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BRANCHING PROCESS APPROACH FOR 2-SAT THRESHOLDS

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Abstract

It is well known that, as n tends to ∞, the probability of satisfiability for a random 2-SAT formula on n variables, where each clause occurs independently with probability $\alpha/2n$, exhibits a sharp threshold at $\alpha = 1$. We study a more general 2-SAT model in which each clause occurs independently but with probability $\alpha_i/2n$, where $i \in \{0, 1, 2\}$ is the number of positive literals in that clause. We generalize the branching process arguments used by Verhoeven (1999) to determine the satisfiability threshold for this model in terms of the maximum eigenvalue of the branching matrix.

Keywords: 2-SAT; satisfiability; phase transition; two-type branching process

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1. Introduction

1.1. Background

The k-satisfiability (or k-SAT for short) problem is a canonical constraint satisfaction problem in theoretical computer science. A k-SAT formula is a conjunction of m clauses, each of which is a disjunction of length k chosen from n Boolean variables and their negations. Given a k-SAT formula, a natural problem is to find an assignment of n variables which satisfies the formula. The decision version of the problem is to determine whether there exists an assignment satisfying the formula.

From the computational complexity perspective, the problem is well understood. The problem is NP-hard for $k \geq 3$ [10] and linear time solvable for $k = 2$ [6]. Much recent interest has been devoted to the understanding of random k-SAT formulae where each clause is chosen independently with the same probability and the expected number of clauses in the formula is αn. This problem lies in the intersection of three different subjects—statistical physics, discrete mathematics, and complexity theory.

In statistical physics, the notion of 'phase transition' refers to a situation where systems undergo some abrupt behavioral change depending on some external control parameter such as temperature. In the context of random k-SAT formulae the natural parameter is the density of the formula α, i.e. the ratio between the number of clauses and the number of variables. Much recent research has been devoted to understanding the critical densities for random k-SAT problems. The most important critical density being that for satisfiability, i.e. the

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threshold at which a formula changes from being satisfiable with high probability to being unsatisfiable with high probability [2], [24]. Other thresholds involve the geometry of the solution space and the performance of various algorithms (see, e.g. [3], [23], and the references therein).

The 2-SAT problem is more amenable to analysis than the k-SAT problem for $k \geq 3$. This is closely related to the fact that the 2-SAT problem can be solved in linear time and that the satisfiability of the 2-SAT problem is equivalent to a clear graph theoretic criteria (see Lemma 1). The threshold for the 2-SAT problem is well known to be $\alpha = 1$ (see [9], [13], [16], and [17]) and detailed information on the scaling window is given in [8]. For k-SAT, $k \geq 3$, problems, there are various bounds and conjectures on the critical threshold for satisfiability, but the thresholds are not known rigorously. See [1], [2], [4], [12], [15], [21], [25], and the references therein.

In this paper we establish the threshold of a more general 2-SAT model where the probability of having a clause in the formula depends on the number of positive and negative variables in the clause. Our proof is based on branching process arguments. Branching process techniques have been used before to study the standard 2-SAT formulae in the unsatisfiability regime ($\alpha > 1$); see, e.g. Verhoeven [27]. We generalize the arguments used by Verhoeven in the two-type branching process setup to analyze the general 2-SAT model. Our main contribution is in demonstrating that branching process arguments extend to a multitype setup. A well-accepted idea in studying random graphs and constraint satisfaction problems is that, since the 'local' structure of the problems is tree-like, processes defined on trees play a key role in analyzing the problems. The classical example is the threshold for the existence of a 'giant' component in random graphs where branching processes play a key role in the proof (see, e.g. [7] and [20]). Some more recent examples include [5], [26], and [28].

A seemingly closely related work is that of Cooper *et al.* [11], who derived the threshold for a random 2-SAT model with given literal degree distribution. We note that the two works are incomparable, since our work is stated in terms of the distribution of clauses of different types and the model in [11] is stated in terms of the degrees of the literals. For example, a random 2-SAT formula with $2n$ positive–negative clauses has the same literal degree distribution as a uniform random formula with $2n$ clauses. It is obvious that while the former is always satisfiable, the latter is not satisfiable with high probability.

1.2. Definitions and statements of the main results

Let $x_1, x_2, \ldots, x_n$ be n Boolean *variables*. Let the negation of x_i be denoted by $\bar{x}_i$. Then the n Boolean variables give us $2n$ *literals* $\{x_1, x_2, \ldots, x_n, \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n\}$. The two literals x_i and $\bar{x}_i$ are called complementary to each other ($x_i = 1$ if and only if $\bar{x}_i = 0$) with the convention that $\bar{\bar{x}}_i = x_i$. We call the n literals x_i, $i = 1, 2, \ldots, n$, *positive literals* and their complementary literals $\bar{x}_i$, $i = 1, 2, \ldots, n$, *negative literals*.

Given a literal u, $\mathrm{vr}(u)$ denotes the corresponding variable; the notation naturally extends to a set of literals S, i.e. $\mathrm{vr}(S) = \{\mathrm{vr}(u) \colon u \in S\}$. Two literals u and v are said to be *strongly distinct* if $u \neq v$ and $u \neq \bar{v}$, or, equivalently, if $\mathrm{vr}(u) \neq \mathrm{vr}(v)$. A *2-clause* (which we will call simply a 'clause' later) is a disjunction $C = x \vee y$ of two strongly distinct literals. In this paper we will not allow $x \vee x$ or $x \vee \bar{x}$ to be valid clauses. A 2-SAT formula is a conjunction $F = C_1 \wedge C_2 \wedge \cdots \wedge C_m$ of 2-clauses $C_1, C_2, \ldots, C_m$. Let $\mathcal{C} = \{C_1, C_2, \ldots, C_m\}$ be the collection of clauses corresponding to F.

As is usual in Boolean algebra, 0 stands for the logical value false, and 1 stands for the logical value true. A 2-SAT formula $F = F(x_1, x_2, \ldots, x_n)$ is said to be *satisfiable* if there exists a truth assignment $\eta = (\eta_1, \eta_2, \dots, \eta_n) \in \{0, 1\}^n$ such that $F(\eta_1, \eta_2, \dots, \eta_n) = 1$. The formula F is called *SAT* if F is satisfiable and called *UNSAT* otherwise.

In the standard model for a random 2-SAT formula we choose each of the possible $4\binom{n}{2}$ clauses independently with probability $\alpha/2n$. In this paper we will study a more general model in which a 2-SAT formula F consists of a random subset $\mathcal{C}$ of clauses such that each clause appears in $\mathcal{C}$ independently and a clause having i positive literals is present in the formula with probability $\alpha_i/2n$, $i = 0, 1, 2$, for some constants $\alpha_i \geq 0$. Of course, taking $\alpha_0 = \alpha_1 = \alpha_2 = \alpha$ we retrieve the standard model.

Let $\boldsymbol{M}$ be the branching matrix given by

$$\boldsymbol{M} = \frac{1}{2}\begin{bmatrix} \alpha_1 & \alpha_0 \\ \alpha_2 & \alpha_1 \end{bmatrix}. \tag{1}$$

Note that though $\boldsymbol{M}$ is not symmetric in general, its eigenvalues are all real and given by $\frac{1}{2}(\alpha_1 \pm \sqrt{\alpha_0\alpha_2})$. Let

$$\rho = \tfrac{1}{2}(\alpha_1 + \sqrt{\alpha_0\alpha_2})$$

denote the largest eigenvalue of $\boldsymbol{M}$. We show that ρ is a crucial parameter for satisfiability. In particular, our main result is the following theorem which establishes that the generalized 2-SAT model undergoes a phase transition from satisfiability to unsatisfiability at $\rho = 1$.

Theorem 1. *Let F be a random 2-SAT formula under the generalized model with parameter ρ.*

(a) *If $\rho < 1$ or $\alpha_0\alpha_2 = 0$, then F is satisfiable with probability tending to 1 as $n \to \infty$.*

(b) *If $\rho > 1$ and $\alpha_0\alpha_2 > 0$, then F is unsatisfiable with probability tending to 1 as $n \to \infty$.*

Remark 1. It is easy to see (following the arguments given in Appendix A of [8]) that the satisfiability threshold for the general 2-SAT model remains the same for a variant of the model where $n\alpha_i$, $i = 0, 1, 2$, clauses are chosen uniformly at random from the set of all clauses with i positive literals. We can also allow clauses of the form $x \vee x$ and $x \vee \bar{x}$.

2. The 2-SAT formula and implication digraph

We exploit the standard representation of a 2-SAT formula as a directed graph (see, e.g. [8]), called the *implication digraph* associated with the 2-SAT formula. This graph has $2n$ vertices, labeled by the $2n$ literals. If the clause $(u \vee v)$ is present in the 2-SAT formula then we draw the two directed edges $\bar{u} \to v$ and $\bar{v} \to u$. The directed edges can be thought of as logical implications since if there is a directed edge from $u \to v$ and $u = 1$, then, for the formula to be satisfiable, it is necessary to have $v = 1$.

By a *directed path* (from u to v) we mean a sequence of vertices $u_0 = u, u_1, u_2, \dots, u_k = v$ such that there is a directed edge from $u_i \to u_{i+1}$ for all $i = 0, 1, \dots, k-1$. The length of this directed path is k. A *contradictory cycle* is a union of two (not necessarily vertex disjoint) directed paths—one starts from a literal u and ends at its compliment $\bar{u}$, and the other starts from $\bar{u}$ and ends at u.

The following lemma connects the concept of satisfiability of the 2-SAT problem to the existence of a contradictory cycle in the implication digraph. For a proof, see [8].

Lemma 1. ([8].) *A 2-SAT formula is satisfiable if and only if its implication digraph contains no contradictory cycle.*

3. Proof of Theorem 1(a)

The proof resembles the first moment arguments given in [9]. The extension to the more general case considered here uses a recursive argument which allows us to deal with the multi-parameter general model.

Definition 1. (*[9].*) Suppose that there exists strongly distinct literals $y_1, y_2, \ldots, y_s$ and $u, v \in \{y_1, y_2, \ldots, y_s, \bar{y}_1, \bar{y}_2, \ldots, \bar{y}_s\}$ such that $(\bar{u} \vee y_1), (\bar{y}_1 \vee y_2), \ldots, (\bar{y}_{s-1} \vee y_s), (\bar{y}_s \vee v) \in \mathcal{C}$, or, equivalently, that there exists a directed path $u \to y_1 \to y_2 \to \cdots \to y_s \to v$ in the implication digraph corresponding to the 2-SAT formula. We call this sequence of literals a bicycle (of length $s+1$). Note that the initial variable u and the final variable v must repeat in a bicycle.

Lemma 2. *If a 2-SAT formula is unsatisfiable then its implication digraph contains a bicycle of length greater than or equal to* 3.

Proof. Suppose that a 2-SAT formula is unsatisfiable. By Lemma 1 we have a contradictory cycle in the implication digraph, say

$$u_0 \to u_1 \to u_2 \to \cdots \to u_l = \bar{u}_0 \to u_{l+1} \to u_{l+2} \to \cdots \to u_k = u_0.$$

The cycle has at least one directed path from a literal to its complement. Choosing one that minimizes the length we get an implication chain formed by a sequence of literals, $u_h \to u_{(h+1) \bmod k} \to u_{(h+2) \bmod k} \to \cdots \to u_{(h+t) \bmod k} = \bar{u}_h$, so that $u_{(h+1) \bmod k}, u_{(h+2) \bmod k}, \ldots, u_{(h+t) \bmod k}$ are strongly distinct. Find the largest $s \geq t$ such that $u_{(h+1) \bmod k}, u_{(h+2) \bmod k}, \ldots, u_{(h+s) \bmod k}$ are strongly distinct. Let v be the element pointed to by $u_{(h+s) \bmod k}$ in the cycle. Then, clearly, $u_h \to u_{(h+1) \bmod k} \to u_{(h+2) \bmod k} \to \cdots \to u_{(h+s) \bmod k} \to v$ is a bicycle of length $s+1$. Since there can be no edge between a literal w and its complement $\bar{w}$, we must have $t \geq 2$ and, therefore, $s \geq 2$.

Lemma 3. *We have*

$$\mathrm{P}(F \textit{ is unsatisfiable}) \leq C \sum_{s=2}^{n} \frac{(2s)^2}{n} [T_{s-1}^{+} + T_{s-1}^{-}], \tag{2}$$

where T_{s-1}^{+} and T_{s-1}^{-} are the expected numbers of directed paths of length $s-1$ started from x_1 and $\bar{x}_1$, respectively, consisting of strongly distinct literals with $T_0^{+} = T_0^{-} = 1$ and $C = [\max(\alpha_0, \alpha_1, \alpha_2)]^2$.

Proof. Let H_s be the number of bicycles of length $s+1$ in the implication digraph of the 2-SAT formula, and let Γ_s be the number of directed paths of s strongly distinct literals in the same digraph. From Lemma 2,

$$\begin{aligned} \mathrm{P}(F \text{ is unsatisfiable}) &\leq \mathrm{P}(\text{there exists a bicycle of length } s+1 \text{ for some } s \geq 2) \\ &\leq \sum_{s=2}^{n} \mathrm{E}(H_s) &(3) \\ &\leq C \sum_{s=2}^{n} \frac{(2s)^2}{(2n)^2} \mathrm{E}(\Gamma_s) &(4) \\ &= C \sum_{s=2}^{n} \frac{s^2}{n} [T_{s-1}^{+} + T_{s-1}^{-}]. &(5) \end{aligned}$$

Step (3) follows from the simple union bound and the Markov inequality. Inequality (4) can be explained as follows:

$$H_s = \sum^{*}_{y_1,\dots,y_s} \sum_{u,v\in\{y_1,\dots,y_s,\bar{y}_1,\dots,\bar{y}_s\}} I((\bar{u} \vee y_1), (\bar{y}_1 \vee y_2), \dots, (\bar{y}_s \vee v) \in \mathcal{C}).$$

Here $\sum^{*}$ means that the sum is taken over the set of all possible strongly distinct literals of size s. Observe that, for a fixed choice of strongly distinct literals, there are $2s$ choices for each u and v, and each clause occurs with probability at most $\max(\alpha_0, \alpha_1, \alpha_2)/2n$. Now taking the expectation and using independence between clauses, we have

$$\begin{aligned}
\mathrm{E}(H_s) &= \sum^{*}_{y_1,\dots,y_s} \sum_{u,v\in\{y_1,\dots,y_s,\bar{y}_1,\dots,\bar{y}_s\}} \mathrm{P}((\bar{y}_1 \vee y_2), \dots, (\bar{y}_{s-1} \vee y_s) \in \mathcal{C})\, \mathrm{P}((\bar{u} \vee y_1) \in \mathcal{C}) \\
&\qquad \times \mathrm{P}((\bar{y}_s \vee v) \in \mathcal{C}) \\
&\le \frac{C(2s)^2}{(2n)^2} \sum^{*}_{y_1,\dots,y_s} \mathrm{P}((\bar{y}_1 \vee y_2), (\bar{y}_2 \vee y_3), \dots, (\bar{y}_{s-1} \vee y_s) \in \mathcal{C}) \\
&= \frac{C(2s)^2}{(2n)^2} \mathrm{E}(\Gamma_s).
\end{aligned}$$

Noting that the quantities T^{+}_{s-1} and T^{-}_{s-1} do not depend on x_1, (5) follows.

Lemma 4. *Write* $\boldsymbol{T}_k = (T^{+}_k, T^{-}_k)^{\top}$. *Then*

$$\boldsymbol{T}_{s-1} \le \boldsymbol{M}^{s-1}\mathbf{1},$$

where $\boldsymbol{M}$ *is defined in (1) and* $\mathbf{1} = (1, 1)^{\top}$.

Proof. For a literal u strongly distinct from x_1, let J_u denote the number of directed paths of length $s-2$ starting from u that consist of strongly distinct literals and do not involve the variable x_1. Then

$$T^{+}_{s-1} = \sum_{\substack{\{u\colon u \text{ literals}\}\\ \mathrm{vr}(u)\neq\mathrm{vr}(x_1)}} \mathrm{E}(J_u \times I((\bar{x}_1 \vee u) \in \mathcal{C})) = \sum_{\substack{\{u\colon u \text{ literals}\}\\ \mathrm{vr}(u)\neq\mathrm{vr}(x_1)}} \mathrm{E}(J_u) \times \mathrm{P}((\bar{x}_1 \vee u) \in \mathcal{C}).$$

The last step follows from the independence of clauses.

A simple coupling argument yields $\mathrm{E}(J_u) \le T^{+}_{s-2}$ or T^{-}_{s-2} depending on whether the literal u is positive or negative. Combining the above facts, we get the following recursive inequality:

$$T^{+}_{s-1} \le nT^{+}_{s-2}\,\mathrm{P}(\bar{x}_1 \vee x_2) + nT^{-}_{s-2}\,\mathrm{P}(\bar{x}_1 \vee \bar{x}_2) \le \frac{\alpha_1}{2}T^{+}_{s-2} + \frac{\alpha_0}{2}T^{-}_{s-2},$$

and, similarly,

$$T^{-}_{s-1} \le \frac{\alpha_2}{2}T^{+}_{s-2} + \frac{\alpha_1}{2}T^{-}_{s-2}.$$

Now the above two equations can be written in a more compact way as

$$\boldsymbol{T}_{s-1} \le \boldsymbol{M}\boldsymbol{T}_{s-2}. \tag{6}$$

Iterating (6), we get $\boldsymbol{T}_{s-1} \le \boldsymbol{M}^{s-1}\boldsymbol{T}_0 = \boldsymbol{M}^{s-1}\mathbf{1}$.

Proof of Theorem 1(a). We are now ready to complete the proof of part (a) of Theorem 1. If $\alpha_0\alpha_2 = 0$ then either all the zero or all the one assignments always satisfy the 2-SAT formula F. So, take $\alpha_0\alpha_2 > 0$. Then $\boldsymbol{M}$ is semisimple (i.e. similar to a diagonal matrix).

By Lemma 4,

$$T^+_{s-1,n} + T^-_{s-1,n} \le \mathbf{1}^\top \boldsymbol{M}^{s-1}\mathbf{1} \le B\rho^{s-1}$$

for some constant B. The last inequality holds since we assume that $\boldsymbol{M}$ is semisimple. Substituting this into (2), we finally have

$$\begin{aligned} \mathrm{P}(F \text{ is unsatisfiable}) &\le \frac{K}{n}\sum_{s=2}^{n} s^2\rho^{s-1} \quad \text{for some constant } K > 0 \\ &= O(n^{-1}) \quad \text{since } \rho < 1. \end{aligned}$$

4. The exploration process

Observe that when $\rho > 1$, we need to find a contradictory cycle in the implication digraph of the random 2-SAT formula with high probability. In order to prove this, we will show that starting from any fixed vertex there is a constant probability that it *implies* a large number of literals in the digraph, meaning that there are directed paths to a large number of vertices from the fixed vertex. To achieve this, we explore the digraph dynamically starting from a fixed literal x under certain rules and keep track of variables that are implied by x at each step. We call this the *exploration process*, which is defined next.

Definition and notation. Given a realization of the 2-SAT formula and an arbitrarily fixed literal x, we will consider an exploration process in its implication digraph starting from x.

- The exploration process describes the evolution of two sets of literals, which will be called the *exposed set* and the *active set*.
- A literal is said to be *alive* in a particular step of the process if it is strongly distinct from those in the exposed set and from those in the active set at that step.
- We maintain two stacks for the literals in the active set, one for positive literals and another for negative literals.
- At each step we pop-up a literal (call it the *current* literal) from one of the two stacks of the active set, depending on some event to be described later, and *expose* it. This means that we look for all the literals that are alive at that time and to which there is a directed edge from the current literal.
- We then put those new literals in the stacks of the active set (positive or negative) in some predetermined order and throw the current literal in the exposed set.
- We go on repeating this procedure until the stack of the active literals becomes empty and the process stops.
- Mathematically, let E_t and A_t respectively denote the set of exposed and active sets of literals at time t. Also, let $U_t = \{\mathrm{vr}(u)\colon \mathrm{vr}(u) \notin \mathrm{vr}(E_t) \cup \mathrm{vr}(A_t)\}$ be the set of alive variables at time t. Set $E_0 = \varnothing$ and $A_0 = \{x\}$. If A_t is nonempty and the literal $l \in A_t$

is exposed from the stack, then we have the following updates at time $t+1$:

$$A_{t+1} = (A_t \setminus \{l\}) \cup \{u \colon u \text{ literal such that } \mathrm{vr}(u) \in U_t \text{ and the clause } (\bar{l} \vee u) \text{ is present}\},$$

$$E_{t+1} = E_t \cup \{l\}.$$

If A_t is empty then so is A_{t+1} and E_{t+1} will be same as E_t.

Note that during the evolution of the process, each clause is examined only once. Also, every literal in $\bigcup_t (A_t \cup E_t)$ can be reached from x via a directed path (consisting of strongly distinct literals).

For a subset S of literals, we can partition it as $S = S^+ \cup S^-$, where S^+ and S^- are the sets of all positive and, respectively, negative literals of S. Let u_t, a_t^+, and a_t^- be the shorthand for $|U_t|$, $|A_t^+|$, and $|A_t^-|$, where $|\cdot|$ denotes the size of a set. Set $a_t := |A_t| = a_t^+ + a_t^-$.

Distribution of the process. The stochastic description of the evolution of the process (u_t, a_t^+, a_t^-), $0 \le t \le n$, for a random 2-SAT formula on n variables can be summarized in the next lemma, whose proof is immediate.

Lemma 5. *Define a triangular array of independent Bernoulli random variables as follows:*

$$W_i^{(t)} \sim \mathrm{Ber}\left(\frac{\alpha_0}{2n}\right); \qquad X_i^{(t)}, Y_i^{(t)} \sim \mathrm{Ber}\left(\frac{\alpha_1}{2n}\right); \qquad Z_i^{(t)} \sim \mathrm{Ber}\left(\frac{\alpha_2}{2n}\right);$$

for $1 \le i \le n$ and $0 \le t \le n$. Let $A_t \ne \varnothing$. Given $H(t)$, the history up to t, and the fact that the current literal at time t is positive, we have

$$u_t - u_{t+1} \overset{\mathrm{D}}{=} \sum_{i=1}^{u_t} \{(W_i^{(t)} + X_i^{(t)}) \wedge 1\} = \mathrm{Bin}\left(u_t, \frac{\alpha_0}{2n} + \frac{\alpha_1}{2n} - \frac{\alpha_0\alpha_1}{4n^2}\right),$$

$$a_{t+1}^+ - a_t^+ \overset{\mathrm{D}}{=} -1 + \sum_{i=1}^{u_t} X_i^{(t)} = -1 + \mathrm{Bin}\left(u_t, \frac{\alpha_1}{2n}\right),$$

$$a_{t+1}^- - a_t^- \overset{\mathrm{D}}{=} \sum_{i=1}^{u_t} W_i^{(t)} = -1 + \mathrm{Bin}\left(u_t, \frac{\alpha_0}{2n}\right),$$

where '$\overset{\mathrm{D}}{=}$' denotes equality in distribution. Similarly, given $H(t)$ and conditional on the event that the current literal at time t is negative, we have

$$u_t - u_{t+1} \overset{\mathrm{D}}{=} \sum_{i=1}^{u_t} \{(Y_i^{(t)} + Z_i^{(t)}) \wedge 1\} = \mathrm{Bin}\left(u_t, \frac{\alpha_1}{2n} + \frac{\alpha_2}{2n} - \frac{\alpha_1\alpha_2}{4n^2}\right),$$

$$a_{t+1}^+ - a_t^+ \overset{\mathrm{D}}{=} -1 + \sum_{i=1}^{u_t} Z_i^{(t)} = -1 + \mathrm{Bin}\left(u_t, \frac{\alpha_2}{2n}\right),$$

$$a_{t+1}^- - a_t^- \overset{\mathrm{D}}{=} \sum_{i=1}^{u_t} Y_i^{(t)} = -1 + \mathrm{Bin}\left(u_t, \frac{\alpha_1}{2n}\right).$$

Definition 2. For the rest of the paper, we fix $T = [\sqrt{n}]$. Let $\tau := \sup\{t \le T \colon u_t \ge u_0 - 2\alpha T\}$, where $\alpha = \max(\alpha_0, \alpha_1, \alpha_2)$. In words, τ is the last time before T such that the decrease in the number of unexposed variables is at most $2\alpha T$.

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We define a *round* of the exploration process as follows. Fix a subset S of variables of size $N \geq (1-\delta/2)n$ for some small $\delta > 0$ such that $(1-\delta)\rho > 1$ and a starting literal, say $x \in S$. First run the exploration process from x on the implication digraph restricted to literals from S up to time τ. If $\tau < T$, stop. Otherwise, delete all the variables in $\mathrm{vr}(E_T \cup A_T) \setminus \mathrm{vr}(x)$ from S to get a new set of variables $S' \subseteq S$. By the definition of τ, $|S'| \geq N - 2\alpha T$. Now again run another independent exploration process starting from $\bar{x}$ up to time τ but on the digraph restricted to literals from $S' \cup \{\bar{x}\}$. As before, we apply the stopping rule $\tau < T$.

Lemma 6. *The (random) set of clauses examined during the evolution of an exploration process up to time T is disjoint from the set $\{(\bar{u} \vee \bar{v})\colon u, v \in A_T\}$. Furthermore, the clauses examined during the evolution of the second exploration process of a round are distinct from the clauses in the set $\{(\bar{u} \vee \bar{v})\colon u, v \in A_T\} \cup \mathcal{D}$, where A_T and $\mathcal{D}$ are respectively the active set at time T and the set of clauses examined during the evolution of the first exploration process.*

Proof. The first statement of the lemma follows from the easy observation that if $u \in A_T$ then, from the very definition of the exploration process, the clause $(\bar{u} \vee w)$ is not examined up to time T for all literals w such that $\bar{w} \notin E_T$.

For the second statement, note that any problematic clause should include literal x or $\bar{x}$ (x is the starting vertex). Now all the clauses involving $\mathrm{vr}(x)$ which are examined during the first exploration process of a round must have the form $(\bar{x} \vee y)$, where literal y is such that $\mathrm{vr}(y) \neq \mathrm{vr}(x)$. But the clauses involving $\mathrm{vr}(x)$ scanned by the second exploration process of the round are all of the form $(x \vee y)$, where literal y is such that $\mathrm{vr}(y) \neq \mathrm{vr}(x)$, and there can be no clause from the set $\{(\bar{u} \vee \bar{v})\colon u, v \in A_T\}$ which contains either x or $\bar{x}$.

Corollary 1. *Given whether each of the clauses in $\{(\bar{u} \vee \bar{v})\colon u, v \in A_T\} \cup \mathcal{D}$ is present in $\mathcal{C}$ or not, the distribution of the evolution of the second exploration process depends only on the number of variables with which the second process starts.*

4.1. Proof of Theorem 1(b) when the α_is are all equal

Before tackling the general situation we pause for a moment to give a quick sketch, after [27], of the unsatisfiability part of the phase transition for the standard 2-SAT model. This will serve as a prelude to the proof for the general case.

Let $\alpha = \alpha_0 = \alpha_1 = \alpha_2 > 1$. Then $\rho = \alpha$. In this special case, we slightly modify our exploration process by demanding that we will always choose the current literal from the set of active literals uniformly. Thus, at each time $t \geq 1$, given its size a_t, A_t is uniformly random over all the literals except x and $\bar{x}$, the starting vertex and its complement.

Since the probabilities for the clauses to be present are all equal, the distribution of the exploration process (u_t, a_t) simplifies. Given $H(t)$, the history up to time t, and $a_t > 0$,

$$u_t - u_{t+1} \stackrel{\mathrm{D}}{=} \mathrm{Bin}(u_t, 2p_n - p_n^2), \qquad a_{t+1} - a_t \stackrel{\mathrm{D}}{=} -1 + \mathrm{Bin}(2u_t, p_n), \qquad \text{where } p_n = \frac{\alpha}{2n}.$$

Note that each of the random variables $(u_t - u_{t+1})$ is stochastically dominated by $\mathrm{Bin}(n, 2p_n)$, which has mean α. Thus, using concentration of the binomial distribution, it is easy to see that, for time $T = [\sqrt{n}]$, the event $\{\tau < T\} = \{\sum_{t=1}^{T-1}(u_{t-1} - u_t) > 2\alpha T\}$ occurs with probability at most $A\exp(-cT)$, where $c > 0$. See Lemma 7 in Section 4.2 for a proof of the more general fact.

Let $\delta > 0$ be as given in Definition 2. If $u_0 \geq (1-\delta/2)n$ then $\{\tau = T\} \subseteq \{u_T \geq (1-\delta)n\}$. When $\tau = T$, the process $\{a_t,\ 1 \leq t \leq T\}$ behaves like a random walk with positive drift on nonnegative integers with 0 as the absorbing state and, hence,

$$\text{there exists } C > 0 \text{ such that } \mathrm{P}(a_T \geq CT,\ \tau = T) \geq \zeta \text{ for some constant } \zeta > 0, \text{ independent of } n. \quad (7)$$

If both u and $\bar{u}$ are in A_T for some literal u, we have a directed path from the starting vertex to its complement using the literals in $E_T \cup A_T$. Otherwise, each pair of literals $u, v \in A_T$ are strongly distinct. There are edges $u \to \bar{v}$ and $v \to \bar{u}$ in the digraph if the clause $(\bar{u} \vee \bar{v})$ is present in the formula. If at least one of these $\binom{CT}{2}$ many clauses is present then we again have a directed path from the starting vertex to its complement using the literals in $E_T \cup A_T$. Let D be the event that there exists a directed path from the starting vertex of the exploration process to its complement in $E_T \cup A_T$. Therefore, by Lemma 6,

$$\mathrm{P}(D \mid a_T \geq CT,\ \tau = T) \geq 1 - \left(1 - \frac{\alpha}{2n}\right)^{\binom{CT}{2}} \geq p,$$

where $p > 0$ is a constant, independent of n. This implies that

$$\mathrm{P}(D, \tau = T) \geq \mathrm{P}(D \mid a_T \geq CT,\ \tau = T)\,\mathrm{P}(a_T \geq CT,\ \tau = T) \geq p\xi > 0.$$

Now, by Corollary 1 we can say that after a round the probability that there is no termination and there exists a contradictory cycle in the variables visited during the round is at least $p^2\zeta^2$.

We continue with another round of the exploration process in the deleted graph containing only unvisited variables. We repeat this process until a round stops due to the stopping rule. If each of the successive rounds does not terminate, we have $\Theta(\sqrt{n})$ rounds of the exploration processes before the event $\{u_t < (1-\delta/2)n\}$ occurs. It is easy to see that the clauses examined in different rounds of the exploration process are all distinct and, hence, the rounds are independent. Thus, the probability that we get no contradictory cycle in all the rounds is at most

$$\begin{aligned}
&\mathrm{P}(\text{no contradictory cycle and no round stops}) + \mathrm{P}(\text{one of the rounds stops}) \\
&\quad \leq (1 - p^2\zeta^2)^{\Theta(\sqrt{n})} + (1 - (1 - A\exp(-cT))^{2\Theta(\sqrt{n})}) \\
&\quad \leq \text{constant} \times \exp(-B\sqrt{n}) \quad \text{for some } B > 0.
\end{aligned}$$

Remark 2. Instead of taking $T = [\sqrt{n}]$ as in the proof, if we choose $T = \Theta(n)$ suitably then it follows from (7) that $a_T \geq \Omega(n)$ for probability at least r for some $r > 0$. Thus, for any literal $y \neq x, \bar{x}$, we get $\mathrm{P}(y \in A_T) \geq p$ for some $p > 0$ and all large enough n. So, the probability that there is a directed path from x_1 to x_2 is at least p. The same holds true for the directed path from x_2 to x_1. These are monotonic events. So, by the Fortuin–Kasteleyn–Ginibre (FKG) inequality [14], they occur simultaneously with probability greater than or equal to p^2. Thus, the chance that there exists a directed path from x_1 to $\bar{x}_1$ is at least p^2. Again, applying the FKG inequality, we have a contradictory cycle with probability at least p^4. Now appealing to Friedgut's theorem for the sharp threshold [15], we can conclude that the formula is UNSAT with probability tending to 1 as $n \to \infty$.

4.2. Associated two-type branching process

Now we return to the general case. Given an exploration process on a subgraph of the implication digraph consisting of $N = \Theta(n)$ many variables starting from any fixed literal, our goal is to couple it with a suitable two-type supercritical branching process up to time

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$T = [\sqrt{n}]$ on a set of high probability. Assume that $N \geq (1 - \delta/2)n$, where $\delta > 0$ is such that $(1 - \delta)\rho > 1$.

On the set where $\{u_t \geq N - 2\alpha T$ for all $t \leq T\}$ for large enough n, $\mathrm{Bin}(u_t, \alpha_i/2n)$ stochastically dominates $\mathrm{Bin}((1 - \delta)n, \alpha_i/2n)$ for all time $t \leq T$. Next we are going to prove that this event happens with high probability.

Lemma 7. *Let T, δ be as above, and let $\alpha = \max(\alpha_0, \alpha_1, \alpha_2)$. Then*

$$\mathrm{P}(\tau < T) \leq 2\exp\left(-\frac{\alpha T}{2}\right).$$

Therefore,

$$\mathrm{P}(u_t \geq (1 - \delta)n \textit{ for all } t \leq T) \geq 1 - 2\exp\left(-\frac{\alpha T}{2}\right)$$

for sufficiently large n.

Proof. Since u_t is decreasing in t and $N - 2\alpha T \geq (1 - \delta)n$ for sufficiently large n, it is enough to prove that $\mathrm{P}(\tau < T) \leq 2\exp(-\alpha T/2)$. Note that $u_0 = N$. Clearly, given u_{t-1} and the current literal type at time t, the random variable $u_{t-1} - u_t$ is conditionally independent of $u_0 - u_1, u_1 - u_2, \dots, u_{t-2} - u_{t-1}$ and is stochastically dominated by $\mathrm{Bin}(2N, \alpha/2n)$ irrespective of the conditioning event. Therefore, the distribution of $u_0 - u_T$ is stochastically dominated by $\mathrm{Bin}(2NT, \alpha/2n)$. By Bernstein's inequality,

$$\mathrm{P}\left(\mathrm{Bin}\left(2NT, \frac{\alpha}{2n}\right) \geq 2\alpha T\right) \leq 2\exp\left(-\frac{\alpha T}{2}\right).$$

Therefore,

$$\mathrm{P}(\tau < T) = \mathrm{P}(u_T < N - 2\alpha T) = \mathrm{P}(u_0 - u_T > 2\alpha T) \leq 2\exp\left(-\frac{\alpha T}{2}\right).$$

Lemma 8. *For each $0 \leq i \leq 2$, there exists a bounded distribution F_i taking values in nonnegative integers and with mean m_i such that*

(a) *for all sufficiently large n,*

$$\mathrm{P}\left(\mathrm{Bin}\left(n(1 - \delta), \frac{\alpha_i}{2n}\right) = k\right) \geq \mathrm{P}(X = k \mid X \sim F_i) \quad \textit{for all } k \geq 1,\ 0 \leq i \leq 2;$$

(b) *if $\boldsymbol{M}_0$ is the branching matrix given by*

$$\boldsymbol{M}_0 = \begin{bmatrix} m_1 & m_0 \\ m_2 & m_1 \end{bmatrix}$$

then $\rho_0 > 1$, where ρ_0 is the maximum eigenvalue of $\boldsymbol{M}_0$.

Proof. Fix some $\beta \in (0, 1)$ so that $(1-\delta)(1-\beta)\rho > 1$. Let $\gamma_i = (1-\delta)\alpha_i/2$ for $i = 0, 1, 2$. Find c large enough so that

$$\sum_{k=1}^{c} \frac{k(1 - \beta/2)\exp(-\gamma_i)\gamma_i^k}{k!} > (1 - \beta)\gamma_i, \qquad i = 0, 1, 2.$$

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For each $0 \leq i \leq 2$, let us now define a truncated (and reweighed) Poisson distribution which takes the value k with probability $(1-\beta/2)\exp(-\gamma_i)\gamma_i^k/k!$ for $1 \leq k \leq c$ and 0 otherwise. Call this distribution F_i. By the choice of c, its mean m_i is greater than $(1-\beta)\gamma_i$. Poissonian convergence says that $\operatorname{Bin}(n(1-\delta), \alpha_i/2n) \xrightarrow{L^1} \operatorname{Poisson}(\gamma_i)$ and part (a) of the lemma follows.

Part (b) of the lemma follows from the fact that $\rho_0 = m_1 + \sqrt{m_0 m_2} \geq (1-\delta)(1-\beta)\rho > 1$.

Definition 3. Consider a supercritical two-type branching process, which we call an F-branching process, with offspring distributions

$$\text{type I} \to (\text{type I}, \text{type II}): \underbrace{(F_1, F_0)}_{\text{independent}},$$

$$\text{type II} \to (\text{type I}, \text{type II}): \underbrace{(F_2, F_1)}_{\text{independent}}$$

Next we define a new process $\boldsymbol{X}(t) = (X_1(t), X_2(t))$ by sequentially traversing the Galton–Watson tree of the F-branching process. We fix a suitable order among the types of node of the tree and, moreover, we always prefer to visit a node of type I to a node of type II. Then we traverse the tree sequentially and at each step we expand the tree by including all the children of the node we visit. Let us denote the number of unvisited or unexplored children of type i in the tree traversed up to time t by $X_i(t)$.

Lemma 9. *There exists a coupling such that* $(a_t^+, a_t^-) \geq \boldsymbol{X}(t)$ *for all* $t \leq \tau$ *and large enough* n.

Proof. Fix n sufficiently large. If the starting vertex of the exploration process is of positive type, we initiate the branching process with one individual of type I. Similarly for the other case. We run in parallel the exploration process where the choice of the current literal type at time t depends on the type of node visited at time t. This can be done because, if $t \leq \tau$, we can always simultaneously choose our random variables in such a way (by Lemmas 5 and 8) that, for every step, the number of active literals generated of each type is no less than the number of unvisited nodes of the corresponding type in the tree grown up to that step. If $\tau < t \leq T$ or if we have no unvisited child left in the tree, then we choose the current literal from the active set in some fixed predetermined procedure.

Next we are going to find a lower bound on the total number of unvisited children after T steps of the above process.

Lemma 10. *Suppose that* $\boldsymbol{X}(t)$ *is as in Definition 3 with* $\boldsymbol{X}(0) = (1, 0)$ *or* $(0, 1)$. *Then there exist* $C > 0$ *and* $\eta > 0$ *such that* $\mathrm{P}(X_1(T) + X_2(T) \geq CT) \geq \eta$.

Proof. Though a proof of the above lemma can be found implicitly in [22], we present it here for the sake of completeness. Recall that the F-branching process is supercritical, as ρ_0, the maximum eigenvalue of $\boldsymbol{M}_0$, is strictly greater than 1.

If we assume that $\alpha_1 > 0$, trivially, this process is positive regular and nonsingular. Thus, by a well-known result (see [18, Theorem 7.1]) on the supercritical multitype branching process, its extinction probability is given by $\boldsymbol{0} \leq \boldsymbol{q} = (q_1, q_2) < \boldsymbol{1}$, where q_i is the probability that the process becomes extinct starting with one object of type i.

Note that if $\alpha_1 = 0$, we no longer have the positive regularity. In this case, though [18, Theorem 7.1] cannot be directly applied, we can argue as follows to get the same conclusion. If the process starts with only one individual of type i, the corresponding branching process can be viewed as a single-type supercritical branching process (made of the individuals of type

i only) if we observe the process only at the even number of steps. So, the probability that it eventually dies out, which is nothing but q_i, is strictly less than 1.

Let us define $\boldsymbol{e}_1 = (1, 0)$ and $\boldsymbol{e}_2 = (0, 1)$. Instead of looking at $\boldsymbol{X}(t)$, which has (0, 0) as an absorbing state, we will consider a new chain $\hat{\boldsymbol{X}}(t)$ starting from $\hat{\boldsymbol{X}}(0) = \boldsymbol{X}(0)$ which is supported on the entire $\mathbb{Z}^2$. Given $\hat{\boldsymbol{X}}(0), \hat{\boldsymbol{X}}(1), \dots, \hat{\boldsymbol{X}}(t)$, define

$$\hat{\boldsymbol{X}}(t+1) \overset{\mathrm{D}}{=} \begin{cases} \hat{\boldsymbol{X}}(t) - \boldsymbol{e}_1 + (F_1, F_0) & \text{if } \hat{X}_1(t) > 0, \\ \hat{\boldsymbol{X}}(t) - \boldsymbol{e}_2 + (F_2, F_1) & \text{otherwise.} \end{cases}$$

We can couple $\boldsymbol{X}(t)$ and $\hat{\boldsymbol{X}}(t)$ together so that $\hat{\boldsymbol{X}}(t) = \boldsymbol{X}(t)$ until $\boldsymbol{X}(t)$ reaches (0, 0).

Let (a, b) be a normalized eigenvector of $\boldsymbol{M}_0$ corresponding to eigenvalue ρ_0 such that $a^2 + b^2 = 1$. Since $\alpha_0, \alpha_2 > 0$, we have both $a > 0$ and $b > 0$. Let

$$Z(t) := aX_1(t) + bX_2(t) \quad \text{and} \quad \hat{Z}(t) := a\hat{X}_1(t) + b\hat{X}_2(t).$$

Let

$$\mathcal{F}_t := \sigma(\hat{\boldsymbol{X}}(0), \hat{\boldsymbol{X}}(1), \dots, \hat{\boldsymbol{X}}(t)) \quad \text{and} \quad \Delta\hat{Z}(t) := \hat{Z}(t+1) - \hat{Z}(t).$$

Since the F_is are bounded, so are the $\Delta\hat{Z}(t)$s. Then

$$\begin{aligned} \mathrm{E}(\Delta\hat{Z}(t) \mid \mathcal{F}_t) &= \begin{cases} (\rho_0 - 1)a & \text{if } \hat{X}_1(t) > 0, \\ (\rho_0 - 1)b & \text{otherwise} \end{cases} \\ &\geq \mu \\ &:= (\rho_0 - 1)\min(a, b) \\ &> 0. \end{aligned}$$

Now we have

$$\begin{aligned} \mathrm{P}\bigg(\hat{Z}(T) \leq \frac{\mu T}{2}\bigg) &\leq \mathrm{P}\bigg(\hat{Z}(T) - \sum_{i=0}^{T-1} \mathrm{E}(\Delta\hat{Z}(i) \mid \mathcal{F}_i) \leq -\frac{\mu T}{2}\bigg) \\ &\leq \frac{\sum_{i=0}^{T-1} \mathrm{E}(\Delta\hat{Z}(i) - \mathrm{E}(\Delta\hat{Z}(i) \mid \mathcal{F}_i))^2}{\mu^2 T^2/4} \\ &= O(T^{-1}). \end{aligned}$$

In the last line we used orthogonality of the increments, boundedness of $\Delta\hat{Z}(t)$, and Chebyshev's inequality. Therefore, we can conclude that

$$\begin{aligned} &\mathrm{P}\bigg(X_1(T) + X_2(T) \geq \frac{\mu T}{2} \;\bigg|\; \boldsymbol{X}(0) = \boldsymbol{e}_i\bigg) \\ &\quad \geq \mathrm{P}\bigg(Z(T) \geq \frac{\mu T}{2} \;\bigg|\; \boldsymbol{X}(0) = \boldsymbol{e}_i\bigg) \\ &\quad \geq \mathrm{P}\bigg(\hat{Z}(T) \geq \frac{\mu T}{2}, \ \boldsymbol{X}(t) \neq 0 \text{ for all } 0 \leq t \leq T \;\bigg|\; \boldsymbol{X}(0) = \boldsymbol{e}_i\bigg) \\ &\quad \geq \mathrm{P}(\boldsymbol{X}(t) \neq 0 \text{ for all } t \geq 0 \mid \boldsymbol{X}(0) = \boldsymbol{e}_i) - \mathrm{P}\bigg(\hat{Z}(T) < \frac{\mu T}{2} \;\bigg|\; \boldsymbol{X}(0) = \boldsymbol{e}_i\bigg) \\ &\quad \geq (1 - q_i) - O(T^{-1}). \end{aligned}$$

In the second inequality we used the fact that $Z(t) = \hat{Z}(t)$ until $\boldsymbol{X}(t)$ reaches (0, 0).

5. Proof of Theorem 1(b)

Lemma 11. *In one round of the exploration process on a subgraph involving $N \geq (1-\delta/2)n$ many variables the probability that*

- *there is no termination due to the stopping rule; and*
- *there exists a contradictory cycle using the variables visited through the round*

is at least κ for some $\kappa > 0$ independent of n.

Proof. From Lemmas 9 and 10, we obtain

$$\begin{aligned} \mathrm{P}(a_T \geq CT,\ \tau = T) &\geq \mathrm{P}(X_1(T) + X_2(T) \geq CT,\ \tau = T) \\ &\geq \mathrm{P}(X_1(T) + X_2(T) \geq CT) - \mathrm{P}(\tau < T) \\ &\geq \eta - 2\exp\left(-\frac{\alpha T}{2}\right) \\ &\geq \zeta > 0. \end{aligned}$$

If both u and $\bar{u} \in A_T$ for some literal u, we have a directed path from the starting vertex to its complement using the literals in $E_T \cup A_T$. Otherwise, for each pair of literals $u, v \in A_T$, which are strongly distinct, there are edges $u \to \bar{v}$ and $u \to \bar{v}$ in the digraph if the clause $(\bar{u} \vee \bar{v})$ is present in the formula. If at least one of these $\binom{CT}{2}$ many clauses is present then we again have a directed path from the starting vertex to its complement using the literals in $E_T \cup A_T$.

Case I: $\alpha_1 > 0$. Let $\alpha_{\min} = \min(\alpha_0, \alpha_1, \alpha_2) > 0$. Let D be the event that there exists a directed path from the starting vertex of the exploration process to its complement in $E_T \cup A_T$. Then, by Lemma 6, there exists $p > 0$ independent of n, such that

$$\mathrm{P}(D \mid a_T \geq CT \text{ and } \tau = T) \geq 1 - \left(1 - \frac{\alpha_{\min}}{2n}\right)^{\binom{CT}{2}} \geq p > 0.$$

Case II: $\alpha_1 = 0$. Now $\alpha_{\min} = 0$ and we cannot prove the above statement. But then, instead of looking at all the $(\bar{u} \vee \bar{v})$ clauses where u and v are strongly distinct clauses belonging to A_T, we only consider those clauses for which u and v have the same parity. Since there are at least $2\binom{CT/2}{2}$ many clauses of this type, we have, similarly to case I,

$$\mathrm{P}(D \mid a_T \geq CT \text{ and } \tau = T) \geq 1 - \left(1 - \frac{\alpha'}{2n}\right)^{2\binom{CT/2}{2}} \geq p > 0,$$

where $\alpha' = \min(\alpha_0, \alpha_2) > 0$. Therefore,

$$\mathrm{P}(D, \tau = T) \geq \mathrm{P}(D \mid a_T \geq CT \text{ and } \tau = T)\,\mathrm{P}(a_T \geq CT \text{ and } \tau = T) \geq p\zeta.$$

The lemma is now immediate from Corollary 1 with $\kappa = p^2\zeta^2$.

Remark 3. From the proof of the above lemma, we have seen that, for large n, with probability at least r for some $r > 0$, we have a directed path in the implication digraph from any literal u to its complement $\bar{u}$. Invoking the FKG inequality, we can say that we can find a contradictory cycle with probability at least $r^2 > 0$. But note that we do not have a ready-made theorem like Friedgut's sharp threshold result for the generalized 2-SAT model. While we believe that tweaking Lemma 4 of [19] may help, we will not pursue this further. Instead we take a different route to bypass the problem.

Proof of Theorem 1(b). We now show how to bootstrap this positive probability event to an event with high probability.

Initially we run a round of the exploration process on the entire set of variables starting from x_1. If the process does not terminate after the first round, out of at least $n - 4\alpha T - 1$ many unvisited variables, we pick up an arbitrary one and run another round of the exploration process in the deleted graph starting from it. We repeat this process $\delta\sqrt{n}/9\alpha \leq \delta n/2(4\alpha T + 1)$ many times, provided that we do not have to stop in any one of these rounds of the exploration process, each time discarding previously visited variables to achieve independence among the different rounds. We thus ensure that in each run of the exploration process, we have at least $(1 - \delta/2)n$ many variables to start with.

We conclude that the probability that we get no contradictory cycle in all the rounds is at most

$$\begin{aligned} &\mathrm{P}(\text{no contradictory cycle and no round stops}) + \mathrm{P}(\text{one of the rounds stops}) \\ &\quad \leq (1-\kappa)^{\delta\sqrt{n}/9\alpha} + \biggl(1 - \biggl(1 - 2\exp\biggl(-\frac{\alpha T}{2}\biggr)\biggr)^{2\delta\sqrt{n}/9\alpha}\biggr) \\ &\quad \leq \text{constant} \times \exp(-B\sqrt{n}) \quad \text{for some } B > 0. \end{aligned}$$

This concludes the proof.

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