remark 3.4). For this, one considers the diagram

$$\begin{CD}
 A^{n+1}(X_{b_1})[N] @>{AJ}>> J_a^{2n+1}(X_{b_1})[N] @>>{C}>> H^{2n+1}(X_{b_1}, \mu_N^{\otimes(n+1)}) \\
 @V{\downarrow \sigma^*[N]}VV @VV{\downarrow \sigma(\mathbb{C})[N]}V @VV{\downarrow \sigma^*}V \\
 A^{n+1}(X_{b_2})[N] @>{AJ}>> J_a^{2n+1}(X_{b_2})[N] @>>{C}>> H^{2n+1}(X_{b_2}, \mu_N^{\otimes(n+1)})
 \end{CD}$$

which is commutative for all integers  $N > 1$  due to the fact that, by construction, the right-hand square is commutative and the fact that the outer square is commutative because the composition of horizontal arrows is the Bloch map, which is functorial with respect to automorphisms of the field (see, for example, [ACMV18, Section 2.3]).

Now, since  $Z$  is defined over  $K$ , there exist, by [ACMV19, Theorem 1], an abelian variety  $A$  over  $K$ , a  $K$ -point  $p \in A(K)$ , and a correspondence  $\mathcal{E} \in \text{CH}^{n+1}(A \times_K X)$  such that  $Z = \mathcal{E}_p - \mathcal{E}_0$ . Since the Abel–Jacobi map is a

regular homomorphism, the base change of  $\mathcal{E}$  along  $b_i$  induces a homomorphism  $\psi_{\mathcal{E},i} : A_{b_i} \rightarrow J_a^{2n+1}(X_{b_i})$  with  $\psi_{\mathcal{E},i}(q) = AJ(\mathcal{E}_q - \mathcal{E}_0)$ , in particular,  $(\psi_{\mathcal{E}})_i(p) = AJ(Z_{b_i})$ . We have then a not *a priori* commutative diagram

$$\begin{CD} A_{b_1} @>\psi_{\mathcal{E},1}>> J_a^{2n+1}(X_{b_1}) \\ @V\sigma^*VV @VV\sigma V \\ A_{b_2} @>\psi_{\mathcal{E},2}>> J_a^{2n+1}(X_{b_2}) \end{CD}$$

However, since the right-hand side of (3.3) is commutative on torsion, this diagram is also commutative on torsion (note that  $\mathcal{E}_q - \mathcal{E}_0$  is torsion in  $A^{n+1}(X_{b_i})$  whenever  $q$  is a torsion point in  $A_{b_i}$ ; for example, [ACMV18, Lemma 3.2]) and since torsion points are dense, the diagram is, in fact, commutative, thereby establishing the desired identity  $\sigma AJ(Z_{b_1}) = AJ(Z_{b_2})$ . □

According to the Bloch–Beilinson philosophy (see, for example, [Gre14, Lecture 3]), if  $X$  is a smooth projective variety defined over a subfield  $K \subseteq \mathbb{C}$ , then the kernel of the Abel–Jacobi map  $AJ : \text{CH}^{n+1}(X)_{\text{hom}} \rightarrow \text{CH}^{n+1}(X_{\mathbb{C}})_{\text{hom}} \rightarrow J^{2n+1}(X_{\mathbb{C}})$  defined on homologically trivial cycles defined over  $K$  should be independent of the choice of embedding  $K \hookrightarrow \mathbb{C}$  after tensoring with  $\mathbb{Q}$ . Even such a concrete consequence of the Bloch and Beilinson conjectures remains wide open. As a noteworthy consequence of Proposition 3.1, we can establish this, with integral coefficients, for algebraically trivial cycle classes.

**COROLLARY 3.3.** *Let  $X$  be a smooth projective variety over a field  $K$  of finite transcendence degree over  $\mathbb{Q}$  and let  $Z \in A^{n+1}(X)$  be an algebraically trivial cycle class. Let  $b : K \hookrightarrow \mathbb{C}$  be an inclusion of fields. Then  $AJ(Z_b) = 0$  for one such embedding if and only if  $AJ(Z_b) = 0$  for all such embeddings.* □

**REMARK 3.4.** In fact, one can also show that the right-hand side of (3.3) is commutative on all cycle classes (that is, not just the ones defined over  $K$ ). In particular, the kernel of the Abel–Jacobi map restricted to algebraically trivial cycles (not necessarily defined over  $K$ ) is independent of the choice of an embedding  $K \hookrightarrow \mathbb{C}$  in the sense that if  $Z \in A^{n+1}(X_{\mathbb{C}})$  is such that  $AJ(Z) = 0$ , then  $AJ(Z^\sigma) = 0$  for all automorphisms  $\sigma \in \text{Aut}(\mathbb{C})$ . For brevity, we have omitted the proof.

**3.2. Distinguished models and distinguished normal functions of very general fibers.** We now focus on the main case of interest in this paper. Let  $F \subseteq \mathbb{C}$  be a field of finite transcendence degree over  $\mathbb{Q}$ . Let  $f : X \rightarrow B$  be a smooth surjective projective morphism of smooth integral schemes of finite type

over  $F$  and let  $n$  be a nonnegative integer. Let  $f : X \rightarrow B$  be the associated map of complex manifolds. Let  $\eta$  be the generic point of  $B$  with residue field  $K$ , which is also of finite transcendence degree over  $\mathbb{Q}$ .

EXAMPLE 3.5 (Distinguished model of a very general fiber). In the notation above, fix an inclusion  $K \subseteq \mathbb{C}$ , let  $X_\eta$  be the generic fiber, and let  $J_{a, X_\eta/K}^{2n+1}$  be the corresponding distinguished model associated with  $(X_C)_{an}$  (see also Proposition 3.1). Now let  $u \in B$  be an  $F$ -very general point; that is,  $u$  corresponds to a closed  $\mathbb{C}$ -point  $u : \text{Spec } \mathbb{C} \rightarrow B_C$ , which is itself a morphism of  $\mathbb{C}$ -schemes, so that the composition  $u : \text{Spec } \mathbb{C} \rightarrow B_C \rightarrow B$  has image the generic point of  $B$ , given by a second inclusion  $i : K \hookrightarrow \mathbb{C}$ . From Proposition 3.1, the distinguished model of  $J^{2n+1}(X_u)$  is (after pullback to  $K$ ) isomorphic over  $K$  to  $J_{a, X_\eta/K}^{2n+1}$ . In fact, the distinguished models of all  $F$ -very general fibers agree up to isomorphism over  $K$ . Put another way, for any  $F$ -very general point  $u \in B$ , corresponding to a point  $u : \text{Spec } \mathbb{C} \rightarrow B_C$ ,

$$((J_{a, X_\eta/K}^{2n+1})_u)_{an} = J_a^{2n+1}(X_u).$$

EXAMPLE 3.6 (Distinguished normal function of a very general fiber). In the same situation as Example 3.5, let  $Z \in \text{CH}^{n+1}(X)$  be such that every Gysin fiber is algebraically trivial. Let  $\delta_Z : \text{Spec } K \rightarrow J_{a, X_\eta/K}^{2n+1}$  be the distinguished normal function (2.1), which *a priori* depends on the inclusion  $K \subseteq \mathbb{C}$ . However, from Proposition 3.1, the distinguished normal function associated with any  $F$ -very general point  $u \in B$  (corresponding to an inclusion  $i : K \hookrightarrow \mathbb{C}$ ) agrees (after pullback to  $K$ ) with  $\delta_Z$ . Put another way, for any  $F$ -very general point  $u \in B$ , corresponding to a point  $u : \text{Spec } \mathbb{C} \rightarrow B_C$ ,

$$((\delta_Z)_u)_{an} = (v_Z)_u : u \rightarrow J_a^{2n+1}(X_u),$$

where  $v_Z : B \rightarrow J^{2n+1}(X/B)$  is the analytic normal function associated with  $Z$ .

REMARK 3.7 (The geometric generic fiber). Another way to frame the relationship between the distinguished models and distinguished normal functions associated with different  $F$ -very general points of  $B$  is to observe that the  $\mathbb{C}$ -scheme  $X_u$  associated with an  $F$ -very general fiber  $X_u$  of  $f : X \rightarrow B$  is isomorphic as a  $K$ -scheme to a geometric generic fiber  $X_{\overline{\mathbb{C}(B)}}$ . In fact, after choosing a  $K$ -isomorphism  $\alpha : \mathbb{C} \rightarrow \overline{\mathbb{C}(B)}$ , we have that  $X_u$  and  $X_{\overline{\mathbb{C}(B)}}$  are isomorphic over  $\alpha$  (see, for example, [Via13, Lemma 2.1]).

REMARK 3.8 (First consequence for zero loci of normal functions: Conjecture 2 in the algebraically  $F$ -motivated case). Already, we obtain a quick proof of much

of Corollary 1, that is, the zero locus of  $\nu_Z$  is a countable union of algebraic subsets of  $B$  defined over  $F$ . Indeed, from Example 3.6, we have that if  $(\nu_Z)_U$  is zero for one  $F$ -very general point  $U \in B$ , then it is zero for every  $F$ -very general point  $U' \in B$  (see also Corollary 3.3). By continuity, if  $\nu_Z$  is not identically zero, then the zero locus of  $\nu_Z$  is contained in the complement of the  $F$ -very general points, which is a countable union of algebraic subsets of  $B$  defined over  $F$ , and not equal to  $B$ . Restricting  $\nu_Z$  to an  $F$ -desingularization of each irreducible component and arguing recursively on each component, one obtains the claim. Note that together with Conjecture 1, one obtains a proof of Corollary 1. We will, however, give below a direct proof of Theorem 1, and, hence, of Corollary 1, that does not rely on the validity of Conjecture 1.

#### 4. Spreading the distinguished model and distinguished normal function

We now consider a family of smooth complex projective varieties and descend the image of the Abel–Jacobi map of a very general fiber to the generic point of the base of the family. We then spread this to an open subset of the base. The following theorem collects some properties of this spread. (Note that the caveat of Remark 5.2 already applies to the abelian scheme  $J_a^{2n+1}(X_U/U)$  constructed below.)

**THEOREM 4.1.** *Let  $F \subseteq \mathbb{C}$  be a subfield of finite transcendence degree over  $\mathbb{Q}$ , let  $f : X \rightarrow B$  be a smooth surjective projective morphism of smooth integral varieties of finite type over  $F$ , and let  $n$  be a nonnegative integer. Let  $f : X \rightarrow B$  be the associated morphism of complex analytic spaces.*

*Let  $\eta$  be the generic point of  $B$  with residue field  $K$ , and fix an inclusion  $K \hookrightarrow \mathbb{C}$ . Let  $X_\eta$  be the generic fiber of  $X$  over  $K$ , let  $J_{a, X_\eta/K}^{2n+1}$  be the distinguished model of  $J_a^{2n+1}((X_C)_{an})$  over  $K$ , and let*

$$\Gamma \in \text{CH}^{\dim(J_{a, X_\eta/K}^{2n+1})+n}(J_{a, X_\eta/K}^{2n+1} \times_K X_\eta) \quad \text{and} \quad M \in \mathbb{Z}_{>0} \tag{4.1}$$

*be the correspondence and integer, respectively, from Theorem 2.1 (see also Proposition 3.1). Spread this data to a Zariski open subset  $U \subseteq B$ . More precisely, let  $U \subseteq B$  be a Zariski open subset over which there is an abelian scheme*

$$g : J_{a, X_U/U}^{2n+1} \rightarrow U$$

*with generic fiber isomorphic to  $J_{a, X_\eta/\eta}^{2n+1}$  and a cycle*

$$\Gamma_U \in \text{CH}^{\dim(J_{a, X_\eta/K}^{2n+1})+n}(J_{a, X_U/U}^{2n+1} \times_U X_U)$$

with  $(\Gamma_U)_\eta = \Gamma$ . Let  $\mathbf{U} \subseteq \mathbf{B}$  be the Zariski open subset corresponding to  $U \subseteq B$  and let  $\Gamma_U$  be the corresponding complex analytic correspondence.

- (1) For every prime number  $\ell$ , the correspondence  $\Gamma_U$  induces a morphism of sheaves on  $U$

$$(\Gamma_U)_* : R^1 g_* \mathbb{Q}_\ell \longrightarrow R^{2n+1}(f|_U)_* \mathbb{Q}_\ell(n), \quad (4.2)$$

which, at the geometric generic point  $\bar{u} : \text{Spec } \bar{K} \rightarrow \text{Spec } K \rightarrow U$ , induces an inclusion of  $\text{Gal}(\bar{K}/K)$ -representations

$$(\Gamma_{U,\bar{u}})_* : H^1((J_{a,X_\eta/K}^{2n+1})_{\bar{K}}, \mathbb{Q}_\ell) \hookrightarrow H^{2n+1}(X_{\bar{K}}, \mathbb{Q}_\ell(n)) \quad (4.3)$$

(with image  $N^n H^{2n+1}(X_{\bar{K}}, \mathbb{Q}_\ell(n))$ , where  $N^\bullet$  denotes the geometric coniveau filtration).

- (2) The correspondence  $\Gamma_U$  induces a morphism of variations of pure integral Hodge structures and thus a morphism of relative complex tori over  $\mathbf{U}$

$$(\Gamma_U)_* : (J_{a,X_U/U}^{2n+1})_{an} \rightarrow J^{2n+1}(X/B)|_U. \quad (4.4)$$

The image of (4.4) is an algebraic relative complex torus  $A_U \subseteq J^{2n+1}(X/B)|_U$  over  $\mathbf{U}$ , induced by an abelian scheme

$$A_U/U,$$

defined over  $F$ , with generic fiber  $(A_U)_\eta$  isomorphic to  $J_{a,X_\eta/K}^{2n+1}$  over  $K$ .

For  $F$ -very general  $u \in \mathbf{U}$ , the morphism (4.4) restricts to a morphism

$$((\Gamma_U)_*)_u : J_a^{2n+1}(X_u) \rightarrow J^{2n+1}(X_u) \quad (4.5)$$

that is given by  $M \cdot i_{a,X_u}^{2n+1}$ , that is,  $M$  times the natural inclusion (where  $M$  is defined in (4.1)). In particular, the image of (4.5), that is, the fiber  $A_{U,u}$ , is equal to  $J_a^{2n+1}(X_u)$ .

- (3) Let  $Z \in \text{CH}^{n+1}(X)$  be a cycle class with every Gysin fiber algebraically trivial, let  $Z_\eta$  be the restriction of  $Z$  to the generic fiber  $X_\eta$ , and let

$$\delta_{Z_\eta} : \text{Spec } K \rightarrow J_{a,X_\eta/K}^{2n+1}$$

be the associated distinguished normal function (see (2.1) and Proposition 3.1). After possibly replacing the Zariski open subset  $U \subseteq B$  with a smaller Zariski open subset, let  $\delta : U \rightarrow J_{a,X_U/U}^{2n+1}$  be the spread of the distinguished normal function and let  $\delta_{an} : \mathbf{U} \rightarrow (J_{a,X_U/U}^{2n+1})_{an}$  denote the



associated morphism of complex analytic spaces. We have the following formula relating the normal function  $\nu_Z$ , the spread  $\delta_{an}$  of the distinguished normal function, and the morphism (4.4):

$$(\Gamma_U)_* \circ \delta_{an} = M \cdot \nu_Z|_U, \quad (4.6)$$

and  $M \cdot \nu_Z|_U$  is algebraic and defined over  $F$ .

*Proof.* (1) Correspondences induce morphisms of sheaves, giving (4.2). Equation (4.3) is just a statement about fibers of correspondences and follows from Theorem 2.1 (but see also [ACMV18, Theorem A]).

(2) Correspondences induce morphisms of Hodge structures, giving (4.4). To show that the image  $A_U$  of (4.4) is algebraic, one shows that the kernel of the morphism of relative complex tori is algebraic. For this, it suffices to check that torsion is preserved by  $\text{Aut}(\mathbb{C}/F)$ , and it is easy to see this holds from the fact that the morphism is induced by an algebraic cycle defined over  $F$  (as in Lemma 2.3); alternatively, it is dominated by the relative algebraic complex torus  $(J_{a, X_U/U})_{an}$ .

The assertion (4.5) is just a statement about fibers of correspondences, and Theorem 2.1 and Example 3.5 provide the needed identification of fibers. One also uses the general observation that the image of the multiplication by  $M$  map is the same complex torus.

The final statement, that the generic fiber  $(A_U)_\eta$  is isomorphic to  $J_{a, X_\eta/K}^{2n+1}$  over  $K$ , can be established as follows. Let  $G$  be the kernel of the  $K$ -isogeny  $J_{a, X_\eta/K}^{2n+1} \rightarrow (A_U)_\eta$ . The isogeny, when pulled back to  $u$ , gives a morphism

$$J_a^{2n+1}(X_u) \longrightarrow A_{U,u} \xrightarrow{\cong} J_a^{2n+1}(X_u)$$

with composition equal to the multiplication by  $M$  map. Thus,  $G$  and  $J_{a, X_\eta/K}^{2n+1}[M]$  are reduced  $K$ -subschemes of  $J_{a, X_\eta/K}^{2n+1}$  with the same  $\mathbb{C}$ -points and are, therefore, the same scheme. It follows that  $(A_U)_\eta \cong J_{a, X_\eta/K}^{2n+1}/J_{a, X_\eta/K}^{2n+1}[M] = J_{a, X_\eta/K}^{2n+1}$ .

(3) The only thing to show is (4.6). Since both sides of the equation are continuous functions, it suffices to prove the assertion for a dense subset of  $U$ , and, in particular, we can focus on  $F$ -very general points  $u \in U$ . The assertion then follows from (2) together with Example 3.6.  $\square$

We now use Theorem 4.1 to prove Theorem 1 over a Zariski dense open subset of the base.

**COROLLARY 4.2.** *Let  $f : X \rightarrow B$  be a smooth surjective projective morphism of complex algebraic manifolds, let  $n$  be a nonnegative integer, and let  $J^{2n+1}(X/B) \rightarrow B$  be the  $(2n + 1)$ th relative Griffiths intermediate Jacobian.*

There is a Zariski open subset  $U \subseteq B$ , a relative algebraic complex subtorus  $J_a^{2n+1}(X_U/U) \subseteq J^{2n+1}(X/B)|_U$  over  $U$  such that for very general  $u \in U$ , the fiber  $J_a(X_U/U)_u \subseteq J^{2n+1}(X_u)$  is the image  $J_a^{2n+1}(X_u)$  of the Abel–Jacobi map  $AJ_{X_u} : A^{n+1}(X_u) \rightarrow J^{2n+1}(X_u)$ , and for any  $Z \in CH^{n+1}(X)$  with every Gysin fiber algebraically trivial:

- (1) the restriction of the normal function  $v_Z|_U : U \rightarrow J^{2n+1}(X/B)|_U$  has image contained in the relative algebraic complex torus  $J_a^{2n+1}(X_U/U)$  and is an algebraic map;
- (2) if, moreover,  $X, B, f$ , and  $Z$  are all defined over a subfield  $F \subseteq \mathbb{C}$ , then so are  $J_a^{2n+1}(X_U/U)$  and the morphisms  $J_a^{2n+1}(X_U/U) \rightarrow U$  and  $v_Z|_U$ .

*Proof.* In case (1), since  $f : X \rightarrow B$  is defined over some field  $F \subseteq \mathbb{C}$  that is finitely generated over  $\mathbb{Q}$ , we may as well make this assumption from the start. In case (2), we may take our field of definition  $F'$  to be contained in the given field  $F$  and can base change to  $F$  at the end, if necessary, and so we may as well assume  $F$  is finitely generated over  $\mathbb{Q}$  in case (2), as well. We are then in the situation of Theorem 4.1, and we will use the notation from that theorem moving forward.

First, we can take  $J_a^{2n+1}(X_U/U) = A_U \subseteq J^{2n+1}(X/B)|_U$ , the image of (4.4). Let  $A_U$  be the corresponding abelian scheme over  $U \subseteq B$ . From Theorem 4.1(2), we have that the generic fiber  $(A_U)_\eta$  is isomorphic over  $K$  to the distinguished model  $J_{a, X_\eta/K}^{2n+1}$ . With cycle class  $Z \in CH^{n+1}(X)$  as in the theorem, let

$$\delta_{Z_\eta} : \text{Spec } K \rightarrow J_{a, X_\eta/K}^{2n+1} \cong (A_U)_\eta \quad (4.7)$$

be the distinguished normal function. This then spreads to an  $F$ -morphism  $\delta : U \rightarrow A_U$ , after possibly replacing  $U$  with a smaller Zariski open subset. The associated complex analytic map  $\delta_{an} : U \rightarrow A_U \subseteq J^{2n+1}(X/B)|_U$  has the property that for  $F$ -very general  $u \in U$ , we have  $(\delta_{an})_u : u \rightarrow A_{U,u} = J_a^{2n+1}(X_u)$  has image  $v_Z(u)$ ; that is, it agrees with the complex analytic normal function. This follows from (4.7) and Example 3.6. Therefore, since  $\delta_{an}$  and  $v_Z|_U$  are continuous and agree on the dense open subset of  $F$ -very general points of  $U$ , they agree on all of  $U$ .  $\square$

## 5. Extending the distinguished model and distinguished normal function

In light of Corollary 4.2, in order to prove Theorem 1, we only need to show that the distinguished model and distinguished normal function extend over the entire base  $B$ . We do this now.

*Proof of Theorem 1.* We use the notation from Theorem 1 and the partial result, Corollary 4.2. As mentioned above, we only need to show that the spread of the distinguished model  $J_a^{2n+1}(X_U/U) \subseteq J^{2n+1}(X/B)|_U$  and distinguished normal function  $\delta_{an} : U \rightarrow J_a^{2n+1}(X_U/U)$  extend over the entire base  $B$  to give algebraic objects over  $F$ . To begin, we switch to the algebraic setting, and let  $J_{a,X_U/U}^{2n+1}$  be the algebraic model of  $J_a^{2n+1}(X_U/U)$  and let  $\delta : U \rightarrow J_{a,X_U/U}^{2n+1}$  be the associated morphism of  $F$ -schemes.

First, we show that  $J_{a,X_U/U}^{2n+1}$  extends to an abelian scheme  $g : \tilde{J}_{a,X/B}^{2n+1} \rightarrow B$  over  $B$ . If  $\dim B = 1$ , we use the inclusion (4.3) and the Néron–Ogg–Shafarevich criterion as in [ACMV17, Lemma 6.1(a)]. If  $\dim B \geq 2$ , using the dimension 1 case, we can extend over the generic points of divisors in the boundary  $B - U$ , and, thus, we can assume that  $\text{codim}_B(B - U) \geq 2$ . The assertion now follows from the Faltings–Chai extension theorem [FC90, Corollary 6.8, page 185].

Next, we show that the relative algebraic complex torus  $\tilde{J}_a^{2n+1}(X/B) := (J_{a,X/B}^{2n+1})_{an}$  induces an algebraic relative subtorus  $J_a^{2n+1}(X/B) \subseteq J^{2n+1}(X/B)$  which extends  $J_a^{2n+1}(X_U/U)$ . For this, we use the basic fact that any morphism of variations of Hodge structures extends over a locus of codimension at least 2. (Indeed, by purity, the natural map  $\pi_1(U, u) \rightarrow \pi_1(B, u)$  is an isomorphism; now use [PS08, Theorem 10.11, page 243] or the proof of [Hai95, Lemma 6.3, page 117].) It now follows that the inclusion  $J_a^{2n+1}(X_U/U) \subseteq J^{2n+1}(X/B)|_U$  extends to a morphism

$$\tilde{J}_a^{2n+1}(X/B) \longrightarrow J^{2n+1}(X/B) \tag{5.1}$$

with finite kernel, which *a priori* may be nontrivial only over  $B - U$ . We define  $J_a^{2n+1}(X/B) \subseteq J^{2n+1}(X/B)$  to be the image of (5.1). Although it is not needed, we note that by Zariski’s main theorem, the morphism of relative algebraic complex tori  $\tilde{J}_a^{2n+1}(X/B) \rightarrow J_a^{2n+1}(X/B)$  is an isomorphism.

The normal function  $v_Z$  has image contained in  $J_a^{2n+1}(X/B)$ , since  $J^{2n+1}(X/B)$  is separated, and  $v_Z|_U$  has image contained in  $J_a^{2n+1}(X/B)$ , which is closed in  $J^{2n+1}(X/B)$ . Finally, it is straight forward to check that  $v_Z$  is algebraic, and defined over  $F$ , since  $v_Z|_U$  is algebraic and defined over  $F$ . For completeness, we include this last assertion as Lemma 5.1. □

LEMMA 5.1. *Let  $X, Y$  be schemes of finite type over  $F \subseteq \mathbb{C}$ , with  $X$  reduced and  $Y$  separated, let  $U \subseteq X$  be a Zariski open subset, let  $f_U : U \rightarrow Y$  be a morphism of  $F$ -schemes, and assume that the associated morphism of analytic spaces  $(f_U)_{an} : U \rightarrow Y$  extends to a morphism  $f : X \rightarrow Y$ . Then  $f_U$  extends to a morphism  $f : X \rightarrow Y$  over  $F$  with  $f_{an} = f$ .*

*Proof.* We may immediately reduce to the case with  $X$  integral and  $Y$  reduced. Now consider the graph  $\Gamma_{f_U} \subseteq U \times_F Y \subseteq X \times_F Y$ , which is closed in  $U \times_F Y$  since  $Y$  is separated. Let  $\Gamma \subseteq X \times_F Y$  be the closure of  $\Gamma_{f_U}$ , which we observe is an integral subscheme. Now let  $\Gamma = \Gamma_{an} \subseteq X \times Y$  be the associated complex analytic space. We have by assumption that  $(\Gamma_{f_U})_{an} = (\Gamma_{\dagger})|_{U \times Y}$ . Now  $\Gamma_{\dagger}$  is the analytic closure of  $(\Gamma_{\dagger})|_{U \times Y}$  in  $X \times Y$ . Since  $(\Gamma_{\dagger})|_{U \times Y} = (\Gamma_{f_U})_{an} \subseteq \Gamma$ , we have  $\Gamma_{\dagger}$  is equal to the analytic closure of  $(\Gamma_{f_U})_{an}$  in  $\Gamma$ . But  $\Gamma_{f_U}$  is a Zariski open subset of an integral scheme  $\Gamma$  of finite type over  $F$ , and so  $\Gamma_{\dagger} = \Gamma$ . (In general, if  $T$  is a locally closed subset of a scheme  $Z/\mathbb{C}$  which is locally of finite type, then  $T$  is dense in  $Z$  if and only if  $\mathbb{T}$  is dense in  $\mathbb{Z}$  [Gro71, Exposé XII, Corollary 2.3].)

Now we just need to conclude that  $\Gamma$  induces a morphism  $X \rightarrow Y$ . It suffices to show that the second projection  $q_1 : \Gamma \rightarrow X$  is an isomorphism. But this follows from the fact that a morphism between the complex analytic spaces associated with two  $F$ -schemes descends to  $F$  if and only if it is  $\text{Aut}(\mathbb{C}/F)$ -equivariant (apply, for example, [Voi13, Section 5.2] to its graph) and the fact that  $q_1 : \Gamma \rightarrow X$  is an isomorphism.  $\square$

REMARK 5.2. The notation  $J_a^{2n+1}(X/B)$  may be slightly misleading, in the sense that the formation of this object is not compatible with base change in  $B$ . While the very general fiber  $J_a^{2n+1}(X/B)_u$  is equal to the image of the Abel–Jacobi map  $J_a^{2n+1}(X_u)$ , in some cases, there is a countably infinite union of algebraic subsets of  $B$  over which the geometric coniveau of the fiber  $N^n H^{2n+1}(X_b, \mathbb{Q})$  jumps. If this is the case, then over these points, the fiber  $J_a^{2n+1}(X/B)_b$  is strictly contained in  $J_a^{2n+1}(X_b)$ . Nonetheless, we feel that  $J_a^{2n+1}(X/B)$  is a good notation in the sense that this is the smallest relative algebraic subtorus of  $J^{2n+1}(X/B)$  that interpolates between the very general  $J_a^{2n+1}(X_u)$ . Moreover, by part (1) of Theorem 1, it serves as a target for every algebraically motivated normal function.

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