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ON THE FUNDAMENTAL THEOREM OF CARD COUNTING, WITH APPLICATION TO THE GAME OF TRENTE ET QUARANTE

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Abstract

A simplified proof of Thorp and Walden's fundamental theorem of card counting is presented, and a corresponding central limit theorem is established. Results are applied to the casino game of trente et quarante, which was studied by Poisson and De Morgan.

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1. Introduction

Thorp and Walden (1973) proved, assuming a fixed player strategy, that 'the "spread" in the distribution of player expectations for partially depleted card packs increases with the depletion of the card pack,' and they termed this result the fundamental theorem of card counting. Their proof relies on the theory of convex contractions of measures and some combinatorial analysis. Our aim here is to provide a simpler proof, one that depends only on exchangeability and Jensen's inequality. This simplification not only makes the theorem easier to understand, but also allows us to investigate the case of a variable player strategy. It also leads to a central limit theorem in the fixed-strategy case, which in turn permits an analysis of the card-counting potential of the casino game of trente et quarante, without the need for Monte Carlo simulation.

We consider a deck of N distinct cards, which for convenience will be assumed to be labeled $1, 2, \dots, N$. We also label the positions of the cards in the deck as follows. With the cards face down, the top card is in position 1, the second card is in position 2, and so on. Thus, the first card dealt is the card in position 1. Let S_N be the symmetric group of permutations of $\{1, 2, \dots, N\}$, and let Π be a uniformly distributed S_N-valued random variable (i.e. all $N!$ possible values are equally likely). We think of $\Pi(i) = j$ as meaning that the card in position i is moved to position j by the permutation Π. If the cards are in the natural order $(1, 2, \dots, N)$ initially, their order after Π is applied is $(\Pi^{-1}(1), \dots, \Pi^{-1}(N))$. We define $X_j := \Pi^{-1}(j)$ for $j = 1, \dots, N$, so that X_j is the label of the card in position j and $(X_1, \dots, X_N)$ is an exchangeable sequence.

It will be convenient to let $\mathcal{F}_n := \sigma(X_1, \dots, X_n)$ for $n = 1, \dots, N$, and to let $\mathcal{F}_0$ be the trivial σ-field. Here $\sigma(X_1, \dots, X_n)$ denotes the smallest σ-field with respect to which $X_1, \dots, X_n$ are measurable.

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An instructive example, mentioned by Thorp and Walden and studied by Griffin (1999, Chapter 4) in connection with card-counting systems, is provided by the following simple game.

Example 1. Assume that N is even and that the player is allowed to bet, at even money, that the next card dealt is odd. If the first n cards have been seen ($0 \le n \le N-1$) and the player bets on the next one, his profit per unit bet is

$$Y_n := 2\,\mathbf{1}_{\{X_{n+1} \text{ is odd}\}} - 1, \tag{1}$$

where $\mathbf{1}_A$ denotes the indicator of the event A, so his conditional expected profit per unit bet is

$$Z_n := \mathrm{E}[Y_n \mid \mathcal{F}_n] = \frac{2}{N-n}\biggl(\tfrac{1}{2}N - \sum_{i=1}^{n} \mathbf{1}_{\{X_i \text{ is odd}\}}\biggr) - 1, \tag{2}$$

being twice the proportion of odd cards in the unseen deck, less 1. (Here and elsewhere, empty sums are 0, by convention.) Observe that we can rewrite this as

$$Z_n = \frac{1}{N-n}\sum_{i=1}^{n}(1 - 2\,\mathbf{1}_{\{X_i \text{ is odd}\}}) = \frac{1}{N-n}\sum_{i=1}^{n}(-1)^{X_i}. \tag{3}$$

The latter formula has practical implications. Suppose that the player assigns to each odd card seen the point value -1, and to each even card seen the point value 1. The *running count* is the sum of these point values over all cards seen and is adjusted each time a new card is seen. The *true count* is the running count divided by the number of unseen cards. Equation (3) says that the true count provides the player with his exact expected profit per unit bet on the next card. This information can be used to select a suitable bet size.

Note that $\mathrm{E}[Z_n] = 0$ for $0 \le n \le N-1$. More importantly, using the fact that

$$\operatorname{cov}(\mathbf{1}_{\{X_i \text{ is odd}\}}, \mathbf{1}_{\{X_j \text{ is odd}\}}) = \frac{\frac{1}{2}N(\frac{1}{2}N-1)}{N(N-1)} - \frac{1}{4} = -\frac{1}{4(N-1)}$$

if $i \ne j$, we calculate from (2) that

$$\begin{aligned}\operatorname{var}(Z_n) &= \frac{4}{(N-n)^2}\operatorname{var}\biggl(\sum_{i=1}^{n}\mathbf{1}_{\{X_i \text{ is odd}\}}\biggr)\\ &= \frac{4}{(N-n)^2}\biggl(\frac{n}{4} - \frac{n(n-1)}{4(N-1)}\biggr)\\ &= \frac{n}{(N-n)(N-1)},\end{aligned} \tag{4}$$

which increases from 0 to 1 as n increases from 0 to $N-1$. The conclusions that $\mathrm{E}[Z_n]$ is constant in n, while $\operatorname{var}(Z_n)$ is increasing in n, are typical of games with a fixed strategy, as we will see in Section 2.

To indicate how a central limit theorem might be useful in this context, suppose that we want to find the probability that the player has an advantage greater than β. Using (4) and a normal approximation with a continuity correction (based on the central limit theorem for samples from a finite population), we find that if n is even, then $\sum_{i=1}^{n}(-1)^{X_i}$ is also even, so

$$\mathrm{P}\{Z_n > \beta\} \approx 1 - \Phi\biggl(\frac{2}{N-n}\{\lfloor\tfrac{1}{2}(N-n)\beta\rfloor + \tfrac{1}{2}\}\sqrt{\frac{(N-n)(N-1)}{n}}\biggr), \tag{5}$$

where Φ is the standard normal distribution function and $\lfloor x \rfloor$ denotes the integer part of x. For example, assuming that one-quarter of the cards in a 312-card deck have been seen (i.e. $N = 312$, $n = \frac{1}{4}N$), the probability that the player has an advantage greater than 1.25% (i.e. $\beta = 0.0125$) is approximately $1 - \Phi(0.391\,603) = 0.347\,676$. Here, the exact probability can be calculated directly from the hypergeometric distribution, and it is 0.347 522.

In Section 3, we establish a central limit theorem in the more general setting described in Section 2, with an eye toward trente et quarante, a game in which $N = 312$.

Example 2. To illustrate the effect of a variable strategy, we continue to assume that N is even, but let us now suppose that the player is allowed to make either of two even-money bets; one that the next card dealt is odd, and the other that the next card dealt is even. An obvious optimal strategy is to bet on odd if the number of odd cards seen is less than or equal to the number of even cards seen (or, equivalently, if the true count, defined in Example 1, is nonnegative), and to bet on even otherwise. If the first n cards have been seen ($0 \le n \le N - 1$) and the player employs this strategy to bet on the next card, his conditional expected profit per unit bet is

$$Z_n := \frac{2}{N-n}\left\{\tfrac{1}{2}N - \min\left(\sum_{i=1}^{n}\mathbf{1}_{\{X_i \text{ is odd}\}}, n - \sum_{i=1}^{n}\mathbf{1}_{\{X_i \text{ is odd}\}}\right)\right\} - 1 \tag{6}$$

instead of (2), this being twice the proportion of odd cards or of even cards in the unseen deck (whichever is greater), less 1. As before, we can rewrite this in the form

$$Z_n = \frac{1}{N-n}\left|\sum_{i=1}^{n}(-1)^{X_i}\right|.$$

The interpretation is as in Example 1: the player bets on odd if the true count is nonnegative, and on even otherwise. In either case, the absolute value of the true count provides the player with his exact expected profit per unit bet on the next card.

It follows from (6) that

$$\mathrm{E}[Z_n] = \frac{2}{N-n}\left(\tfrac{1}{2}N - \sum_{k=0}^{n}\min(k, n-k)\binom{N}{n}^{-1}\binom{\frac{1}{2}N}{k}\binom{\frac{1}{2}N}{n-k}\right) - 1. \tag{7}$$

Although it is not clear how to write this in closed form, it is clear that $\mathrm{E}[Z_n]$ is no longer constant in n. One way to see this is to note that $Z_0 = 0$, $Z_1 = 1/(N-1)$, and $Z_{N-1} = 1$. These three cases, and only these, have $\operatorname{var}(Z_n) = 0$. This also shows that $\operatorname{var}(Z_n)$, which could also be expressed in a way similar to (7), is no longer increasing (or even nondecreasing) in n. Thus, the conclusions that hold for a fixed strategy may fail for a variable strategy. Nevertheless, it is possible to obtain a weaker form of the fundamental theorem in this setting, and we do this in Section 4.

It should be mentioned that Jostein Lillestol (see Székely (2003)) found the surprisingly simple formula

$$\mathrm{E}[Z_0 + Z_1 + \cdots + Z_{N-1}] = \sum_{n=1}^{N/2}\binom{N}{2n}^{-1}\binom{\frac{1}{2}N}{n}\binom{\frac{1}{2}N}{n}.$$

The analogue of (5) is

$$\mathrm{P}\{Z_n > \beta\} \approx 2\left[1 - \Phi\left(\frac{2}{N-n}\{\lfloor\tfrac{1}{2}(N-n)\beta\rfloor + \tfrac{1}{2}\}\sqrt{\frac{(N-n)(N-1)}{n}}\right)\right] \tag{8}$$

for n even. In the special case of $N = 312$, $n = \frac{1}{4}N$, and $\beta = 0.0125$, this becomes $2(1 - \Phi(0.391\,603)) = 0.695\,352$. The exact probability is 0.695 044.

Example 3. Example 2 is rather special, in that it permits betting on opposite sides of the same proposition. Here, we provide a generalization that is perhaps more typical. Fix a positive integer K, assume that N is divisible by $2K$, and define $A := \{1, 3, \dots, 2K - 1\}$ (the odd positive integers less than $2K$). Let B be a subset of $\{0, 1, 2, \dots, 2K - 1\}$ of cardinality $|B| = K$ and with $B \neq A$, and define $L := |A \cap B|$, so that $0 \leq L \leq K - 1$.

Let us now suppose that the player is allowed to make either of two even-money bets; one that the next card dealt is odd (or, equivalently, is congruent mod $2K$ to an element of A), and the other that the next card dealt is congruent mod $2K$ to an element of B. If the first n cards have been seen ($0 \leq n \leq N - 1$) and the player employs an obvious optimal strategy to bet on the next card (i.e. bet on odd unless the other bet is more favorable), his conditional expected profit per unit bet is $Z_n := \max(Z_n^A, Z_n^B)$, where

$$Z_n^A := \frac{1}{N-n} \sum_{i=1}^{n} (1 - 2\,\mathbf{1}_{\{X_i \,(\mathrm{mod}\; 2K) \in A\}})$$

and

$$Z_n^B := \frac{1}{N-n} \sum_{i=1}^{n} (1 - 2\,\mathbf{1}_{\{X_i \,(\mathrm{mod}\; 2K) \in B\}}).$$

A straightforward calculation, which we omit, shows that

$$\rho := \operatorname{corr}(Z_n^A, Z_n^B) = 2\frac{L}{K} - 1.$$

Example 2 is the special case of $K = 1$ and $L = 0$, in which $Z_n^B = -Z_n^A$ and $Z_n = |Z_n^A|$. If $L \neq 0$, the situation is more complicated, and the card counter must keep two counts (or what is called a two-parameter count) to track his conditional expected profit per unit bet. A bivariate central limit theorem is available (via the Cramér–Wold device), and one can approximate $\mathrm{P}\{Z_n > \beta\}$ using the facts that, for $(V_1, V_2) \sim N(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$ (bivariate normal),

$$\mathrm{P}\{\max(V_1, V_2) > v\} = 1 - \mathrm{P}\{V_1 \leq v, V_2 \leq v\}, \tag{9}$$

and the joint distribution function of (V_1, V_2) is available (in MATHEMATICA®).

For example, suppose that $K = 2$ and $L = 1$. Then $\rho = 0$, so V_1 and V_2 in (9) are independent. Therefore the analogue of (5) and (8) is

$$\mathrm{P}\{Z_n > \beta\} \approx 1 - \Phi\bigg(\frac{2}{N-n}\{\lfloor \tfrac{1}{2}(N-n)\beta \rfloor + \tfrac{1}{2}\}\sqrt{\frac{(N-n)(N-1)}{n}}\bigg)^2 \tag{10}$$

for n even. In the special case of $N = 312$, $n = \frac{1}{4}N$, and $\beta = 0.0125$, this becomes $1 - \Phi(0.391\,603)^2 = 0.574\,473$. The exact probability, from the multivariate hypergeometric distribution, is 0.574 307.

2. The case of a fixed strategy

Consider a game that requires up to m cards to complete a round. (The deck is reshuffled if fewer than m cards remain.) Let us assume also that the player employs a fixed strategy, i.e.

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one that does not depend on the cards already seen. We let $X_1, \dots, X_N$ and $\mathcal{F}_0, \dots, \mathcal{F}_N$ be as in Section 1. If the first n cards have been seen $(0 \le n \le N - m)$ and the player bets on the next round, his profit per unit bet has the form

$$Y_n := f(X_{n+1}, \dots, X_{n+m}) \tag{11}$$

for a suitable nonrandom function f (thereby generalizing (1)), so his conditional expected profit per unit bet is

$$Z_n := \mathrm{E}[Y_n \mid \mathcal{F}_n] = \mathrm{E}[f(X_{n+1}, \dots, X_{n+m}) \mid \mathcal{F}_n]. \tag{12}$$

Our version of the fundamental theorem of card counting can be stated as follows.

Theorem 1. *Under the above assumptions,* $\{Z_n, \mathcal{F}_n, n = 0, \dots, N - m\}$ *is a martingale. In particular,*

$$\mathrm{E}[Z_0] = \cdots = \mathrm{E}[Z_{N-m}] \tag{13}$$

and

$$0 = \mathrm{var}(Z_0) \le \cdots \le \mathrm{var}(Z_{N-m}). \tag{14}$$

Let I *be an interval such that* $\mathrm{P}\{Z_0 \in I, \dots, Z_{N-m} \in I\} = 1$. *If* $\varphi\colon I \to \mathbb{R}$ *is convex then* $\{\varphi(Z_n), \mathcal{F}_n, n = 0, \dots, N - m\}$ *is a submartingale. In particular,*

$$\mathrm{E}[\varphi(Z_0)] \le \cdots \le \mathrm{E}[\varphi(Z_{N-m})]. \tag{15}$$

Finally, for each inequality in (14), the inequality is strict unless both sides are 0*. If* φ *is strictly convex then, for each inequality in (15), the inequality is strict unless both sides of the corresponding inequality in (14) are* 0*.*

Remark 1. This does not imply the fundamental theorem of Thorp and Walden (1973), because their theorem was formulated in terms of convex contractions of measures. However, it does imply the informal statement of their theorem, quoted in the opening sentence of this paper. Thorp and Walden emphasized the case of (15) in which $\varphi(u) := |u - u_0|^\alpha$, where u_0 is arbitrary and $\alpha \ge 1$. They did not explicitly mention (13), (14), or the martingale property, but (13) and (14) are implicit in their work. Specifically, they pointed out that convex contractions are mean preserving, although there is a seemingly contradictory statement in their paper, namely that 'average player expectation is non-decreasing (even increasing under suitable hypotheses) with increasing depletion.' A similar statement appears in Griffin (1976), but it refers not to $\mathrm{E}[Z_n]$ but to $\mathrm{E}[Z_n \mathbf{1}_{\{Z_n > 0\}}]$, and it is likely that this is what Thorp and Walden had in mind.

It should be noted that the martingale $\{Z_n, \mathcal{F}_n\}$ differs from the usual stochastic model of a fair game. For example, the player who bets $B_{n+1} := \mathbf{1}_{\{Z_n > 0\}}$ at trial $n + 1$ can enjoy a considerable advantage over the house.

Proof of Theorem 1. First, the martingale property is a consequence of the fact that

$$Z_n = \mathrm{E}[Y_{N-m} \mid \mathcal{F}_n], \qquad n = 0, 1, \dots, N - m,$$

which holds by virtue of the exchangeability of $X_1, \dots, X_N$. From this follow (13), the submartingale property of $\{\varphi(Z_n), \mathcal{F}_n\}$, and (15). Taking $\varphi(u) := u^2$ in (15) and using (13) implies (14).

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Now let us assume that φ is strictly convex. Fix $n \in \{0, \dots, N-m-1\}$ and suppose that $\mathrm{E}[\varphi(Z_n)] = \mathrm{E}[\varphi(Z_{n+1})]$. Then

$$\mathrm{E}[\varphi(\mathrm{E}[Z_{n+1} \mid \mathcal{F}_n])] = \mathrm{E}[\mathrm{E}[\varphi(Z_{n+1}) \mid \mathcal{F}_n]],$$

so, by the condition for equality in Jensen's inequality, the conditional distribution of Z_{n+1} given $\mathcal{F}_n$ is degenerate. By the definition of conditional expectation, there exists a nonrandom function h_{n+1} such that $Z_{n+1} = h_{n+1}(X_1, \dots, X_{n+1})$. Furthermore, h_{n+1} is a symmetric function of its variables. Since the conditional distribution of $h_{n+1}(X_1, \dots, X_{n+1})$ given $\mathcal{F}_n$ is degenerate, the symmetry of h_{n+1} implies that Z_{n+1} is constant and, hence, that its variance is 0. The stated conclusions follow.

3. A central limit theorem

Continuing under the assumptions of Section 2, we can rewrite (12) as

$$\begin{aligned}
Z_n &= \mathrm{E}[f(X_{n+1}, \dots, X_{n+m}) \mid \mathcal{F}_n] \\
&= \frac{1}{(N-n)_m} \sum_{j_1, \dots, j_m \text{ distinct in } \{n+1, \dots, N\}} \mathrm{E}[f(X_{j_1}, \dots, X_{j_m}) \mid \mathcal{F}_n] \\
&= \mathrm{E}\biggl[\frac{1}{(N-n)_m} \sum_{j_1, \dots, j_m \text{ distinct in } \{n+1, \dots, N\}} f(X_{j_1}, \dots, X_{j_m}) \biggm| \mathcal{F}_n\biggr] \\
&= \frac{1}{(N-n)_m} \sum_{j_1, \dots, j_m \text{ distinct in } \{n+1, \dots, N\}} f(X_{j_1}, \dots, X_{j_m}) \\
&= \binom{N-n}{m}^{-1} \sum_{n+1 \le j_1 < \cdots < j_m \le N} f^*(X_{j_1}, \dots, X_{j_m}),
\end{aligned} \tag{16}$$

where $(N-n)_m = (N-n) \cdots (N-n-m+1)$ and f^* is the symmetrized version of f:

$$f^*(i_1, \dots, i_m) := \frac{1}{m!} \sum_{\pi \in S_m} f(\pi(i_1), \dots, \pi(i_m)).$$

The second equality in (16) uses exchangeability, while the fourth uses the fact that the unordered set of random variables $\{X_{n+1}, \dots, X_N\}$ (the unseen deck) is $\mathcal{F}_n$-measurable and, therefore, that any symmetric function of $X_{n+1}, \dots, X_N$ is $\mathcal{F}_n$-measurable.

Thus, Z_n is a U-statistic with symmetric kernel f^* of degree m, based on a sample of size $N-n$ (namely $X_{n+1}, \dots, X_N$) taken, without replacement, from the finite population $\{1, 2, \dots, N\}$. A central limit theorem in this setting was first proved by Nandi and Sen (1963); we follow Lee (1990). A related invariance principle is also known, but is not needed here. In the remainder of this section, $X_1, \dots, X_N$ and $Z_0, \dots, Z_{N-m}$ all depend on N, but we do not make this explicit in the notation. We assume that the nonrandom functions f and f^* do not depend on N.

Let us define

$$f_1^*(i) := \mathrm{E}[f^*(X_1, \dots, X_m) \mid X_1 = i], \qquad i = 1, \dots, N,$$

and

$$\overline{\sigma}_{1,N}^2 := \mathrm{var}(f_1^*(X_1)).$$

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Theorem 2. *Assume that*

$$\lim_{N\to\infty} N^{-1/2} \max_{1\le i\le N} |f_1^*(i) - \mathrm{E}[f^*(X_1,\ldots,X_m)]| = 0 \tag{17}$$

and

$$\lim_{N\to\infty} \overline{\sigma}_{1,N}^2 =: \sigma^2 > 0.$$

Then, as $N \to \infty$ *and* $n \to \infty$ *with* $n/N \to \alpha \in (0,1)$,

$$N^{1/2}(Z_n - \mathrm{E}[Z_0]) \xrightarrow{\mathrm{D}} N(0, m^2\sigma^2\alpha/(1-\alpha)), \tag{18}$$

where '$\xrightarrow{\mathrm{D}}$' *denotes convergence in distribution.*

Remark 2. In card-counting applications, f (and therefore f^*) is bounded, so (17) is automatically true. Notice that the variance of the limit in (18) is increasing in α, as Theorem 1 suggests it should be.

Proof of Theorem 2. By assumption, $(N-n)/N \to 1-\alpha$, so Theorem 1 of Section 3.7.4 of Lee (1990) tells us that $(N-n)^{1/2}(Z_n - \mathrm{E}[Z_0]) \xrightarrow{\mathrm{D}} N(0, m^2\sigma^2\alpha)$, and this is equivalent to (18).

It is known (Lee (1990, p. 64)) that

$$\mathrm{var}(Z_n) = \frac{m^2 n}{(N-n)(N-1)}\overline{\sigma}_{1,N}^2 + o(N^{-1}), \tag{19}$$

at least if f^* is bounded and $N, n \to \infty$ with $n/N \to \alpha \in (0,1)$. (Note that the error term is 0 in the case of Example 1.) Although exact formulae for $\mathrm{var}(Z_n)$ are available (Lee (1990, p. 64)), they may be difficult to evaluate in practice. Moreover, the approximation suggested by (19), which is implicit in (18), may introduce significant bias in any normal approximation based on Theorem 2.

It must be kept in mind that the main purpose of card counting is to identify favorable situations and vary bet size accordingly. A statistic simpler than Z_n may suffice for this purpose. Let us define

$$e(j) := \mathrm{E}[f(X_2,\ldots,X_{m+1}) \mid X_1 = j] - \mathrm{E}[f(X_1,\ldots,X_m)], \qquad j = 1,\ldots,N.$$

These numbers are the so-called *effects of removal*. They arise from the following hypothesis. Let D be the set of cards $\{1,2,\ldots,N\}$, and assume that to each card $i \in D$ there is associated a number $c(i)$ such that, with U denoting the set of unseen cards, the player's expected profit E_U per unit bet on the next round is given by

$$E_U = \frac{1}{|U|}\sum_{i\in U} c(i). \tag{20}$$

Letting $\mu := E_D$ be the full-deck expectation, we find that

$$\begin{aligned} c(j) &= \sum_{i\in D} c(i) - \sum_{i\in D-\{j\}} c(i) \\ &= N\mu - (N-1)E_{D-\{j\}} \\ &= \mu - (N-1)(E_{D-\{j\}} - \mu) \\ &= \mu - (N-1)e(j), \qquad j = 1,\ldots,N. \end{aligned} \tag{21}$$

Since $\sum_{j=1}^{N} e(j) = 0$, it follows that, when $X_1, \ldots, X_n$ have been seen, the player's conditional expected profit per unit bet on the next round is

$$\tilde{Z}_n := \frac{1}{N-n} \sum_{j=n+1}^{N} \{\mu - (N-1)e(X_j)\} = \mu + \frac{1}{N-n} \sum_{j=1}^{n} (N-1)e(X_j). \tag{22}$$

Notice that this generalizes (3), and the interpretation is similar.

However, we emphasize that the derivation of (22) is based on hypothesis (20), which is really only an approximation. (Example 1 is unusual in that, there, the approximation is exact.) To justify the approximation, Griffin (1999, appendix to Chapter 3) showed that the quantities $\mu - (N-1)e(j)$ in (21) are the least-squares estimators of the parameters $c(j)$ in the linear model

$$E_U = \frac{1}{|U|} \sum_{i \in U} c(i) + \varepsilon_U, \qquad U \subset D,\ |U| = N-n,$$

for fixed $1 \leq n \leq N-m$. (Here ε_U is the usual error term in the linear model.) As Griffin put it, 'here we appeal to the method of least squares not to estimate what is assumed to be linear, but to best approximate what is almost certainly not quite so.'

The next theorem provides an asymptotic justification of the approximation.

Theorem 3. *Under the assumptions of Theorem 2, as $N \to \infty$ and $n \to \infty$ with $n/N \to \alpha \in (0,1)$, $N^{1/2}(Z_n - \mathrm{E}[Z_0])$ and $N^{1/2}(\tilde{Z}_n - \mu)$ are asymptotically equivalent in the sense that*

$$N\,\mathrm{E}[(Z_n - \mathrm{E}[Z_0] - (\tilde{Z}_n - \mu))^2] \to 0. \tag{23}$$

In particular,

$$N^{1/2}(\tilde{Z}_n - \mu) \xrightarrow{\mathrm{D}} N(0, m^2\sigma^2\alpha/(1-\alpha)). \tag{24}$$

More precisely, letting $\sigma_{e,N}^2 := \mathrm{var}(e(X_1))$, we have

$$\frac{\tilde{Z}_n - \mu}{\sigma_{e,N}\sqrt{n(N-1)/(N-n)}} \xrightarrow{\mathrm{D}} N(0,1). \tag{25}$$

Remark 3. The principal result is (23); (25) could be obtained directly from the central limit theorem for samples from a finite population. The reason for saying 'more precisely' is that the left-hand side of (25) has variance 1 – not just asymptotic variance 1 – and, therefore, a normal approximation based on (25) is likely to be more accurate than one based on (24).

Proof of Theorem 3. Lee's (1990, Section 3.7.4) proof of the central limit theorem for U-statistics based on samples from a finite population includes the result that, in our notation,

$$(N-n)\,\mathrm{E}\Bigg[\Bigg(Z_n - \mathrm{E}[Z_0] - \frac{m}{N-n} \sum_{j=n+1}^{N} \{f_1^*(X_j) - \mathrm{E}[f_1^*(X_j)]\}\Bigg)^2\Bigg] \to 0. \tag{26}$$

Lee (1990, p. 151) also showed that, again in our notation,

$$e(X_j) = -\frac{m}{N-m}\{f_1^*(X_j) - \mathrm{E}[f_1^*(X_j)]\}. \tag{27}$$

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Noting that $\sum_{j=1}^{N} e(X_j) = 0$, we can rewrite (26) as

$$N\,\mathrm{E}\biggl[\biggl(Z_n - \mathrm{E}[Z_0] - \frac{N-m}{N-1}\frac{1}{N-n}\sum_{j=1}^{n}(N-1)e(X_j)\biggr)^2\biggr] \to 0,$$

which is equivalent to

$$N\,\mathrm{E}\biggl[\biggl(Z_n - \mathrm{E}[Z_0] - \frac{N-m}{N-1}(\tilde{Z}_n - \mu)\biggr)^2\biggr] \to 0.$$

This implies (23), which, together with Theorem 2, gives (24).

Finally, we note that

$$\mathrm{var}\biggl(\sum_{j=1}^{n} e(X_j)\biggr) = \frac{N-n}{N-1} n\sigma_{e,N}^2.$$

From this and (24), we get (25). Incidentally, the asymptotic equivalence of (24) and (25) follows from (27):

$$\sigma_{e,N}^2 = \biggl(\frac{m}{N-m}\biggr)^2 \bar{\sigma}_{1,N}^2.$$

4. The case of a variable strategy

Here we assume that the player has a number of strategies available and, therefore, a number of choices of f in (11). With m as in Section 2, let us denote by S the set of such functions f. We assume that S is finite. If the first n cards have been seen ($0 \le n \le N-m$) and the player bets on the next round, we assume that he chooses the optimal $f \in S$; it will depend on $X_1, \dots, X_n$, so we denote it by $f_{X_1,\dots,X_n}$. Here, optimality means that

$$\mathrm{E}[f_{X_1,\dots,X_n}(X_{n+1},\dots,X_{n+m}) \mid \mathcal{F}_n] \ge \mathrm{E}[g_{X_1,\dots,X_n}(X_{n+1},\dots,X_{n+m}) \mid \mathcal{F}_n] \tag{28}$$

for every possible choice $g_{X_1,\dots,X_n} \in S$. The player's profit per unit bet is

$$Y_n := f_{X_1,\dots,X_n}(X_{n+1},\dots,X_{n+m}),$$

so his conditional expected profit per unit bet is

$$Z_n := \mathrm{E}[Y_n \mid \mathcal{F}_n] = \mathrm{E}[f_{X_1,\dots,X_n}(X_{n+1},\dots,X_{n+m}) \mid \mathcal{F}_n].$$

Theorem 4. *Under the above assumptions, $\{Z_n, \mathcal{F}_n, n = 0, \dots, N-m\}$ is a submartingale. In particular,*

$$\mathrm{E}[Z_0] \le \cdots \le \mathrm{E}[Z_{N-m}]. \tag{29}$$

More generally, let I be an interval such that $\mathrm{P}\{Z_0 \in I, \dots, Z_{N-m} \in I\} = 1$. If $\varphi\colon I \to \mathbb{R}$ is convex and nondecreasing, then $\{\varphi(Z_n), \mathcal{F}_n,\ n = 0, \dots, N-m\}$ is a submartingale. In particular,

$$\mathrm{E}[\varphi(Z_0)] \le \cdots \le \mathrm{E}[\varphi(Z_{N-m})]. \tag{30}$$

Finally, if φ is strictly convex and increasing then, for $n = 0, \dots, N-m-1$, $\mathrm{E}[\varphi(Z_n)] = \mathrm{E}[\varphi(Z_{n+1})]$ if and only if Z_n and Z_{n+1} are constant and equal.

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Remark 4. For example, (30) holds with $\varphi(u) := \{(u - u_0)_+\}^\alpha$, where u_0 is arbitrary and $\alpha \geq 1$; in particular, $\mathrm{E}[Z_n^2]$ is increasing in n in Example 2 (with $I = [0, 1]$, $u_0 = 0$, and $\alpha = 2$). However, taking $u_0 = 1/(N-1)$ and $\alpha = 2$ in Example 2, we see that (30) may fail with $\varphi(u) := |u - u_0|^\alpha$. Recall from this example that (13) and (14) may also fail in the variable-strategy case.

Proof of Theorem 4. The submartingale property of $\{Z_n, \mathcal{F}_n\}$ follows by noting that, for $n = 0, \dots, N - m - 1$,

$$\begin{aligned}
\mathrm{E}[Z_{n+1} \mid \mathcal{F}_n] &= \mathrm{E}[\mathrm{E}[f_{X_1,\dots,X_{n+1}}(X_{n+2}, \dots, X_{n+m+1}) \mid \mathcal{F}_{n+1}] \mid \mathcal{F}_n] \\
&\geq \mathrm{E}[\mathrm{E}[f_{X_1,\dots,X_n}(X_{n+2}, \dots, X_{n+m+1}) \mid \mathcal{F}_{n+1}] \mid \mathcal{F}_n] \\
&= \mathrm{E}[f_{X_1,\dots,X_n}(X_{n+2}, \dots, X_{n+m+1}) \mid \mathcal{F}_n] \\
&= \mathrm{E}[f_{X_1,\dots,X_n}(X_{n+1}, \dots, X_{n+m}) \mid \mathcal{F}_n] \\
&= Z_n,
\end{aligned}$$

where the inequality uses (28) and the fact that $f_{X_1,\dots,X_n}$ is of the form $g_{X_1,\dots,X_{n+1}}$, and the penultimate equality uses the exchangeability of $X_1, \dots, X_N$. From this follow (29), the submartingale property of $\{\varphi(Z_n), \mathcal{F}_n\}$, and (30).

Now let us assume that φ is strictly convex and increasing. Fix $n \in \{0, \dots, N - m - 1\}$ and suppose that $\mathrm{E}[\varphi(Z_n)] = \mathrm{E}[\varphi(Z_{n+1})]$. Then

$$\mathrm{E}[\varphi(Z_n)] = \mathrm{E}[\varphi(\mathrm{E}[Z_{n+1} \mid \mathcal{F}_n])] = \mathrm{E}[\mathrm{E}[\varphi(Z_{n+1}) \mid \mathcal{F}_n]]. \tag{31}$$

The argument in the proof of Theorem 1 applies to the second equality in (31), resulting in $\mathrm{var}(Z_{n+1}) = 0$. Moreover, the first equality in (31), together with the increasing property of φ and $Z_n \leq \mathrm{E}[Z_{n+1} \mid \mathcal{F}_n]$, tells us that $\mathrm{var}(Z_n) = 0$. The stated conclusion follows.

5. Application to trente et quarante

Trente et quarante (also known as rouge et noir) is a casino game played with six standard 52-card decks mixed together, resulting in a 312-card deck. Suits do not matter but colors do. Aces have value one, picture cards have value 10, and every other card has value equal to its nominal value. Two rows of cards are dealt. In the first row, called Black, cards are dealt until the total value is 31 or greater. In the second row, called Red, the process is repeated. Thus, each row has associated with it a total between 31 and 40, inclusive.

Four even-money bets are available, called red, black, color, and inverse. Let us define the *winning row* (Red or Black) to be the one with the smaller total if the two totals are different; of course, the other row is the *losing row*. A bet on red wins or loses if Red is the winning or losing row, respectively. A bet on black wins or loses if Black is the winning or losing row, respectively. A bet on color wins or loses if the color of the first card dealt to Black is the same as the color of the winning or losing row, respectively. A bet on inverse wins or loses if the color of the first card dealt to Black is different from the color of the winning or losing row, respectively. For all four bets, a *push* occurs if the Red and Black totals are equal and greater than 31; in this case no money changes hands. If the Red and Black totals are both equal to 31, half the amount of the bet is lost.

Associated with each of the four even-money bets is an insurance bet for 1% of the original bet. It pays off the loss in the case of a tie at 31, though the bet itself is retained by the casino (just as an insurance company retains the premium when it pays off a claim). The insurance bet is lost if the original bet is won or lost, and is pushed if the original bet is pushed. A drawback to taking insurance is that it restricts one's bets to 100 times the smallest unit of currency accepted, and integer multiples thereof.

Of historical interest is the problem of finding the probabilities of the ten possible totals, 31 to 40. This was first done by D. M. Florence in a 1739 monograph (see Todhunter (1865)), assuming a nonstandard deck composition; namely, a 40-card deck obtained from the standard 52-card deck by removing the eights, nines, and tens. Todhunter (1865, Article 358) described Florence's effort contemptuously: 'The problem is solved by examining all the cases which can occur, and counting up the number of ways. The operation is most laborious, and the work is perhaps the most conspicuous example of misdirected industry which the literature of Games of Chance can furnish.' The authors have not been able to locate a copy of Florence's work and, so, cannot comment on its accuracy.

Huyn (1788, pp. 28–29) proposed a solution, assuming sampling with replacement from a standard deck, but it is inaccurate. Noting that, for a total of 40, the last card must have value 10 (four denominations); for a total of 39 the last card must have value 9 or 10 (five denominations); … for a total of 31 the last card must have value 1, 2, …, or 10 (13 denominations), Huyn concluded that the probability of a total of i is

$$P(i) = \tfrac{1}{85}(44 - i), \qquad 31 \le i \le 40,$$

since $4 + 5 + \cdots + 13 = 85$. The argument was sufficiently plausible that several subsequent authors (Grégoire (1853, pp. 37–38), Gall (1883, p. 96), Silberer (1910, pp. 72–73), Scrutator (1924, pp. 84–85), and Scarne (1974, p. 518)) adopted it as their own. For good measure, Scarne (1974, p. xx), the self-proclaimed 'world's foremost gambling authority,' added that he was the first to evaluate these probabilities.

Poisson (1825) not only pointed out the error in Huyn's work, but he found two correct expressions for the probabilities in question, assuming sampling without replacement from the 312-card deck. For example, he showed that

$$P(31) = \text{coefficient of } t^{31} \text{ in}$$
$$313\int_0^1 (1 - y + yt)^{24} \cdots (1 - y + yt^9)^{24}(1 - y + yt^{10})^{96}\,\mathrm{d}y.$$

However, because of his lack of a computer, he was able to evaluate the probabilities only in an asymptotic case corresponding to sampling with replacement. (An English translation of Poisson (1825) is available from the authors.)

Independently, De Morgan (1838, Appendix 1) evaluated the ten probabilities assuming sampling with replacement. Bertrand (1888, pp. 35–38) and Boll (1936, Chapter 14) treated the same case in their analyses. De Morgan's argument was simpler than Poisson's: he noted, for example, that $P(31)$ can be found from the recursion

$$P(n) = \tfrac{1}{13}(P(n-1) + \cdots + P(n-9)) + \tfrac{4}{13}P(n-10),$$

where $n = 1, 2, \ldots, 31$, $P(n) := 0$ if $n < 0$, and $P(0) := 1$. Bertrand's argument was identical. Of course, this approach does not work when sampling without replacement.

Thorp and Walden (1973) addressed the problem under the correct assumptions (sampling without replacement from the 312-card deck) but only approximated the probabilities in question by limiting consideration to at most eight cards. Here we evaluate the exact probabilities, rounded to nine decimal places, possibly for the first time. Almost certainly, this calculation could not have been done prior to the computer era.

Let us define a *trente-et-quarante sequence* to be a finite sequence $(a_1, \dots, a_K)$ of positive integers, none of which exceeds 10, and at most 24 of which are equal to 1, such that

$$a_1 + \cdots + a_{K-1} \leq 30 \quad \text{and} \quad a_1 + \cdots + a_K \geq 31.$$

Clearly, if $(a_1, \dots, a_K)$ is such a sequence, then its length K satisfies $4 \leq K \leq 28$. The number of trente-et-quarante sequences can be evaluated by noting that, given any such sequence, each permutation of the terms that fixes the last term results in another trente-et-quarante sequence. We therefore define $p_{10}(k)$ to be the set of partitions of the positive integer k with no part greater than 10. Such a partition can be described as $(k_1, \dots, k_{10})$, with $k_i \geq 0$ being the multiplicity of part i; note that $\sum_{i=1}^{10} ik_i = k$. It follows that the number of trente-et-quarante sequences is

$$\sum_{k=21}^{30} \sum_{(k_1,\dots,k_{10}) \in p_{10}(k):\, k_1 \leq 24} \binom{k_1 + \cdots + k_{10}}{k_1, \dots, k_{10}} (10 - (31 - k) + 1 - \delta_{k,30}\delta_{k_1,24}),$$

where $\delta_{i,j}$ is the Kronecker delta. This is readily computable, because the double sum contains only 18 096 terms. We find that there are 9569 387 893 trente-et-quarante sequences.

Similar reasoning gives the probabilities of the ten trente-et-quarante totals, assuming a full initial deck of 312 cards. Let the initial counts of the ten card values be $n_1 = \cdots = n_9 = 24$ and $n_{10} = 96$, with $N := n_1 + \cdots + n_{10} = 312$. Then, for $i = 31, \dots, 40$, the total i occurs with probability

$$P(i) := \sum_{k=i-10}^{30} \sum_{(k_1,\dots,k_{10}) \in p_{10}(k)} \binom{k_1 + \cdots + k_{10}}{k_1, \dots, k_{10}} \frac{(n_1)_{k_1} \cdots (n_{10})_{k_{10}} (n_{i-k} - k_{i-k})}{(N)_{k_1 + \cdots + k_{10} + 1}}. \tag{32}$$

The condition $k_1 \leq 24$ can be omitted here, because $(n_1)_{k_1} = 0$ if $k_1 > n_1 = 24$. These numbers, too, are easy to compute (the double sum for $P(31)$ has 18 115 terms, of which 19 are 0), and we summarize the results in Table 1. Note that the approximate probabilities of Thorp and Walden (1973), based on an analysis of trente-et-quarante sequences of length eight or less, are accurate to within about 0.000 065.

The distribution of the length of a trente-et-quarante sequence may also be of some interest. Boll (1936, p. 200) listed the probabilities of sequence lengths up to 13, assuming sampling with replacement, while Thorp and Walden (1973) evaluated the probabilities of sequence lengths up to eight, assuming sampling without replacement. They noted the surprisingly large discrepancies between Boll's figures and theirs (e.g. Boll gave 0.174 53 for the probability that a sequence has length four, versus their 0.260 817), and concluded: 'Numbers based on the infinite deck approximation may be in considerable error.' Actually, Boll's figures are simply wrong. The source of his error can be inferred from Boll (1945, Figures 2 and 3). In Table 2, we list the probabilities of sequence lengths up to ten in both cases (without and with replacement), showing that the infinite-deck approximation is reasonably good, as might be expected.

The game of trente et quarante involves an ordered pair of trente-et-quarante sequences $(a_1, \dots, a_K)$ and $(b_1, \dots, b_L)$, with at most 24 of the $K + L$ terms equal to 1 and at most 24 of them equal to 2. Clearly, if $(a_1, \dots, a_K)$ and $(b_1, \dots, b_L)$ is such a pair, then $8 \leq K + L \leq 44$. We assumed in Sections 2 to 4 that a round is not begun unless there are enough cards to complete it, but here we replace the implicit phrase 'with certainty' by the explicit phrase

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TABLE 1: The probabilities of the ten trente-et-quarante totals, assuming that cards are dealt from the full 312-card deck.

Total	Probability without replacement	Probability with replacement
31	0.148 057 777	0.148 060 863
32	0.137 826 224	0.137 905 177
33	0.127 576 652	0.127 512 672
34	0.116 865 052	0.116 891 073
35	0.106 151 668	0.106 049 464
36	0.094 992 448	0.094 998 365
37	0.083 858 996	0.083 749 795
38	0.072 302 455	0.072 317 327
39	0.060 800 856	0.060 716 146
40	0.051 567 873	0.051 799 118

TABLE 2: The (incomplete) distribution of the length of a trente-et-quarante sequence, assuming that cards are dealt from the full 312-card deck.

Length	Probability without replacement	Probability with replacement
4	0.260 817 415	0.262 105 669
5	0.367 049 883	0.365 194 065
6	0.239 624 080	0.238 220 738
7	0.096 878 765	0.097 334 300
8	0.028 043 573	0.028 883 109
9	0.006 268 155	0.006 731 994
10	0.001 127 778	0.001 288 923
11 or greater	0.000 190 349	0.000 241 203

'with high probability.' Regarding K and L as random variables, we have calculated that $\mathrm{P}\{K+L>20\}<0.000\,000\,606$, so we can safely take $m=20$ in Sections 2 to 4.

Let us now find the joint distribution of the Black total and the Red total. Because sampling is without replacement, we cannot assume independence, although doing so gives a reasonable first approximation. Arguing as in (32), for $i, j = 31, \dots, 40$, the totals i for Black and j for Red occur with probability

$$P(i,j) := \sum_{k=i-10}^{30} \sum_{l=j-10}^{30} \sum_{(k_1,\dots,k_{10})\in p_{10}(k)} \sum_{(l_1,\dots,l_{10})\in p_{10}(l)} \times \binom{k_1+\cdots+k_{10}}{k_1,\dots,k_{10}} \binom{l_1+\cdots+l_{10}}{l_1,\dots,l_{10}} \frac{(n_1)_{k_1+l_1}\cdots(n_{10})_{k_{10}+l_{10}}}{(N)_{k_1+\cdots+k_{10}+l_1+\cdots+l_{10}}} \times \frac{(n_{i-k}-k_{i-k}-l_{i-k})(n_{j-l}-k_{j-l}-l_{j-l}-\delta_{i-k,j-l})}{(N-k_1-\cdots-k_{10}-l_1-\cdots-l_{10})_2}. \tag{33}$$

It is clear from (33) (or from a simple exchangeability argument) that the joint distribution is symmetric; of course, both marginals are given by (32). The evaluation of (33) requires

TABLE 3: Some probabilities in trente et quarante, and the house advantage, assuming sampling without replacement from the full 312-card deck.

Event	Probability
Red total < Black total	$p := 0.445\,200\,543$
Red total > Black total	$p := 0.445\,200\,543$
Red total = Black total ≥ 32	$q := 0.087\,707\,543$
Red total = Black total = 31	$r := 0.021\,891\,370$
Red, Black, Color, or Inverse	**House advantage**
without insurance, pushes included	$\frac{1}{2}r = 0.010\,945\,685$
without insurance, pushes excluded	$\frac{1}{2}r/(1-q) = 0.011\,998\,000$
with insurance, pushes included	$0.01(1-q)/1.01 = 0.009\,032\,599$
with insurance, pushes excluded	$0.01/1.01 = 0.009\,900\,990$

a fair amount of computing power, inasmuch as the quadruple sum for $P(31, 31)$ contains $18\,115^2 = 328\,153\,225$ terms. Nevertheless, our program for the joint distribution runs in less than 30 minutes on a 1.8 Ghz Apple® Mac® G5. We record the most important conclusions from this computation in Table 3.

We can now address the card-counting potential of trente et quarante, using Theorem 3. Let D be the set of 312 cards (the full deck), and let U be an arbitrary subset of D. Let P_U denote conditional probability given that U is the unseen deck, with all possible permutations of the cards of U equally likely, and let Y_1 denote the next card to be dealt. Let R, B, C, and I be the events that red, black, color, and inverse win, respectively; let T be the event that Red and Black tie at 32 or more; and let T_{31} be the event that Red and Black tie at 31. Thorp and Walden (1973) made the important observations that $\mathrm{P}_U(R) = \mathrm{P}_U(B)$, regardless of U, whereas it is not necessarily true that $\mathrm{P}_U(C) = \mathrm{P}_U(I)$; nevertheless, the latter equality is true if each card value has an equal number of red and black representatives in U. (It is possible that Gall (1883) was aware of this as well, based on his pages 233–234.) More generally, conditioning on Y_1, we find that

$$\mathrm{P}_U(C) - \mathrm{P}_U(I) = \sum_{i=1}^{10} (\mathrm{P}_U\{Y_1 = \text{red } i\} - \mathrm{P}_U\{Y_1 = \text{black } i\}) \times [\mathrm{P}_U(R \mid \{Y_1 = i\}) - \mathrm{P}_U(B \mid \{Y_1 = i\})].$$

Here we are using the fact that, since $\mathrm{P}_U(R \mid \{Y_1 = \text{red } i\}) = \mathrm{P}_U(R \mid \{Y_1 = \text{black } i\})$, both probabilities are equal to $\mathrm{P}_U(R \mid \{Y_1 = i\})$; of course, the same is true with B in place of R. In particular, for the color bet, the effects of red removals are

$$\begin{aligned} &\mathrm{P}_{D-\{\text{red } j\}}(C) - \mathrm{P}_{D-\{\text{red } j\}}(I) - \tfrac{1}{2}\,\mathrm{P}_{D-\{\text{red } j\}}(T_{31}) - (\mathrm{P}_D(C) - \mathrm{P}_D(I) - \tfrac{1}{2}\,\mathrm{P}_D(T_{31})) \\ &\quad = -\tfrac{1}{311}[\mathrm{P}_{D-\{j\}}(R \mid \{Y_1 = j\}) - \mathrm{P}_{D-\{j\}}(B \mid \{Y_1 = j\})] - \tfrac{1}{2}(\mathrm{P}_{D-\{j\}}(T_{31}) - \mathrm{P}_D(T_{31})), \end{aligned} \tag{34}$$

while the effects of black removals are

$$\begin{aligned} &\mathrm{P}_{D-\{\text{black } j\}}(C) - \mathrm{P}_{D-\{\text{black } j\}}(I) - \tfrac{1}{2}\,\mathrm{P}_{D-\{\text{black } j\}}(T_{31}) - (\mathrm{P}_D(C) - \mathrm{P}_D(I) - \tfrac{1}{2}\,\mathrm{P}_D(T_{31})) \\ &\quad = \tfrac{1}{311}[\mathrm{P}_{D-\{j\}}(R \mid \{Y_1 = j\}) - \mathrm{P}_{D-\{j\}}(B \mid \{Y_1 = j\})] - \tfrac{1}{2}(\mathrm{P}_{D-\{j\}}(T_{31}) - \mathrm{P}_D(T_{31})). \end{aligned} \tag{35}$$

TABLE 4: The effects of removal, multiplied by 311, for the color bet in trente et quarante. For the inverse bet, effects for red cards and black cards are interchanged.

	Without insurance		With insurance	
Card value	Red card	Black card	Red card	Black card
1	0.051 988	0.006 896	0.022 942	−0.022 151
2	0.044 022	−0.006 993	0.025 593	−0.025 422
3	0.040 616	−0.011 962	0.026 236	−0.026 342
4	0.035 853	−0.015 102	0.025 329	−0.025 626
5	0.028 053	−0.016 000	0.021 816	−0.022 237
6	0.016 415	−0.012 616	0.014 292	−0.014 739
7	0.003 811	−0.008 802	0.006 105	−0.006 508
8	−0.010 233	−0.002 965	−0.003 766	0.003 503
9	−0.027 035	0.004 986	−0.016 032	0.015 988
10	−0.045 133	0.014 899	−0.029 889	0.030 143

Here, we are using the facts that

$$\mathrm{P}_{D-\{\text{red } j\}}(A \mid \{Y_1 = j\}) = \mathrm{P}_{D-\{j\}}(A \mid \{Y_1 = j\}) \quad \text{for } A = B, R,$$
$$\mathrm{P}_{D-\{\text{red } j\}}(T_{31}) = \mathrm{P}_{D-\{j\}}(T_{31});$$

similar results hold with black j in place of red j.

For the inverse bet, the results are analogous, but with C and I interchanged, so the sign of the term with coefficient $\frac{1}{311}$ is changed in (34) and (35). Even simpler, the effects of red and black removals are interchanged.

For the color bet with insurance, the term $-\frac{1}{2}(\mathrm{P}_{D-\{j\}}(T_{31}) - \mathrm{P}_D(T_{31}))$ in (34) and (35) is replaced by $0.01(\mathrm{P}_{D-\{j\}}(T) - \mathrm{P}_D(T))$. The same is true of the inverse bet with insurance. (Here, 'per unit bet' in the definition of Z_n means 'per unit bet on color or inverse only.')

Evaluation of these quantities requires an easy modification of the program used for Table 3. Results are summarized in Table 4.

Of course, in practice, the quantities $E_i := (N-1)e(i)$ of (22) (or of Table 4) are replaced by integers F_i that are highly correlated with the given numbers. Writing (22) as

$$\tilde{Z}_n = \mu + \frac{1}{N-n}\sum_{j=1}^{n} E_{X_j},$$

we can approximate $\tilde{Z}_n$ by

$$Z_n^F := \mu + \gamma_F\biggl(\frac{1}{N-n}\sum_{j=1}^{n} F_{X_j}\biggr), \tag{36}$$

where the regression coefficient γ_F is given by

$$\gamma_F = \frac{\sum_{i=1}^N E_i F_i}{\sum_{i=1}^N F_i^2}.$$

As in Example 1, the quantity within parentheses in (36) is called the true count. It must be adjusted by the constants γ_F and μ to estimate the player's advantage. A level-k counting system

TABLE 5: The best level-1 card-counting systems for the color bet in trente et quarante. For the inverse bet, point values for red cards and black cards are interchanged.

Card value	Without insurance		With insurance	
	Red card	Black card	Red card	Black card
1	1	0	1	−1
2	1	0	1	−1
3	1	0	1	−1
4	1	0	1	−1
5	1	0	1	−1
6	0	0	1	−1
7	0	0	0	0
8	0	0	0	0
9	−1	0	−1	1
10	−1	0	−1	1
Correlation with Table 4	0.971 931		0.974 264	

uses integers whose absolute values are at most k (with equality at least once). Table 5 gives what the authors believe to be the best level-1 counting system in two cases: (a) the player bets on color (or inverse), never with insurance; and (b) the player bets on color (or inverse), always with insurance. (For simplicity, we do not consider the case in which the player sometimes takes insurance and other times does not.) In both cases considered, the correlation with the effects of removal is greater than 0.97.

The conclusions are perhaps unexpected. The player who always takes insurance may use a one-parameter counting system to track his advantage at both the color bet and the inverse bet. This situation is analogous to that of Example 2. On the other hand, the player who never takes insurance must use a two-parameter counting system (one parameter for the red cards, the other for the black cards) to track his advantage at both the color bet and the inverse bet. This situation is analogous to that of Example 3 and, of the results there, especially (10).

May (2004) was apparently the first to discover our with-insurance system: 'The best one-level system for counting these two bets [color and inverse] counts red A–6 and black 9–K as $+1$, with black A–6 and red 9–K as -1. When the count is above 23, the color bet is favorable. When it is below -23, inverse is favorable.' No further details were provided.

We next use a normal approximation to approximate the probability that the player's approximate advantage exceeds a certain level as a function of the number of unseen cards. This is complicated by the fact that the player has two bets to choose from, color and inverse.

Observe that (25) can be restated as

$$\frac{1}{(N-n)\sigma_E}\sum_{j=1}^{n} E_{X_j}\sqrt{\frac{(N-n)(N-1)}{n}} \xrightarrow{\mathrm{D}} N(0,1),$$

where $\sigma_E^2 = \mathrm{var}(E_{X_1})$. Similar reasoning gives

$$\frac{1}{(N-n)\sigma_F}\sum_{j=1}^{n} F_{X_j}\sqrt{\frac{(N-n)(N-1)}{n}} \xrightarrow{\mathrm{D}} N(0,1),$$

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where $\sigma_F^2 = \mathrm{var}(F_{X_1})$. Since F_{X_1} is integer valued, we can improve any normal approximation with a continuity correction.

We begin with the case in which the player bets on color or inverse, and always takes insurance. Let Z_n^C and Z_n^I be the analogues of Z_n^F for the color and inverse bets and, similarly, let C_i and I_i correspond to F_i, γ_C and γ_I to γ_F, and σ_C and σ_I to σ_F. Then $I_i = -C_i$, $\gamma_C = \gamma_I > 0$, and $\sigma_C^2 = \sigma_I^2$, so that

$$\begin{aligned}
&\mathrm{P}\{\max(Z_n^C, Z_n^I) > \beta\}\\
&\quad = \mathrm{P}\bigg\{\max\bigg(\mu + \frac{\gamma_C}{N-n}\sum_{j=1}^{n} C_{X_j}, \mu + \frac{\gamma_C}{N-n}\sum_{j=1}^{n} I_{X_j}\bigg) > \beta\bigg\}\\
&\quad = \mathrm{P}\bigg\{\bigg|\sum_{j=1}^{n} C_{X_j}\bigg| > \bigg\lfloor (N-n)\frac{\beta-\mu}{\gamma_C}\bigg\rfloor + \frac{1}{2}\bigg\}\\
&\quad \approx 2\bigg[1 - \Phi\bigg(\frac{1}{(N-n)\sigma_C}\bigg\{\bigg\lfloor (N-n)\frac{\beta-\mu}{\gamma_C}\bigg\rfloor + \frac{1}{2}\bigg\}\sqrt{\frac{(N-n)(N-1)}{n}}\bigg)\bigg]. \qquad (37)
\end{aligned}$$

Here $N = 312$, $\mu = -0.01(1-q) = -0.009\,123$ (from Table 3), $\gamma_C = 0.024\,767$, and $\sigma_C^2 = \frac{11}{13}$. With 20 cards left ($n = 292$), the estimated probability that the player has the advantage ($\beta = 0$ in (37)) is 0.060. We hasten to add that this number is based on three approximations, namely (20), (36), and (37).

We now turn to the case in which the player bets on color or inverse, and never takes insurance. Using the same notation as before (Z_n^C, Z_n^I, C_i, I_i, γ_C, γ_I, σ_C, σ_I), we have $\gamma_C = \gamma_I > 0$, $\sigma_C^2 = \sigma_I^2$, and $\mathrm{corr}(C_{X_1}, I_{X_1}) = 0$, so that

$$\begin{aligned}
&\mathrm{P}\{\max(Z_n^C, Z_n^I) > \beta\}\\
&\quad = \mathrm{P}\bigg\{\max\bigg(\mu + \frac{\gamma_C}{N-n}\sum_{j=1}^{n} C_{X_j}, \mu + \frac{\gamma_C}{N-n}\sum_{j=1}^{n} I_{X_j}\bigg) > \beta\bigg\}\\
&\quad = 1 - \mathrm{P}\bigg\{\sum_{j=1}^{n} C_{X_j} \le \bigg\lfloor (N-n)\frac{\beta-\mu}{\gamma_C}\bigg\rfloor + \frac{1}{2}, \sum_{j=1}^{n} I_{X_j} \le \bigg\lfloor (N-n)\frac{\beta-\mu}{\gamma_C}\bigg\rfloor + \frac{1}{2}\bigg\}\\
&\quad \approx 1 - \Phi\bigg(\frac{1}{(N-n)\sigma_C}\bigg\{\bigg\lfloor (N-n)\frac{\beta-\mu}{\gamma_C}\bigg\rfloor + \frac{1}{2}\bigg\}\sqrt{\frac{(N-n)(N-1)}{n}}\bigg)^2.
\end{aligned}$$

Here $N = 312$, $\mu = -0.010\,946$ (from Table 3), $\gamma_C = 0.040\,810$, and $\sigma_C^2 = \frac{5}{13}$. With 20 cards left, the estimated probability that the player has the advantage is 0.040.

The results are consistent with the findings of Thorp and Walden (1973): no card-counting system at trente et quarante can yield a ‘practically important player advantage.’

Nevertheless, the preceding discussion provides a more complete understanding of trente et quarante. When the directors of the Monte Carlo Casino were advised by General Pierre Polovtsoff (1937, p. 189), President of the International Sporting Club, that an Italian gang was exploiting a weakness in the game, they responded, ‘Impossible! Trente-et-quarante has been played here for eighty years, and it is inconceivable that anyone can have discovered anything about it that we do not already know.’

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