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ESSENTIAL IDEALS, HOMOMORPHICALLY CLOSED CLASSES AND THEIR RADICALS

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Abstract

Olson and Jenkins defined $\mathcal{E}(\mathfrak{M})$ to be the class of all rings each nonzero homomorphic image of which contains either a nonzero $\mathfrak{M}$-ideal or an essential ideal where $\mathfrak{M}$ is any class of rings. $\mathcal{E}(\mathfrak{M})$ was proven to be a radical class and various classes $\mathfrak{M}$ were considered. Here the class $\mathcal{E}(\mathfrak{M})$ is partitioned into two classes: $\mathcal{K}$ the class of all rings each nonzero homomorphic image of which has a proper essential ideal and the class $\mathcal{K}(\mathfrak{M})$ of all rings each nonzero homomorphic image of which contains an $\mathfrak{M}$-ideal. It is shown that $\mathcal{K}$ is a radical class and under certain conditions $\mathcal{K}(\mathfrak{M})$ is also a radical class. Various properties placed on $\mathfrak{M}$ yield several well-known radical classes and an infinite number of supernilpotent nonspecial radical classes is constructed.

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In [10], Olson and Jenkins defined $\mathcal{E}(\mathfrak{M})$ to be the class of all rings each nonzero homomorphic image of which contains either a nonzero $\mathfrak{M}$-ideal or an essential ideal where $\mathfrak{M}$ is any class of rings. $\mathcal{E}(\mathfrak{M})$ was proven to be a radical class and various classes $\mathfrak{M}$ were considered. In this article two subclasses of $\mathcal{E}(\mathfrak{M})$ are considered; $\mathcal{K}$ the class of all rings each nonzero homomorphic image of which has a proper essential ideal and the class $\mathcal{U}\mathcal{S}\mathfrak{M}$ of all rings each nonzero homomorphic image of which contains an $\mathfrak{M}$-ideal. It is shown that $\mathcal{K}$ is a radical class and under certain conditions $\mathcal{U}\mathcal{S}\mathfrak{M}$ is also a radical class. Various properties placed on $\mathfrak{M}$ yield several known radical classes and an infinite number of supernilpotent nonspecial radical classes is constructed. All rings considered will

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be associative and simple rings will be prime. The major knowledge of radical theory required for our purposes can be found in [3] and [13].

Currently there are two ways of interpreting essential ideals; one where the ring is an essential ideal of itself and another where only proper ideals may be essential. For our purposes, if $\mathcal{I}$ is an essential ideal in $\mathcal{R}$ then $\mathcal{I}$ meets all nonzero ideals of $\mathcal{R}$ and thus $\mathcal{R}$ is essential in itself. When $\mathcal{I}$ must be a proper essential ideal it will be stated. In Section 1, we consider only proper essential ideals, whereas in Section 2 we consider essential ideals.

1. The $\mathcal{H}$ radical class

Let $\mathcal{M}$ be any class of rings. In [10], Olson and Jenkins showed that

$$\mathcal{E}(\mathcal{M}) = \{R \mid \text{every nonzero homomorphic image has either a nonzero } \mathcal{M}\text{-ideal or an essential ideal}\}$$

is a radical class. We now consider the subset $\mathcal{H}$ of $\mathcal{E}(\mathcal{M})$.

DEFINITION 1. Let

$$\mathcal{H} = \{R \mid \text{every nonzero homomorphic image of which has an essential ideal}\}.$$

From the definition it is clear that simple rings, as well as rings with maximal ideals, cannot belong to $\mathcal{H}$. Let $\mathcal{S} = \{R \mid R$ is either simple or a prime order zero ring$\}$ and $\mathcal{T} = \{$all direct sums of members of $\mathcal{S}\}$. From [10], we need the following result:

THEOREM 1. *The following are equivalent for any ring R.*

(1) *R has no essential ideals.*

(2) *Each ideal of R is a direct summand of R.*

(3) *R is a member of $\mathcal{T}$.*

With $\mathcal{S}$ as defined above, let $\mathcal{U}\mathcal{S}$ be the upper radical determined by $\mathcal{S}$. We now show that $\mathcal{H}$ is an upper radical class.

THEOREM 2. *$\mathcal{H}$ is a radical class and $\mathcal{H} = \mathcal{U}\mathcal{S}$.*

PROOF. From Theorem 1 and the definition of $\mathcal{H}$ it follows that $\mathcal{H} = \mathcal{U}\mathcal{T}$ so since $\mathcal{T}$ is a hereditary class $\mathcal{H} = \mathcal{U}\mathcal{T}$ is a radical class. Also $\mathcal{S} \subseteq \mathcal{T}$ so $\mathcal{U}\mathcal{T} \subseteq \mathcal{U}\mathcal{S}$. Conversely, if $R \in \mathcal{U}\mathcal{S}$ then R, having no image in $\mathcal{S}$, cannot have an image in $\mathcal{T}$. Thus $R \in \mathcal{U}\mathcal{T}$ and so $\mathcal{H} = \mathcal{U}\mathcal{T} = \mathcal{U}\mathcal{S}$.

EXAMPLE. Consider the sequence $\aleph_0, \aleph_1, \aleph_2, \ldots$. Then $\aleph_\omega = \bigcup \aleph_n > \aleph_n$ for all positive integers n. Suppose V is a vector space over a division ring and that the dimension of V is $\aleph_\omega$. Let R be the ring of all linear transformations of V of rank $< \aleph_\omega$. Then R is an ideal of the ring L of all linear transformations of V. Now if an ideal of R contains a linear transformation of a certain rank, then it also contains all linear transformations of smaller rank [6]. Therefore, it follows that the only ideals of R are of the type:

$$I_n = \{\text{all linear transformations of rank} < \aleph_n,\ n \text{ a nonnegative integer}\}.$$

This implies that R has no maximal ideals and hence $R \in \mathcal{H}$. Furthermore, R contains as an ideal the simple ring $I_0 \in \mathcal{S}\mathcal{H}$ so $\mathcal{H}$ is not hereditary.

If β_s denotes the upper radical class determined by the class of simple rings it is clear that $\mathcal{H} \subsetneqq \beta_s$. A partial solution as to the position of $\mathcal{H}$ in the diagram of well-known radical classes is given in the following proposition and discussion.

PROPOSITION 1. *$\mathcal{H} \subsetneqq \beta_s$ and $\mathcal{H} \cap \mathcal{P} \neq 0$ for any supernilpotent radical class $\mathcal{P}$.*

PROOF. The ring $W = \{2x/(2y+1) \mid (2x, 2y+1) = 1, x, y \in \mathbf{Z}\}$ [3, Example 10] belongs to β_s. All the ideals of W are of the form $W = (2) \supset (2)^2 \supset \cdots$. Since W can be mapped onto a nonzero simple ring, it follows that $W \notin \mathcal{H}$. Furthermore, the ring p^∞ [3, Example 1] is a zero ring and, therefore, belongs to every supernilpotent radical class $\mathcal{P}$. Since p^∞ has no maximal ideals we have $p^\infty \in \mathcal{H}$. Hence $\mathcal{H} \cap \mathcal{P} \neq 0$.

In [8] van Leeuwen gave the following diagram relating the radical classes. We have only extended it to include the Behrens radical J_B. We follow the standard notation where the first row is the lower Baer, Levitzki, nil, Jacobson, Behrens and Brown-McCoy radicals respectively. The second row consists of the upper radicals determined by subdirectly irreducible rings with idempotent hearts, Levitzki semisimple hearts, nil semisimple hearts, Jacobson semisimple hearts, idempotent hearts with idempotent elements and G respectively. The last row gives the upper radicals determined by all simple prime, simple prime N-semisimple, simple primitive and simple J_B-semisimple rings.

$$\begin{array}{ccccccccccc}
\beta & \subset & L & \subset & N & \subset & J & \subset & J_B & \subset & G \\
\cap & & \cap & & \cap & & \cap & & \| & & \| \\
\beta_\phi & = & L_\phi & \subseteq & N_\phi & \subset & J_\phi & \subset & (J_B)_\phi & \subset & G_\phi \\
\cap & & \cap & & \cap & & \cap & & \cap & & \| \\
H \subset \beta_s & = & L_s & \subseteq & N_s & \subset & J_s & \subset & (J_B)_s & \subset & G_s
\end{array}$$

The strictness of the inclusions was shown in [8]. The example following Theorem 2 is $(J_B)_\phi$-semisimple yet $(J_B)_s$-radical. In Example 3 [4] all the proper one-sided

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ideals of the ring X are J_s-semisimple, but $(J_B)_s$-radical. The radical $\mathcal{H}$ is independent with all the radicals of the first two rows of the diagram except for G. In order to see this, any simple zero ring is β-radical but $\mathcal{H}$-semisimple. The ring T in the example following Theorem 2 is $\mathcal{H}$-radical, but J_B-semisimple.

We note that if $\mathfrak{M}$ is a class of rings having no member of $\mathcal{S}$ then

PROPOSITION 2. *If $\mathfrak{M}$ is any class of rings containing no member of $\mathcal{S}$, then $\mathcal{E}(\mathfrak{M}) = \mathcal{H}$.*

PROOF. From Definition 1 and Theorem 2 we have $\mathcal{U}\mathcal{S} = \mathcal{H} \subseteq \mathcal{E}(\mathfrak{M})$. Now let $A \in \mathcal{E}(\mathfrak{M})$ and suppose $A \notin \mathcal{H} = \mathcal{U}\mathcal{S}$. This implies that A has a nonzero image, say A/I, where $A/I \in \mathcal{S}$. Since A/I has no essential ideals and no $\mathfrak{M}$-ideals, we have $A/I \notin \mathcal{E}(\mathfrak{M})$. This contradicts $A \in \mathcal{E}(\mathfrak{M})$ and $\mathcal{E}(\mathfrak{M})$ is a radical class. Thus $\mathcal{H} = \mathcal{E}(\mathfrak{M})$.

2. The class $\mathcal{U}\mathcal{S}\mathfrak{M}$

DEFINITION 2. Let $\mathfrak{M}$ be any class of rings. Then $\mathcal{U}\mathcal{S}\mathfrak{M} = \{R \mid$ each nonzero homomorphic image of which contains an $\mathfrak{M}$-ideal.$\}$

THEOREM 3. *If $\mathfrak{M}$ is any hereditary class of rings containing no or all nilpotent rings then $\mathcal{S}\mathfrak{M}$ is a hereditary class and so $\mathcal{U}\mathcal{S}\mathfrak{M}$ is radical.*

PROOF. Suppose it were possible for $0 \neq J \lhd I \lhd R \in \mathcal{S}\mathfrak{M}$ with $J \in \mathfrak{M}$. If J' is the ideal generated by J in R then $J' \neq J$ for $R \in \mathcal{S}\mathfrak{M}$ and so has no $\mathfrak{M}$-ideals and since $(J')^3 \subseteq J$ it follows from the hereditary property of $\mathfrak{M}$ that $R \in \mathcal{S}\mathfrak{M}$ is contradicted unless $(J')^3 = 0$. Now $J \subset J'$ so $\mathfrak{M}$ could not be a class with no nilpotent rings. But if $\mathfrak{M}$ contains all nilpotent rings then $J' \in \mathfrak{M}$ again contradicting $R \in \mathcal{S}\mathfrak{M}$. Hence $\mathcal{S}\mathfrak{M}$ is hereditary and it follows that $\mathcal{U}\mathcal{S}\mathfrak{M}$ is radical.

In order to show that the radical class of Theorem 3 is hereditary we need the following lemma. It is for this reason that in this section we allow a ring to be essential in itself.

LEMMA 2 [5]. *Let I be any nonzero ideal of a ring R. Then there exists a homomorphic image of R containing an isomorphic copy I' of I such that I' is essential in this image.*

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LEMMA 3. *Let* $\mathfrak{M}$ *be a hereditary class of rings and* $0 \neq K \lhd I \lhd R$ *with* $I/K \in \mathbb{S}\mathfrak{M}$. *If* $K \lhd R$ *then* $R \notin \mathfrak{U}\mathbb{S}\mathfrak{M}$.

PROOF. Since $K \lhd R$ we have $I/K \lhd R/K$. By Lemma 2 there exists a nonzero homomorphic image of R/K, say R/P, containing an isomorphic copy I'/P of I/K as an essential ideal. If $R \in \mathfrak{U}\mathbb{S}\mathfrak{M}$ then R/P has a nonzero $\mathfrak{M}$-ideal, say K/P, by definition. But then since I'/P is essential in R/P and I'/P is isomorphic to $I/K \in \mathbb{S}\mathfrak{M}$ we have $0 \neq I'/P \cap K/P$ is an ideal of $I'/P \in \mathbb{S}\mathfrak{M}$ and is an ideal of $K/P \in \mathfrak{M}$ a contradiction. Hence, $R \notin \mathfrak{U}\mathbb{S}\mathfrak{M}$.

Recall, a radical class R is called *supernilpotent* if R is hereditary and contains all nilpotent rings [3] and is called *subidempotent* if R is hereditary and all rings are idempotent [1].

THEOREM 4. *let* $\mathfrak{M}$ *be a hereditary class of rings. If* $\mathfrak{M}$ *contains all nilpotent rings then* $\mathfrak{U}\mathbb{S}\mathfrak{M}$ *is a supernilpotent radical and if* $\mathfrak{M}$ *contains no nilpotent rings* $\mathfrak{U}\mathbb{S}\mathfrak{M}$ *is a subidempotent radical.*

PROOF. Let $\mathfrak{M}$ be hereditary and contain all nilpotent rings. Let $R \in \mathfrak{U}\mathbb{S}\mathfrak{M}$ and suppose $0 \neq I \lhd R$ and $I \notin \mathfrak{U}\mathbb{S}\mathfrak{M}$. Then there exists a nonzero homomorphic image I/K of I with no nonzero $\mathfrak{M}$-ideals. Since $\mathfrak{M}$ contains all nilpotent rings we have that I/K is a semiprime ring. This implies that K is an ideal of R. But then by Lemma 3, $R \notin \mathfrak{U}\mathbb{S}\mathfrak{M}$, a contradiction. Thus $I \in \mathfrak{U}\mathbb{S}\mathfrak{M}$ and $\mathfrak{U}\mathbb{S}\mathfrak{M}$ is supernilpotent.

Now suppose $\mathfrak{M}$ contains no nilpotent rings. First we show that $\mathfrak{U}\mathbb{S}\mathfrak{M}$ in this case is a class of hereditarily idempotent rings. Suppose $R \in \mathfrak{U}\mathbb{S}\mathfrak{M}$ has a nonzero ideal I with $I \neq I^2$. According to [1], R can be mapped onto a nonzero subdirectly irreducible ring R/J with nilpotent heart H/J. Since $R/J \in \mathfrak{U}\mathbb{S}\mathfrak{M}$, R/J has a nonzero $\mathfrak{M}$-ideal T/J. But $0 \neq H/J \lhd T/J \in \mathfrak{M}$, a contradiction to $\mathfrak{M}$ being hereditary and containing no nilpotent rings. Therefore, $R \in \mathfrak{U}\mathbb{S}\mathfrak{M}$ implies that R is hereditarily idempotent. Now let $R \in \mathfrak{U}\mathbb{S}\mathfrak{M}$ and suppose $0 \neq I \lhd R$ and $I \notin \mathfrak{U}\mathbb{S}\mathfrak{M}$. Then there exists a nonzero homomorphic image I/K of I with no $\mathfrak{M}$-ideals. Since, in this case, R is hereditarily idempotent we have $0 \neq K \lhd R$. Again by Lemma 3, $R \notin \mathfrak{U}\mathbb{S}\mathfrak{M}$, a contradiction so $I \in \mathfrak{U}\mathbb{S}\mathfrak{M}$ and $\mathfrak{U}\mathbb{S}\mathfrak{M}$ is subidempotent.

If $\mathfrak{M}$ is any homomorphically closed class of rings and the lower radical class determined by $\mathfrak{M}$ is $\mathfrak{L}\mathfrak{M} = \{R \mid$ every nonzero homomorphic image of which has a accessible subring in $\mathfrak{M}\}$ we have as corollary to Theorem 4

COROLLARY. (a) *If* $\mathfrak{M}$ *is homomorphically closed*, $\mathfrak{M} \subseteq \mathfrak{U}\mathcal{S}\mathfrak{M} \subseteq \mathfrak{L}\mathfrak{M}$.

(b) *If* $\mathfrak{M}$ *is hereditary and homomorphically closed and in addition contains no or all nonzero nilpotent rings then* $\mathfrak{U}\mathcal{S}\mathfrak{M} = \mathfrak{L}\mathfrak{M}$.

PROOF. (a) Clear.

(b) From Theorem 4 we have that $\mathfrak{U}\mathcal{S}\mathfrak{M}$ is a radical class. Since $\mathfrak{L}\mathfrak{M}$ is the smallest radical class containing $\mathfrak{M}$ we conclude from (a) that $\mathfrak{U}\mathcal{S}\mathfrak{M} = \mathfrak{L}\mathfrak{M}$.

Recall that β is the lower Baer radical, that is the lower radical determined by the class of all nilpotent rings. This can also be thought of as the class of all rings each nonzero homomorphic image of which has a nonzero nilpotent idea. We also let F denote the Blair radical [2] which is the class of all hereditarily idempotent rings.

THEOREM 5. (a) *If* $\mathfrak{M}$ *is the class of all nilpotent rings, then* $\mathfrak{U}\mathcal{S}\mathfrak{M} = \beta$.

(b) *If* $\mathfrak{M}$ *is the class of all semiprime rings, then* $\mathfrak{U}\mathcal{S}\mathfrak{M} = \mathfrak{F}$.

(c) *If* $\mathfrak{M}$ *is any hereditary class of rings containing no nilpotent rings, then* $\mathfrak{U}\mathcal{S}\mathfrak{M} \subseteq \mathfrak{F}$.

PROOF. (a) Clear.

(c) Follows from Theorem 4.

(b) If $\mathfrak{M}$ is the class of all semiprime rings, we have that $\mathfrak{U}\mathcal{S}\mathfrak{M} \subseteq \mathfrak{F}$ from (c). Since $\mathfrak{F}$ is a hereditary radical class of semiprime rings we have that every nonzero image of a ring from $\mathfrak{F}$ has a nonzero ideal, considered as a ring, which is semiprime. Hence $\mathfrak{F} \subseteq \mathfrak{U}\mathcal{S}\mathfrak{M}$.

Now let $\mathfrak{Y}$ denote the upper radical class determined by all fields. We know of many supernilpotent nonspecial radical classes. Those in [11] and [12] contain $\mathfrak{Y}$ properly. Those of [9] are contained in $\mathfrak{Y}$ properly whereas those of [7] are independent of $\mathfrak{Y}$. We propose to construct an infinite number of supernilpotent nonspecial radical classes of the latter type. For this purpose consider the ring A constructed in [9]. Three of the properties of the ring A are:

(1) Every ideal of A contains zero divisors.

(ii) The only prime image of A is $\mathbf{Z}_2$.

(ii) A is Boolean ring.

Let $\mathfrak{P}$ be any hereditary class of prime rings containing $\mathbf{Z}_2$. Consider $\mathcal{C} = \{$all nilpotent rings, $\mathfrak{P}\}$. Then

THEOREM 6. $\mathfrak{U}\mathcal{S}\mathcal{C}$ *is a supernilpotent nonspecial radical class.*

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PROOF. From Theorem 4 it follows that $\mathcal{U}\mathcal{S}\mathcal{C}$ is a supernilpotent radical class. Let $\mathcal{X}$ be the class of all prime $\mathcal{U}\mathcal{S}\mathcal{C}$ semisimple rings. $\mathcal{U}\mathcal{X}$ is then the smallest special radical class containing $\mathcal{U}\mathcal{S}\mathcal{C}$. The only prime image of the ring A is $\mathbf{Z}_2$ and hence $A \in \mathcal{U}\mathcal{X}$. A has no nilpotent ideals since it is Boolean. Since every ideal of A has zero divisors, we conclude that A cannot have prime rings as ideals and hence $A \notin \mathcal{U}\mathcal{S}\mathcal{C}$. So $\mathcal{U}\mathcal{S}\mathcal{C}$ is nonspecial.

In order to construct supernilpotent radical classes independent of $\mathcal{Y}$ consider the following: let E be any prime ring that cannot be mapped onto a field, for example any simple ring which is not a field will do such as the quaternions, and $\mathcal{B}$ any hereditary class of prime rings containing $\mathbf{Z}_2$ but not E nor any ideals of E. Consider now $\mathcal{K} = \{\text{all nilpotent rings}, \mathcal{B}\}$. Then

COROLLARY 2. *$\mathcal{U}\mathcal{S}\mathcal{K}$ is a supernilpotent nonspecial radical class independent of* $\mathcal{Y}$.

PROOF. From Theorem 6 we conclude that $\mathcal{U}\mathcal{S}\mathcal{K}$ is supernilpotent and nonspecial. Furthermore $\mathbf{Z}_2 \in \mathcal{U}\mathcal{S}\mathcal{K}$ but $\mathbf{Z}^2 \notin \mathcal{Y}$. The ring E above is in $\mathcal{Y}$ but $E \notin \mathcal{U}\mathcal{S}\mathcal{K}$.

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References

[1] V. A. Andrunakievic, 'Radicals of associative rings. I,' *Mat. Sb.* **44** (1958), 179–212 (Russian).
[2] R. L. Blair, 'A note on *f*-regularity in rings,' *Proc. Amer. Math. Soc.* **6** (1955), 511–515.
[3] N. J. Divinsky, *Rings and radicals* (Univ. of Toronto Press, 1965).
[4] N. Divinsky, J. Krempa and A. Sulinski, 'Strong radical properties of alternative and associative rings,' *J. Algebra* **17** (1971), 369–388.
[5] G. A. P. Heyman and H. J. le Roux, 'On upper and complementary radicals,' *Math. Japon.*, to appear.
[6] N. Jacobson, *Structure of rings*, (Amer. Math. Soc. Colloq. Pub. 37, Providence, R.I., 1956).
[7] W. G. Leavitt, 'A minimally embeddable ring,' *Period. Math. Hungar.* **12** (1981), 129–140.
[8] L. C. A. van Leeuwen, 'On a generalization of Jenkins radical,' *Arch. Mat. (Basel)* **22** (1971), 155–160.
[9] L. C. A. van Leeuwen and T. L. Jenkins, 'A supernilpotent non-special radical class,' *Bull. Austral. Math. Soc.* **9** (1973), 343–348.
[10] D. M. Olson and T. L. Jenkins, 'Upper radicals and essential ideals,' *J. Austral. Math. Soc.*, submitted.
[11] Ju. M. Rjabuhin, 'On hypernilpotent and special radicals,' *Studies in Algebra and Math. Analysis, Izdat. "Kartja Moldovenjaske,"* (Kishinev, 1965).

[12] Robert L. Snider, 'Lattices of radicals,' *Pacific J. Math.* **40** (1972), 207–220.
[13] R. Wiegandt, *Radical and semisimple classes of rings*, (Queen's papers in pure and applied math. 37, Kingston, Ontario, 1974).

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