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General Relativistic Electrodynamics and Pulsar Theory

L. Mestel

Astronomy Centre, University of Sussex, Falmer, Brighton, BN1 9QH, England

Abstract. Muslimov & Tsygan (1986, 1990, 1991, 1992) have shown how the dragging of inertial frames modifies the Goldreich-Julian charge density. The consequences are noted for different models of the electron outflow near the neutron star's surface.

Space-time outside a rotating neutron star is described by the Kerr metric. When the ratio $J/Mcr_g \ll 1$, where J is the angular momentum of the star and $r_g = 2GM/c^2$ = the Schwarzschild radius, the metric can be approximated by the Schwarzschild metric plus the nondiagonal terms that yield the dragging of inertial frames (the Lense-Thirring effect). Macdonald & Thorne (1982) use the set of fundamental ZAMOs (zero-angular momentum observers), each rotating with respect to the inertial frame at infinity with the local angular velocity

$$\omega \approx \frac{2G\mathbf{J}}{c^2 r^3} = \frac{2GM\mathbf{\Omega} R^2 j}{c^2 r^3} = j\left(\frac{r_g}{r}\right)\left(\frac{R^2}{r^2}\right)\mathbf{\Omega},$$

where j is a number ≈ 0.4 for more-or-less uniformly dense stars. The effects of general relativity (g-r) are contained in two functions: $\omega = (jr_g R^2/r^3)\Omega$, and the red shift $\alpha = (1 - r_g/r)^{1/2}$, relating time by a local clock with time by the Schwarzschild clock at infinity.

Maxwell's equations for $\mathbf{E}$, $\mathbf{B}$, as measured by ZAMO observers, reduce in an axisymmetric steady state to

$$\nabla \cdot \mathbf{B} = 0, \qquad \nabla \times \alpha\mathbf{B} = \frac{4\pi}{c}\alpha\mathbf{j}, \tag{1), (2}$$

$$\rho_e = \frac{\nabla \cdot \mathbf{E}}{4\pi}, \qquad \nabla \times \alpha\mathbf{E} = -\frac{1}{c}(\mathbf{B}\cdot\nabla\omega)(\varpi\mathbf{t}), \tag{3), (4}$$

where ρ_e = actual charge density, $\mathbf{t}$ is the unit toroidal vector $\mathbf{e}_\phi$, and ϖ is the distance from the axis. The operator ∇ is that appropriate to the Schwarzschild spatial metric

$$dl^2 = \frac{dr^2}{(1 - r_g/r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

Since magnetospheric currents are weak near the star, $\mathbf{B}$ is given by the solution of (1) and (2) for $\mathbf{j} = 0$. With a dipolar angular structure assumed, then (e.g. Muslimov & Tsygan 1992, hereafter MT)

$$B_r = B_0\frac{f(\eta)}{f(1)}\frac{1}{\eta^3}\cos\theta, \qquad B_\theta = \frac{1}{2}\frac{B_0\alpha\sin\theta}{\eta^3}\left[-\frac{2f(\eta)}{f(1)} + \frac{3}{(1-\varepsilon/\eta)f(1)}\right], \tag{5}$$

where $\eta = \frac{r}{R}$, $\varepsilon = r_g/R \approx 0.3\,M_\odot/R_6$, $\alpha = (1 - r_g/r)^{1/2}$,

$$f(\eta) = -3\left(\frac{\eta}{\varepsilon}\right)^3 \left[\ln\left(1 - \frac{\varepsilon}{\eta}\right) + \frac{\varepsilon}{\eta}\left(1 + \frac{\varepsilon}{2\eta}\right)\right] \approx 1 + \frac{3}{4}\frac{\varepsilon}{\eta} + \cdots, \tag{6}$$

and the approximation applies for small ε/η.

Eqn (4) yields the generalization of the familiar pulsar equation:

$$\alpha \mathbf{E} = -\frac{1}{c}\left[(\mathbf{\Omega} - \omega) \times \mathbf{r}\right] \times \mathbf{B} - \nabla\psi, \tag{7}$$

— i.e. the "corotational field" $-(\mathbf{\Omega} \times \mathbf{r}) \times \mathbf{B}/c$ is now replaced by $-[(\mathbf{\Omega} - \omega) \times \mathbf{r}] \times \mathbf{B}/c$, so that the angular velocity entering is measured relative to the local frame rotating with ω.

The "non-corotational" potential ψ satisfies (from (3) & (7))

$$\nabla \cdot \left(\frac{\nabla\psi}{\alpha}\right) = -4\pi(\rho_e - \rho_{GJ}), \tag{8}$$

where ρ_{GJ} (called by MT "ρ_{eff}" — not to be confused with our ρ_e) is

$$\rho_{GJ} = -\frac{1}{4\pi c}\nabla \cdot \left[\frac{1}{\alpha}(\Omega - \omega) r \sin\theta\, \mathbf{t} \times \mathbf{B}\right]. \tag{9}$$

With (5) substituted, in the pulsar polar caps this reduces to

$$\rho_{GJ} = -\frac{\Omega B_0}{2\pi c}\frac{1}{\alpha\eta^3}\frac{f(\eta)}{f(1)}\left(1 - \frac{\kappa}{\eta^3}\right)\cos\theta, \tag{10}$$

where $\kappa = \dfrac{2GJ}{\Omega R^3 c^2} = j\left(\dfrac{r_g}{R}\right) \approx 0.4\varepsilon \approx 0.12\left(\dfrac{M_\odot}{R_6}\right)$.

In steady outflow of single sign charges along field lines, $\mathbf{v} = k\mathbf{B}$, and

$$\frac{\alpha\rho_e v}{B} = \text{constant on field-streamlines.} \tag{11}$$

(Note the factor α: from (2),

$$0 = \nabla \cdot (\alpha \mathbf{j}) = \nabla \cdot (\alpha\rho_e \mathbf{v}) = \nabla \cdot (\alpha\rho_e k\mathbf{B}) = \mathbf{B} \cdot \nabla(\alpha\rho_e k),$$

which implies that $\alpha\rho_e k = \alpha\rho_e v/B$ is constant on streamlines.) Now write this in terms of (ρ_e/ρ_{GJ}): from (10) and (5)

$$\alpha\frac{\rho_e}{\rho_{GJ}}\frac{\rho_{GJ} v}{B} = \frac{\alpha\left(\dfrac{\rho_e}{\rho_{GJ}}\right)\left(\dfrac{v}{c}\right) c\left[-\dfrac{\Omega B_0}{2\pi c}\dfrac{1}{\alpha\eta^3}\dfrac{f(\eta)}{f(1)}\left(1 - \dfrac{\kappa}{\eta^3}\right)\cos\theta\right]}{B_0\dfrac{f(\eta)}{f(1)}\dfrac{1}{\eta^3}\cos\theta} \tag{12}$$

$$= \left(\frac{\rho_e}{\rho_{GJ}}\right)\left(\frac{v}{c}\right)\left(\frac{-\Omega}{2\pi}\right)\left(1 - \frac{\kappa}{\eta^3}\right) = \text{constant on streamlines,} \tag{13}$$

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where the difference between B and the component B_z along $\mathbf{\Omega}$ is for the moment ignored.

One can now ask: can one have a "GJ flow" — a non-relativistic outflow of gas with the GJ density, with $\psi \approx 0$, satisfying continuity and not violating special relativity?

Put $\rho_e/\rho_{GJ} = 1$ in (13): this requires

$$v/c \propto 1/(1-\kappa/\eta^3). \tag{14}$$

So with this assumption ($\rho_e = \rho_{GJ}$), the g-r correction allows such flow with v/c decreasing outwards initially. Further out, the usual effects of field line curvature (one way or the other) need to be included.

By contrast MT follow the approach of Jon Arons and colleagues (Fawley et al. 1977; Scharlemann et al. 1978): they assume that v stays equal to c. (They do not discuss the initial acceleration by a super-GJ density from $v < c$ to $v \approx c$, an essential feature of the Arons et al. solution). The continuity condition (11) then requires $\alpha\rho_e \propto B$ on a streamline. They find

$$\rho_e = -\frac{\Omega B_0}{2\pi c}\frac{1}{\alpha\eta^3}\frac{f(\eta)}{f(1)}(A(\xi)), \tag{15}$$

where ξ is a normalized θ-coordinate, constant on field lines, and $A(\xi)$ is a function determined by the solution which approaches $1-\xi^2$ asymptotically.

The boundary conditions are $\psi = 0$ on the star and on the sides of the flow domain (bounded by the domain of field lines that close within the l-c), and $\mathbf{E}\cdot\mathbf{B} = 0$ on the star and also at "infinity". As with the Arons et al. solution, it is this outer boundary condition — requiring screening of the aligned electric field *before* pair production occurs — that fixes the value of the current density $j = \rho_e c$.

As pointed out by J. Kuijpers (privated communication 1996), even though MT bypassed the initial sub-relativistic flow, where the density must be super-GJ, in their solution there remains a slight super-GJ density near $\eta = 1$;

$$\rho_e \approx -\frac{\Omega B_0}{2\pi c}\frac{1}{\alpha\eta^3}\frac{f(\eta)}{f(1)}[(1-\kappa) + \text{small positive term}] \tag{16}$$

which is close to but slightly greater in modulus than ρ_{GJ}. Further out, their solution is "starved", like that of Arons et al.:

$$\frac{\rho_e}{\rho_{GJ}} = (1-\kappa)\left(\frac{\rho_e}{\rho_{GJ}}\right)_{\text{star}} < 1. \tag{17}$$

The parallel electric field they construct has the value

$$E \sim \left(\frac{\Omega R}{c}\right)^2 B_0 \left(\frac{GM}{Rc^2}\right) \tag{18}$$

at a height $\approx$ the polar cap width. As noted, the value of the current is fixed by the far boundary condition. A larger current would yield $|E_{\|}|$ increasing outwards; a smaller current would yield a quasi-GJ system, with $v < c$ (no

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relativistic acceleration), and with the spatial oscillations found by Mestel & Pryce (1985), Shibata (1991) and Mestel & Shibata (1995).

The curvature of the field lines does not come into the MT discussion. It will enter further out if the flow does in fact opt for the sub-relativistic G-J flow, as given by (14). Again, spontaneous acceleration must occur along "away" field lines, as in Mestel & Shibata (1995), and screening of the electric field must then be effected by pair production followed by relative shift of the positive and negative electrons.

Shinpei Shibata has done some calculations which illustrate these conclusions. He solves the equations for ψ and so for γ, ignoring both field line curvature and the "3-dimensional effect" due to the finite width of the beam, but starting (like Arons *et al.*) with a non-relativistic flow and super-GJ density near the star. The equation for γ is

$$-\frac{d^2\gamma}{dl^2} = -\frac{j}{\sqrt{1-1/\gamma^2}} + j_0. \tag{19}$$

where l is length along a field line scaled in terms of a plasma wave-length. In the case without g-r, $j_0 = 1$; with g-r, $j_0 = (1-\kappa/\eta^3)$, where $\eta = 1$ is again the star's surface and $\kappa \approx 0.12$. The Arons-type solution is again *monotonic* in γ and ψ and satisfies $E_{||} = 0$ both at the star *and at infinity.* This last condition requires that j have the critical value $j_0(\infty)$. As found by MT, by reducing somewhat the GJ term j_0 near the star, the g-r correction yields significantly greater γ-values. There is also a case, with the current below this critical value, which exhibits the predicted oscillations superposed on GJ flow; the effect of the g-r correction is then quite modest. Which flow the system adopts will be determined as part of the global solution for the whole magnetosphere, including the effects of pair production (cf. Arons, this volume).

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