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CHARACTERIZATIONS OF HYPERSURFACES OF TYPE A_2 IN A COMPLEX PROJECTIVE SPACE

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Abstract

In this paper, we characterize hypersurfaces of type A_2 in a complex projective space in terms of their geodesics.

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1. Introduction

When we study Riemannian submanifolds, we can get information on their shapes by investigating their geodesics. It is known that a hypersurface M^n isometrically immersed into a space form $\widetilde{M}^{n+1}(c)$ of constant sectional curvature c is totally umbilic if and only if every geodesic on M^n is mapped to a circle in $\widetilde{M}^{n+1}$. On the contrary, in a complex n-dimensional complex projective space $\mathbb{C}P^n$ of constant holomorphic sectional curvature 4, there exist no real hypersurfaces all of whose geodesics are mapped to circles in $\mathbb{C}P^n$.

Among real hypersurfaces in $\mathbb{C}P^n$ the following hypersurfaces are quite important:

(A_1) a geodesic sphere of radius r $(0 < r < \pi/2)$ in $\mathbb{C}P^n$;

(A_2) a tube of radius r $(0 < r < \pi/2)$ around a totally geodesic Kähler submanifold $\mathbb{C}P^k$ in $\mathbb{C}P^n$ with $1 \leqq k \leqq n-2$.

In this paper, we say these real hypersurfaces are of type A_1 and of type A_2, respectively. Hypersurfaces of type A_1 have two distinct constant principal curvatures in $\mathbb{C}P^n$. It is well known that $\mathbb{C}P^n$ does *not* admit totally umbilic real hypersurfaces and that a real hypersurface M^{2n-1} in $\mathbb{C}P^n$ $(n \geqq 3)$ is of type A_1 if and only if M has at most two distinct principal curvatures at each point of M. These tell us that

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hypersurfaces of type A_1 are the simplest examples of real hypersurfaces in $\mathbb{C}P^n$. Hypersurfaces of type A_2 have three distinct constant principal curvatures in $\mathbb{C}P^n$. However, both hypersurfaces of type A_1 and of type A_2 have many *common* nice properties, which enrich the theory of real hypersurfaces (see [5]). Moreover, it is known that they are typical examples of naturally reductive Riemannian homogeneous manifolds. Hence they are nice examples from both the viewpoints of extrinsic geometry (that is, submanifold theory) and of intrinsic geometry.

The aim of this paper is to distinguish between hypersurfaces of type A_2 and hypersurfaces of type A_1 in $\mathbb{C}P^n$. We characterize only hypersurfaces of type A_2 by studying shapes of their geodesics in $\mathbb{C}P^n$ (Theorems 3.1 and 4.3).

2. Real hypersurfaces of type A_1 and A_2

Let M^{2n-1} be an orientable real hypersurface of $\mathbb{C}P^n$ and $\mathcal{N}$ a unit normal vector field on M in $\mathbb{C}P^n$. The Riemannian connections $\widetilde{\nabla}$ of $\mathbb{C}P^n$ and ∇ of M are related by the following formulas which are the so-called Gauss formula and the Weingarten formula, respectively:

$$\widetilde{\nabla}_X Y = \nabla_X Y + \langle AX, Y\rangle \mathcal{N} \quad \text{and} \quad \widetilde{\nabla}_X \mathcal{N} = -AX \tag{2.1}$$

for vector fields X, Y on M, where $\langle\, ,\, \rangle$ denotes the Riemannian metric on M induced from the Fubini–Study metric on $\mathbb{C}P^n$ and A is the shape operator of M in $\mathbb{C}P^n$. It is known that M admits an almost contact metric structure $(\phi, \xi, \eta, \langle\, ,\, \rangle)$ induced from the Kähler structure J of $\mathbb{C}P^n$. The vector field ξ defined by $\xi = -J\mathcal{N}$ is called the characteristic vector field. The tensor field ϕ of type $(1, 1)$ and the 1-form η on M are given by

$$\langle \phi X, Y\rangle = \langle JX, Y\rangle \quad \text{and} \quad \eta(X) = \langle \xi, X\rangle = \langle JX, \mathcal{N}\rangle,$$

and satisfy $\phi^2 X = -X + \eta(X)\xi$ and $\phi\xi = 0$. By use of the Weingarten formula we find

$$\nabla_X \xi = \phi AX. \tag{2.2}$$

Eigenvalues and eigenvectors of the shape operator A are called *principal curvatures* and *principal curvature vectors*, respectively. We now recall some properties of hypersurfaces of type A_1 and A_2. We denote by V_λ the eigenspace of A associated with eigenvalue λ. When M is a hypersurface of type A_1, the tangent bundle TM is decomposed as $TM = \mathbb{R}\xi \oplus V_{\cot r}$ and $A\xi = (2\cot 2r)\xi$. When M is a hypersurface of type A_2, its tangent bundle is decomposed as

$$TM = \mathbb{R}\xi \oplus V_{\cot r} \oplus V_{-\tan r}$$

with $\dim V_{\cot r} = 2n - 2k - 2$, $\dim V_{-\tan r} = 2k$ and $A\xi = (2\cot 2r)\xi$. These real hypersurfaces of type A_1 and A_2 are characterized by the property of their shape operators in the following manner [5].

LEMMA 2.1. *Let M be a real hypersurface of $\mathbb{C}P^n$. Then the following four conditions are equivalent:*

(1) *M is locally congruent to a hypersurface of either type A_1 or type A_2;*
(2) *the shape operator A of M in $\mathbb{C}P^n$ satisfies $\langle(\nabla_X A)X, X\rangle = 0$ for each vector X on M;*
(3) *$\phi A = A\phi$ holds everywhere on M;*
(4) *$\|\nabla A\|^2 = 4(n-1)$ holds everywhere on M.*

3. Extrinsic geodesics on hypersurfaces

In this section we study real hypersurfaces of type A_2 by paying attention to the existence of extrinsic geodesics. A geodesic on a submanifold M in $\widetilde{M}$ is said to be an *extrinsic geodesic* if it is also a geodesic on $\widetilde{M}$.

For a geodesic γ on a real hypersurface M in $\mathbb{C}P^n$, we define its *structure torsion* ρ_γ by $\rho_\gamma = \langle\dot\gamma, \xi_\gamma\rangle$. By use of (2.2) we have

$$\rho'_\gamma = \langle\dot\gamma, \nabla_{\dot\gamma}\xi\rangle = \langle\dot\gamma, \phi A\dot\gamma\rangle = -\langle A\phi\dot\gamma, \dot\gamma\rangle,$$

hence we obtain

$$\rho'_\gamma = \tfrac{1}{2}\langle\dot\gamma, (\phi A - A\phi)\dot\gamma\rangle. \tag{3.1}$$

Therefore, when M is a hypersurface of type A_1 or of type A_2, we see by Lemma 2.1 that ρ_γ is constant along γ.

THEOREM 3.1. *A connected real hypersurface M of $\mathbb{C}P^n$ is of type A_2 with radius r $(0 < r < \pi/2)$ if and only if it satisfies the following two conditions.*

(1) *At each point $x \in M$ there exists an orthonormal basis $v_1, v_2, \ldots, v_{2n-1}$ of T_xM such that every geodesic γ_{ij} of M through x in the direction $v_i + v_j$ $(1 \leqq i \leqq j \leqq 2n-1)$ has constant structure torsion $\rho_{\gamma_{ij}}$.*
(2) *At some point $x \in M$ there exists an extrinsic geodesic γ of M through x orthogonal to ξ_x.*

PROOF. Suppose that M satisfies the condition (1). By (3.1) we see that

$$0 = \rho'_{\gamma_{ii}}(0) = \tfrac{1}{2}\big\langle\dot\gamma_{ii}, (\phi A - A\phi)\dot\gamma_{ii}\big\rangle(0) = \tfrac{1}{2}\big\langle v_i, (\phi A - A\phi)v_i\big\rangle,$$
$$0 = \rho'_{\gamma_{jk}}(0) = \tfrac{1}{2}\big\langle\dot\gamma_{jk}, (\phi A - A\phi)\dot\gamma_{jk}\big\rangle(0) = \tfrac{1}{4}\big\langle v_j + v_k, (\phi A - A\phi)(v_j + v_k)\big\rangle$$

for $1 \leqq i \leqq 2n-2$ and $1 \leqq j < k \leqq 2n-2$. These, together with the fact that $\phi A - A\phi$ is symmetric, imply that $\langle v_i, (\phi A - A\phi)v_j\rangle = 0$ for $1 \leqq i \leqq j \leqq 2n-2$. We hence find that $\phi A = A\phi$ and M is a hypersurface of type either A_1 or A_2 by Lemma 2.1. Thus the condition (1) characterizes hypersurfaces of types A_1 and A_2.

What we have to do is to check the condition (2) for hypersurfaces of types A_1 and A_2. We take an arbitrary geodesic γ on a hypersurface M whose initial vector is

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orthogonal to the characteristic vector. When M is a hypersurface of type A_1, as we have $\langle A\dot{\gamma}(0), \dot{\gamma}(0)\rangle = \cot r > 0$, we find by the Gauss formula that γ is not a geodesic in $\mathbb{C}P^n$. When M is a hypersurface of type A_2, we denote $\dot{\gamma}(0) = (\sin\theta)v + (\cos\theta)w$ with unit vectors $v \in V_{\cot r}$ and $w \in V_{-\tan r}$. We then have

$$\langle A\dot{\gamma}(0), \dot{\gamma}(0)\rangle = \sin^2\theta \cot r - \cos^2\theta \tan r.$$

By use of Lemma 2.1, we see that $\langle A\dot{\gamma}, \dot{\gamma}\rangle$ is constant along γ. Thus by the Gauss formula we find that γ is an extrinsic geodesic if and only if its initial vector is given as $\dot{\gamma}(0) = (\sin r)v + (\cos r)w$ (that is, $\theta = r$). We hence obtain the desired conclusion. □

REMARK. The constancy condition on structure torsions in the condition (1) in Theorem 3.1 can be weakened to a condition $\rho'_{\gamma_{ij}}(0) = 0$ at an initial point.

We should point out that the orthogonality to the characteristic vector field in the condition (2) in Theorem 3.1 is important.

THEOREM 3.2. *A connected real hypersurface M of $\mathbb{C}P^n$ is either a hypersurface of type* A_1 *with radius r ($\pi/4 \leqq r < \pi/2$) or a hypersurface of type* A_2 *with radius r ($0 < r < \pi/2$) if and only if it satisfies the following two conditions.*

(1) *At each point $x \in M$ there exists an orthonormal basis $v_1, v_2, \ldots, v_{2n-1}$ of T_xM such that every geodesic γ_{ij} of M through x in the direction $v_i + v_j$ $(1 \leqq i \leqq j \leqq 2n-1)$ has constant structure torsion $\rho_{\gamma_{ij}}$.*
(2) *There exists an extrinsic geodesic on M.*

PROOF. By virtue of the discussion in the proof of Theorem 3.1, we have only to check the condition (2) for hypersurfaces of type A_1. We take an arbitrary geodesic γ on a hypersurface M of type A_1 and denote its initial vector as

$$\dot{\gamma}(0) = (\cos\psi)v + (\sin\psi)\xi_{\gamma(0)}$$

with $0 \leqq |\psi| \leqq \pi/2$ and $v \in V_{\cot r}$. We then have

$$\langle A\dot{\gamma}(0), \gamma(0)\rangle = \cos^2\psi \cot r + 2\sin^2\psi \cot 2r = 1 - \sin^2\psi \tan^2 r,$$

which is null when $\pi/4 \leqq r < \pi/2$ and $\sin\psi = \cot r$. Since $\langle A\dot{\gamma}, \dot{\gamma}\rangle$ is constant along γ by Lemma 2.1, we obtain the conclusion. □

4. Extrinsic shapes of geodesics

In the previous section we studied extrinsic geodesics on real hypersurfaces. We are hence interested in extrinsic shapes of other geodesics. For a smooth curve γ on a submanifold M in $\widetilde{M}$, regarding it as a curve on $\widetilde{M}$, we call it the extrinsic shape of γ.

We give here some terminology. A smooth curve γ on a Riemannian manifold M parameterized by its arc length is called a *helix of proper order d* if there exist a field of orthonormal frames $\{V_1 = \dot{\gamma}, V_2, \ldots, V_d\}$ along γ and positive constants $k_1, \ldots, k_{d-1}$ satisfying the following system of ordinary differential equations:

$$\nabla_{\dot{\gamma}} V_i = -k_{i-1} V_{i-1} + k_i V_{i+1}, \quad i = 1, \ldots, d,$$

where V_0, V_{d+1} are null vector fields, $k_\emptyset = k_d = \emptyset$ and $\nabla_{\dot{\gamma}}$ denotes the covariant differentiation along γ. The constants k_i $(1 \leqq i \leqq d-1)$ and a field of orthonormal frames $\{V_1, \ldots, V_d\}$ are called the *curvatures* and the *Frenet frame* of γ, respectively. A curve is called a *helix of order d* if it is a helix of proper order h $(\leqq d)$. A helix of order 1 is nothing but a geodesic and a helix of order 2 is called a circle. Needless to say, a geodesic is regarded as a circle of null curvature.

Extrinsic shapes of geodesics were studied in the preceding papers [1, 2].

LEMMA 4.1 (Adachi *et al.* [2]). *Let γ be a geodesic with structure torsion ρ_γ on a hypersurface M of type* A_1 *with radius r $(0 < r < \pi/2)$ in $\mathbb{C}P^n$. Then the extrinsic shape of γ is as follows.*

(1) *When the radius r satisfies $\pi/4 \leqq r < \pi/2$, if $\rho_\gamma = \pm \cot r$, then it is an extrinsic geodesic.*

(2) *When $r \neq \pi/4$, if $\rho_\gamma = \pm 1$, then the extrinsic shape is a circle of positive curvature* $2|\cot 2r|$ *which lies on a totally geodesic complex curve $\mathbb{C}P^1$ in $\mathbb{C}P^n$.*

(3) *If $\rho_\gamma = 0$, then the extrinsic shape is a circle of positive curvature* $\cot r$ *which lies on a totally real totally geodesic real projective plane $\mathbb{R}P^2$ of constant sectional curvature* 1 *in $\mathbb{C}P^n$.*

(4) *Generally, if $\rho_\gamma = \sin\psi$ $(0 < |\psi| < \pi/2, \sin\psi \neq \cot r)$, then the extrinsic shape is a helix of proper order* 4 *whose curvatures are described as*

$$k_1 = |\cot r - \tan r \sin^2 \psi|, \quad k_2 = \tan r |\sin\psi| \cos\psi, \quad k_3 = \cot r.$$

This helix lies on a totally geodesic Kähler surface $\mathbb{C}P^2$ in $\mathbb{C}P^n$.

LEMMA 4.2 (Adachi [1]). *Let γ be a geodesic with null structure torsion on a hypersurface of type* A_2 *with radius r $(0 < r < \pi/2)$ in $\mathbb{C}P^n$. If the initial vector $\dot{\gamma}(0)$ is of the form* $(\cos\theta)v + (\sin\theta)w$ $(0 \leqq \theta \leqq \pi/2)$ *with unit vectors $v \in V_{\cot r}$ and $w \in V_{-\tan r}$, then its extrinsic shape is as follows.*

(1) *When* $\cos\theta = \sin r$, *it is an extrinsic geodesic.*

(2) *When $\theta = 0$ (respectively, $\theta = \pi/2$), the extrinsic shape is a circle of positive curvature* $\cot r$ *(respectively,* $\tan r$*). This circle lies on a totally real totally geodesic $\mathbb{R}P^2$ in $\mathbb{C}P^n$.*

(3) *When* $\cos\theta \neq \sin r$, *the extrinsic shape is a helix of proper order* 3 *whose curvatures are described as*

$$k_1 = 2|\cos^2\theta - \sin^2 r| / \sin 2r, \quad k_2 = 2\sin\theta\cos\theta / \sin 2r.$$

This helix lies on a totally real totally geodesic (*real*) *three-dimensional real projective space $\mathbb{R}P^3$ of constant sectional curvature* 1 *in $\mathbb{C}P^n$.*

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These lemmas show that extrinsic shapes of geodesics with null structure torsion on a hypersurface of type A_1 are helices of proper order 2 and that those on a hypersurface of type A_2 are helices of proper order of either 1, 2 or 3. One can also characterize hypersurfaces of type A_2 by such a property.

THEOREM 4.3. *A connected real hypersurface M^{2n-1} of $\mathbb{C}P^n$ is of type* A_2 *if and only if M satisfies the following two conditions.*

(1) *At each point $x \in M$ there exists an orthonormal basis $v_1, v_2, \ldots, v_{2n-1}$ of T_xM such that every geodesic γ_{ij} of M through x in the direction $v_i + v_j$ $(1 \leqq i \leqq j \leqq 2n-1)$ has constant structure torsion $\rho_{\gamma_{ij}}$.*

(2) *There exists a geodesic* (*on* M) *whose extrinsic shape is a helix of proper order* 3.

We should note that a characterization much like Theorem 4.3 also holds for tubes around totally geodesic $\mathbb{C}H^k$ $(1 \leqq k \leqq n-2)$ in a complex hyperbolic space $\mathbb{C}H^n$. However, we cannot characterize such tubes by the property of the existence of extrinsic geodesics, because there exist no extrinsic geodesics on such tubes in $\mathbb{C}H^n$.

5. Ruled real hypersurfaces

It is an interesting problem to weaken the condition on the constancy of structure torsions. Is it possible to reduce the number of geodesics with constant structure torsion in the condition (1) of Theorems 3.1, 3.2 and 4.3?

A real hypersurface M is said to be a *ruled real hypersurface* in $\mathbb{C}P^n$ if the holomorphic distribution T^0M defined by $T^0_xM = \{v \in T_xM \mid v \perp \xi_x\}$ for $x \in M$ is integrable and each of its maximal integral submanifolds is a totally geodesic complex hypersurface $\mathbb{C}P^{n-1}$ in $\mathbb{C}P^n$.

PROPOSITION 5.1. *If the initial vector of a geodesic γ on a ruled real hypersurface M in $\mathbb{C}P^n$ is orthogonal to the characteristic vector, then it has constant structure torsion and is an extrinsic geodesic.*

PROOF. Let M_x be the maximal integral submanifold through a point $x \in M$ for the holomorphic distribution T^0M. We take a geodesic σ on M_x with $\sigma(0) = x$ and $\dot\sigma(0) \perp \xi_x$. Since M_x is totally geodesic in $\mathbb{C}P^n$, the curve σ is also a geodesic in the ambient space $\mathbb{C}P^n$. Hence σ, considered as a curve on our real hypersurface M, is a geodesic on M. Therefore the uniqueness theorem for geodesics guarantees that γ is an extrinsic geodesic with null structure torsion. □

By using this property for geodesics on ruled real hypersurfaces, we can provide a characterization of such hypersurfaces (for details, see [3]).

PROPOSITION 5.2. *A real hypersurface M in $\mathbb{C}P^n$ is ruled if and only if at each point $x \in M$ there exist orthonormal vectors $v_1, v_2, \ldots, v_{2n-2}$ orthogonal to ξ_x such that every geodesic γ_{ij} of M through x in the direction $v_i + v_j$ $(1 \leqq i \leqq j \leqq 2n-2)$ is also a geodesic in the ambient space $\mathbb{C}P^n$.*

These propositions show that we cannot reduce the number of geodesics with constant structure torsion to characterize hypersurfaces of type A_2.

At the end of this paper we summarize some fundamental results on ruled real hypersurfaces. Each ruled real hypersurface in $\mathbb{C}P^n$ is constructed in the following manner. We take an arbitrary regular (real) curve γ, and attach a totally geodesic complex hypersurface $M_{\gamma(s)}$ at each point $\gamma(s)$ which is holomorphically isometric to $\mathbb{C}P^{n-1}$ and whose tangent space is orthogonal to the complex one-dimensional subspace $\mathbb{C}\dot{\gamma}(s)$ in $T_{\gamma(s)}\mathbb{C}P^n$. We then get a ruled real hypersurface $M = \bigcup_s M_{\gamma(s)}$.

We treat a ruled real hypersurface locally, because generally this hypersurface has self-intersections and singularities. We set differentiable functions μ, ν on a ruled real hypersurface M associated with its shape operator A by $\mu = \langle A\xi, \xi\rangle$ and $\nu = \|A\xi - \mu\xi\|$. Then on an open dense subset $M_1 = \{x \in M \mid \nu(x) \neq 0\}$ of M the shape operator A of M satisfies the following equalities with a unit vector field U orthogonal to ξ:

$$A\xi = \mu\xi + \nu U, \quad AU = \nu\xi, \quad AX = 0$$

for an arbitrary tangent vector X orthogonal to ξ and U. When we study ruled real hypersurfaces, we usually omit points where ξ is principal and suppose that ν does not vanish everywhere, namely a ruled hypersurface M is usually supposed to satisfy $M_1 = M$. Here we consider the vector field ϕU (see [4, (18) and (19)]). By direct computation we find that on each integral curve of the vector field ϕU the function ν satisfies the ordinary differential equation $\phi U\nu = \nu^2 + 1$, so that $\nu(s) = \tan s$. On the other hand, we see that $\nabla_{\phi U}\phi U = 0$. These imply that every geodesic γ with $\dot{\gamma}(0) = \phi U$ is defined only in the open interval $(-\pi/2, \pi/2)$. Therefore, we conclude that every ruled real hypersurface of $\mathbb{C}P^n$ is *not* complete, so that in particular they are not homogeneous real hypersurfaces. That is, every ruled real hypersurface of $\mathbb{C}P^n$ is not an orbit of some subgroup of the projective unitary group $PU(n+1)$. On the contrary, both hypersurfaces of type A_1 and of type A_2 are compact homogeneous real hypersurfaces of $\mathbb{C}P^n$. We should note that at an arbitrary point of each homogeneous real hypersurface in $\mathbb{C}P^n$ there exists an orthonormal frame $\{v_1, \ldots, v_{2n-1}\}$ such that the geodesic γ_i with initial vector v_i has constant structure torsion for $1 \leqq i \leqq 2n-1$.

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