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ON RING EXTENSIONS OF FSG RINGS

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A ring R is called right FSG if every finitely generated right R-subgenerator is a generator. In this note we consider the question of when a ring extension of a given right FSG ring is right FSG and the converse. As a consequence we obtain some results about right FSG group rings.

1. INTRODUCTION

In this note all rings are associative with identities and all modules are unitary. For a ring R, the category of all right (left) R-modules is denoted by Mod-R (R-Mod). Let M_R be a right R-module. A module N is called M-generated (or M generates N) if there exists a set A and an epimorphism $M^{(A)} \to N$, where $M^{(A)}$ is the direct sum of $|A|$ copies of M ($|A|$ denotes the cardinality of the set A). When A is finite, we say that N is M-finitely generated. N is called M-cogenerated (or M cogenerates N) if there exist a set A and a monomorphism $N \to M^A$ is the direct product of $|A|$ copies of M. When A is finite, we say that N is M-finitely cogenerated. For a module M_R, we denote by $\sigma[M]$ the full subcategory of Mod-R whose objects are submodules of M-generated modules (see [**11**]).

For a right R-module M, the trace ideal of M in R is denoted by trace(M). By definition, $\operatorname{trace}(M) = \sum\{\operatorname{im}\varphi : \varphi \in \operatorname{Hom}_R(M, R_R)\}$ (see [**11**, p.154]).

A module M_R is called faithful if $\{a \in R : Ma = 0\} = 0$. Then M is faithful if and only if M cogenerates every projective right R-module. Dually, a module M_R is called cofaithful if M generates every injective right R-module (see [**1**, p.217]). It follows that M is cofaithful if and only if there exists a finite subset $\{m_1, \dots, m_n\}$ of elements of M such that $\{x \in R : m_1x = \dots = m_nx = 0\} = 0$. By Lemma 1 below we see that M_R is cofaithful if and only if $\sigma[M_R] = $ Mod-R. A module M_R with this property is called a subgenerator of Mod-R (see Wisbauer [**11**, p.118]). Therefore, instead of cofaithful right R-modules we shall use the terminology "right R-subgenerators".

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Let R be a ring and G a group. Then by $R[G]$ we denote the group ring of R over G.

A ring R is called right FPF if every finitely generated faithful right R-module is a generator. For details about FPF rings we refer to Faith and Page [**6**] and Faith and Pillay [**7**]. We introduce the family of right FSG rings as a generalisation of the class of right self-injective rings and the class of right FPF rings: A ring R is called right FSG if every finitely generated right R-subgenerator is a generator. Basic results about FSG rings were obtained in [**8**].

Let A and B be rings. If A is a subring of B with common identity, then we say that B is a ring extension of A. In this paper, we shall consider conditions under which a ring extension B of a given right FSG ring A becomes right FSG and conversely.

2. Results

First we list some known results used in this section.

Lemma 1. *Let $M_R \in$ Mod-R. Then the following conditions are equivalent:*

(i) *M_R is a cofaithful module.*
(ii) *There exists a finite set $\{m_1, \dots, m_n\}$ of elements of M such that $\{x \in R : m_1 x = \dots = m_n x = 0\} = 0$,*
(iii) *There exists a positive integer n such that R_R can be embedded into M^n.*
(iv) *M generates every injective right R-module.*
(v) $\sigma[M] =$ Mod-R.
(vi) *Cyclic submodules of $M^{(\mathbb{N})}$ form a set of generators in* Mod-R.

Proof: See [**1**, Exercise 18.25,p.217], [**2**, Proposition 4.5.4] and [**11**, 15.3]. ∎

Recall that a ring R is strongly right bounded if every nonzero right ideal contains a nonzero ideal. A commutative ring is strongly right (and left) bounded.

Lemma 2. *If R is a strongly right bounded right FSG ring then R is right FPF. In particular any commutative FSG ring is FPF.*

Proof: Let R be a strongly right bounded right FSG ring and let M be a finitely generated faithful right R-module, say $M = x_1 R + \dots + x_n R$. Set $A = r(\{x_1, \dots, x_n\})$. If $A \neq 0$, there is a nonzero ideal B of R such that $B \subseteq A$. Then $MB = (x_1 R + \dots + x_n R)B = x_1 B + \dots + x_n B = 0$, a contradiction. Hence $A = 0$ and so M is a subgenerator and then a generator of Mod-R. This means that R is right FPF. ∎

The following result provides sufficient conditions for a ring extension of a right FSG to be right FSG.

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Theorem 3. *Let B be a ring extension of A such that:*

(a) *B is finitely generated as a right A-module,*

(b) *B generates $B \otimes_A B$ as B-bimodules.*

If A is right FSG so is B.

Proof: Let X be a finitely generated right B-subgenerator. It is easy to see that X is a finitely generated right A-subgenerator. By assumption, X_A is a generator. We shall prove that X is a generator in Mod-R.

From (b), we have an exact sequence as B-bimodules:

$$\bigoplus_I B \longrightarrow B \otimes_A B \longrightarrow 0 \tag{1}$$

for some index set I.

By tensoring (1) with X_B, we have the following commutative diagram in Mod-B with exact rows:

$$\begin{array}{ccccc}
X\otimes_B(\oplus_I B) & \longrightarrow & X\otimes_B(B\otimes_A B) & \longrightarrow & 0 \\
\wr\| & & \wr\| & & \\
\oplus_I(X\otimes_B B) & \longrightarrow & (X\otimes_B B)\otimes_A B & \longrightarrow & 0 \\
\wr\| & & \wr\| & & \\
\oplus_I X & \longrightarrow & X\otimes_A B & \longrightarrow & 0
\end{array}$$

It follows from this that X generates $X \otimes_A B$.

Since X is a generator in Mod-A, we obtain an exact sequence in Mod-A:

$$X^n \longrightarrow A \longrightarrow 0 \tag{2}$$

for some positive integer n.

Tensoring (2) with ${}_A B$, we have the following commutative diagram with exact rows in Mod-B:

$$\begin{array}{ccccc}
X^t\otimes_A B & \longrightarrow & A\otimes_A B & \longrightarrow & 0 \\
\wr\| & & \wr\| & & \\
(X\otimes_A B)^t & \longrightarrow & B & \longrightarrow & 0.
\end{array}$$

Hence $X \otimes_A B$ is a generator in Mod-B. This proves that X is a generator in Mod-B. Thus B is a right FSG ring. ∎

Similar to [5], we have a result about group rings.

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Proposition 4. *Let R be a ring and G a finite group. If R is right FSG then $R[G]$ is right FSG.*

Proof: Let $G = \{g_1, \dots, g_n\}$ and let M be a finitely generated subgenerator in Mod-$R[G]$. It is easy to see that M is a finitely generated right R-module. Moreover, M is a subgenerator in Mod-R. Indeed, let $\{m_1, \dots, m_t\} \subset M$ such that $r_{R[G]}(\{m_1, \dots, m_t\}) = 0$. Then if $c \in r_R(\{m_1, \dots, m_t\})$, that is, $m_1 c = \dots = m_t c = 0$, it follows that $c = 0$. By assumption, M is a generator in Mod-R.

We have:

$$\operatorname{trace}_{R[G]} M = \bigoplus_{i=1}^{n} \operatorname{trace}_{g_i R} M = \bigoplus_{i=1}^{n} g_i R = R[G],$$

proving that M is a generator in Mod-$R[G]$. ∎

Now we present an example of a right and left FSG ring whose centre is not FSG. Another example of the subring of elements fixed by a finite group of automorphisms of an FSG ring which need not be FSG is presented here.

Example 5: (Clark [**3**, 4]). Let K be a field of two elements and G the quaternion group of order eight, that is

$$G = \langle a, b \colon a^4 = b^4 = 1, a^2 = b^2, b^{-1}ab = a^{-1} \rangle.$$

Then we have the group ring $R = K[G]$. Since G is finite, R is self-injective by a result of Renault [**9, 10**]. Thus R is FSG by [**2**, Proposition 4.5.6].

By [**3**], C, the centre of R is not FPF. By Lemma 2, C is not FSG.

Also, by [4], G has automorphism group

$$S_4 = \langle g, h \colon g^4 = h^2 = (gh)^3 = 1 \rangle$$

where $g(a) = a$, $g(b) = ab$, $h(a) = b$, and $h(b) = a$. Now let F denote the group of automorphisms of R obtained by extending linearly to R the action of S_4 on G. Then it is easy to check that:

$$R^F := \{r \in R : \forall g \in F(g(r) = r)\} = \{0, 1, a^2, 1 + a^2, w, 1 + w, a^2 + w, 1 + a^2 + w\}$$

where $w = a + a^3 + b + ab + a^2 b + a^3 b$. Moreover, R^F is commutative and not FPF. By Lemma 2, R^F is not FSG.

However, in the following results we shall consider the FSG ring extension of a ring A with some additional condition for which A becomes a FSG ring.

Recall a module M_R is called a torsionless module if for each non-zero element x in M there exists an R-homomorphism f from M to R_R such that $f(x) \neq 0$. For example, every projective module is torsionless.

Let R be a ring. For a subset X of R, we denote by $V_R(X)$ the subring of R consisting of all r in R such that $rx = xr$ for all x in X.

THEOREM 6. *Let B be a ring extension of A such that:*

(a) *B is torsionless as a left A-module,*

(b) *B is a generator in* Mod-A,

(c) *$V_B(A)$ generates B as a A-module.*

Then if B is a right FSG ring, so is A.

PROOF: Let B be a ring extension of A such that B and A satisfy (a), (b) and (c). Assume that B is right FSG and Y is a finitely generated subgenerator in Mod-A. Set $X = Y \otimes_A B$. Then X is a finitely generated right B-module. Moreover, X is a subgenerator in Mod-B. Indeed, since Y_A is a subgenerator, there exists $\{y_1, y_2, \cdots, y_n\} \subset Y$ such that $r_A(\{y_1, y_2, \cdots, y_n\}) = 0$. Assume that

$$(y_1 \otimes 1)b = \ldots = (y_n \otimes 1)b = 0$$

for some b in B. Let f be any homomorphism from B to A in A-Mod. Then $y_1 f(b) = \ldots = y_n f(b) = 0$. Hence $f(b) = 0$. By (a), $b = 0$. This proves that

$$r_B(\{y_1 \otimes 1, \cdots, y_n \otimes 1\}) = 0,$$

that is, X is a subgenerator in Mod-B. By assumption, X is a generator in Mod-B. Hence X is a generator in Mod-A by (b).

By(c) we have an exact sequence of A-bimodules:

(3) $$\bigoplus_I A \longrightarrow B \longrightarrow 0$$

for some index set I.

Tensoring (3) with Y_A gives the following commutative diagram with exact rows in Mod-A.

$$\begin{array}{ccccc} Y\otimes_A(\oplus_I A) & \longrightarrow & Y\otimes_A B & \longrightarrow & 0 \\ \wr\| & & \| & & \\ \oplus_I(Y\otimes_A A) & \longrightarrow & Y\otimes_A B & \longrightarrow & 0 \\ \wr\| & & \| & & \\ \oplus_I Y & \longrightarrow & Y\otimes_A B & \longrightarrow & 0, \end{array}$$

that is, Y generates $X = Y \oplus_A B$ in Mod-A. It follows that Y is a generator in Mod-A. Hence A is right FSG. $\square$

Corollary 7. *Let B be a ring extension of A such that A finitely generates B as an A-bimodule. If B is right (respectively left) FSG, then A is right (respectively left) FSG.*

Proof: Since A finitely generates B as A-bimodules, there exist $v_1, \cdots, v_n$ in $V_B(A)$ and $f_1, \cdots, f_n$ in $\mathrm{Hom}({}_AB_A, {}_AA_A)$ such that

$$\sum_{i=1}^{n} f_i(b)v_i = b,$$

for all b in B. Let C be the centre of A. Let us define mappings:

$$\begin{gathered} g\colon V_B(A) \otimes_C A \longrightarrow B \\ v \otimes a \longmapsto va \\ h\colon B \longrightarrow V_B(A) \otimes_C A \\ b \longmapsto \sum_{i=1}^{n} v_i \otimes f_i(b). \end{gathered}$$

Then g and h are mutually inverse mappings. Since $f_i(V_B(A)) \subset C$, $V_B(A)$ is a finitely generated projective C-module, hence it is a generator. It follows that B is a generator as a right A-module. By the same argument as in proving Theorem 6, we obtain that if B is right FSG then so is A. Similarly, if B is left FSG then so is A. ∎

Corollary 8. *Let R be a ring and G a group. If the group ring $R[G]$ is right (respectively left) FSG, then R is right (respectively left) FSG. Moreover, if G is finite, then R is right (respectively left) FSG if and only if $R[G]$ is right (respectively left) FSG.*

Proof: By Theorem 6 and Proposition 4. ∎

Concerning this Corollary 8 we note that Connell [5] proved that, if G is a finite group then the group ring $R[G]$ is right self-injective if and only if R is right self-injective.

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