On the Latent Roots of Compound Determinants and Brill's Determinants.

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§ 1. *Introduction.*

The roots of the equation in λ

$$\begin{vmatrix} a_{11} - \lambda & a_{12} \ldots\ldots\ldots a_{1n} \\ a_{21} & a_{22} - \lambda \ldots.. a_{2n} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n1} & a_{n2} \ldots\ldots\ldots a_{nn} - \lambda \end{vmatrix} = 0$$

were named by Sylvester the *latent roots* of the determinant $| a_{11} a_{22} \ldots a_{nn} |$. So early as 1852, Sylvester showed[*] that if any determinant D is given, we can at once write down a determinant whose latent roots are the squares of the latent roots of D: this determinant is in fact the square of D, the process of squaring being performed by multiplying rows into columns: so that, *e.g.* if the latent roots of $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ are λ_1 and λ_2, then the roots of $\begin{vmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{vmatrix}$ are λ_1^2 and λ_2^2. Spottiswoode had also shown[†] in 1851 that the latent roots of the reciprocal of a determinant are the reciprocals of the latent roots of the determinant itself. Both these theorems were soon found to be particular cases of a general theorem which was enunciated by Sylvester thus: *The latent roots of any function of a matrix are respectively the same functions of the latent roots of the matrix itself.*

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[*] First in connexion with axisymmetric determinants, in *Nouv. Ann.* 1852: *Coll. Papers* 1, p. 364.

[†] *Elementary theorems relating to determinants*, London, 1851.

In order to complete the definition of the terms used in the title of this paper, we must now explain what *compound determinants* and *Brill's determinants* are.

First, as regards compound determinants. Let D be any determinant of order n. Consider all those minors of D which are of order m: they are in number $\binom{n}{m}^{2}$* : and arrange them so that in any one row (or column) those minors stand which are contained in the same m rows (or columns) of D. The determinant thus formed, which was first considered by Cauchy,† is called the mth compound of D. The principal known theorem regarding compound determinants is due to Sylvester,‡ and asserts that *the mth compound of any determinant D is equal to*

$$D^{\binom{n-1}{m-1}}.$$

Next, as regards Brill's determinants. In 1870, Brill,§ in the course of some investigations regarding those members of a pencil of curves, which touch a given curve, came upon the determinant

$$\begin{vmatrix} x_1^{2} & x_2^{2} & x_3^{2} & 2x_2\,x_3 & 2x_3\,x_1 & 2x_1\,x_2 \\ y_1^{2} & y_2^{2} & y_3^{2} & 2y_2\,y_3 & 2y_3\,y_1 & 2y_1\,y_2 \\ z_1^{2} & z_2^{2} & z_3^{2} & 2z_2\,z_3 & 2z_3\,z_1 & 2z_1\,z_2 \\ y_1\,z_1 & y_2\,z_2 & y_3\,z_3 & y_2\,z_3 + z_2\,y_3 & y_3\,z_1 + z_3\,y_1 & y_1\,z_2 + z_1\,y_2 \\ z_1\,x_1 & z_2\,x_2 & z_3\,x_3 & z_2\,x_3 + x_2\,z_3 & z_3\,x_1 + x_3\,z_1 & z_1\,x_2 + x_1\,z_2 \\ x_1\,y_1 & x_2\,y_2 & x_3\,y_3 & x_2\,y_3 + y_2\,x_3 & x_3\,y_1 + y_3\,x_1 & x_1\,y_2 + y_1\,x_2 \end{vmatrix}$$

which he proved equal to

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}^{4}.$$

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* $\binom{n}{m}$ means the number of combinations of n things m at a time, i.e. $\dfrac{n!}{m!\,(n-m)!}$.

† *Journal de l'Ec. Pol.*, cah. 17, p. 93.

‡ *Phil. Mag.* (4) **1** (1851), pp. 295, 415.

§ *Math. Ann.* **3**, p. 458.

It is evident that this determinant of Brill's is the eliminant of the θ's between the equations

$$\left\{ \begin{array}{l} (x_1\,\theta_1 + x_2\,\theta_2 + x_3\,\theta_3)^2 = 0 \\ (y_1\,\theta_1 + y_2\,\theta_2 + y_3\,\theta_3)^2 = 0 \\ (z_1\,\theta_1 + z_2\,\theta_2 + z_3\,\theta_3)^2 = 0 \\ (y_1\,\theta_1 + y_2\,\theta_2 + y_3\,\theta_3)\,(z_1\,\theta_1 + z_2\,\theta_2 + z_3\,\theta_3) = 0 \\ (z_1\,\theta_1 + z_2\,\theta_2 + z_3\,\theta_3)\,(x_1\,\theta_1 + x_2\,\theta_2 + x_3\,\theta_3) = 0 \\ (x_1\,\theta_1 + x_2\,\theta_2 + x_3\,\theta_3)\,(y_1\,\theta_1 + y_2\,\theta_2 + y_3\,\theta_3) = 0 \end{array} \right.$$

and that it is therefore only one of an infinite set of determinants which may be found by taking a set of linear equations

$$\left\{ \begin{array}{l} x_{11}\,\theta_1 + x_{12}\,\theta_2 + \ldots + x_{1n}\,\theta_n = 0 \\ x_{21}\,\theta_1 + x_{22}\,\theta_2 + \ldots + x_{2n}\,\theta_n = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ x_{n1}\,\theta_1 + x_{n2}\,\theta_2 + \ldots + x_{nn}\,\theta_n = 0 , \end{array} \right.$$

forming all combinations of these equations which are homogeneous of degree m in the θ's, and then eliminating the θ's determinantally. Brill's original determinant given above represents the case $n = 3, m = 2$. When there are n variables θ_1, θ_2, $\ldots\theta_n$, and the linear equations are raised to the power m, we shall call the determinant the mth *Brill's determinant formed from the determinant* $|\,x_{11}\,x_{22}\,x_{33}\ldots x_{nn}\,|$.

The object of the present paper is *to find the latent roots of all compound determinants and Brill's determinants, in terms of the latent roots of the determinant from which they are formed.*

§ 2. *The latent roots of compound determinants.*

We propose to show first that *the latent roots of the mth compound of a determinant D are the products, m at a time, of the latent roots of D.* To prove this, suppose that λ_1, λ_2, $\ldots\lambda_m$ are m distinct latent roots of a determinant $D = |\,a_{11}\,a_{22}\,a_{33},\ldots a_{nn}\,|$: so that the values of $(x_{11}, x_{12}, \ldots x_{1n})$, which are not all zero, can be found to satisfy the n equations

$$a_{p1}\,x_{11} + a_{p2}\,x_{12} + \ldots + a_{pn}\,x_{1n} = \lambda_1\,x_{1p} \quad (p = 1, 2, \ldots n) \ \ldots\ldots(1)$$

and values of $(x_{21}, x_{22}, \ldots, x_{2n})$, which are not all zero and are

not all identical respectively with $(x_{11}, x_{12}, \ldots x_{1n})$, can be found to satisfy the n equations

$$a_{p1} x_{21} + a_{p2} x_{22} + \ldots + a_{pn} x_{2n} = \lambda_2 x_{2p} \qquad (p = 1, 2, \ldots n) \ldots\ldots (2)$$

and so on. These equations (1), (2), etc., show that the determinant

$$\lambda_1 \lambda_2 \ldots \lambda_m \, | \, x_{11} \, x_{22} \ldots x_{mm} \, | \, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

is the product of the two arrays

$$
\begin{array}{cccc}
a_{11} & a_{12} & a_{13} \ldots a_{1n} \\
a_{21} & a_{22} & a_{23} \ldots a_{2n} \\
\ldots\ldots\ldots & \ldots\ldots\ldots\ldots \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
a_{m1} & a_{m2} & a_{m3} \ldots a_{mn}
\end{array}
$$

and

$$
\begin{array}{cccc}
x_{11} & x_{12} & x_{13} \ldots x_{1n} \\
x_{21} & x_{22} & x_{23} \ldots x_{2n} \\
\ldots\ldots & \ldots\ldots\ldots\ldots \\
\ldots\ldots\ldots\ldots\ldots\ldots \\
x_{m1} & x_{m2} & x_{m3} \ldots x_{mn}
\end{array}
$$

and is therefore equal to

$$\left.
\begin{array}{l}
\text{a sum of products of determinants formed from} \\
\text{these arrays, the first of these products being, } e.g. \\
| \, a_{11} \, a_{12} \ldots a_{mm} \, | \, \cdot \, | \, x_{11} \, x_{22} \ldots x_{mm} \, |
\end{array}
\right\} \quad \ldots (4)$$

Equating (3) to (4), we have a linear equation

$$\lambda_1 \lambda_2 \ldots \lambda_m \, X_1 = A_1 X_1 + A_2 X_2 + A_3 X_3 + \ldots$$

in which the variables $X_1, X_2, X_3, \ldots$ are all the $\binom{n}{m}$ determinants that can be formed by taking m columns of the array

$$
\begin{array}{ccc}
x_{11} & x_{12} \ldots x_{1n} \\
x_{21} & x_{22} \ldots x_{2n} \\
\ldots\ldots\ldots\ldots \\
\ldots\ldots\ldots\ldots \\
x_{m1} & x_{m2} \ldots x_{mn},
\end{array}
$$

while the coefficients $A_1, A_2, A_3, \ldots$ are all the $\binom{n}{m}$ minors that can be formed from the first m lines of the determinant $| \, a_{11} \, a_{22} \, a_{33} \ldots a_{nn} \, |$.

In the same way we may obtain other linear equations in which the variables are the same quantities X_1, X_2, X_3, ..., and the coefficients are (on the left-hand side) $\lambda_1 \lambda_2 \ldots \lambda_m$ and (on the right-hand side) all the $\binom{n}{m}$ determinants that can be formed from some other m lines of the determinant $| a_{11} a_{22} \ldots a_{nn} |$. We obtain altogether $\binom{n}{m}$ of these linear equations. Since the variables X_1, X_2, X_3, ... in these equations are not all zero, we may eliminate them in the usual way, by equating to zero the determinant of the coefficients. But it is obvious that this determinant is simply the mth compound of D, with the quantity $\lambda_1 \lambda_2 \ldots \lambda_m$ subtracted from each term in the principal diagonal. Therefore the quantity $\lambda_1 \lambda_2 \ldots \lambda_m$ is one of the latent roots of the mth compound of D. Similarly, any other product of m distinct latent roots of D is one of the latent roots of the mth compound of D. But when the latent roots of D are all distinct, the number of distinct products of this kind is $\binom{n}{m}$: and the number of latent roots of the mth compound is equal to its order, which is also $\binom{n}{m}$. Therefore *the latent roots of the mth compound of D are precisely the products, m at a time, of the latent roots of D.*[*] The principle of continuity shows that this statement is still true even if the latent roots of D are not all distinct.

It may be remarked that this result furnishes a simple proof of Sylvester's theorem on compound determinants. For any determinant is evidently equal to the product of its own latent roots : so if Δ denotes the mth compound of D, we have D equal to the product of its own latent roots, and Δ equal to the product of the $\binom{n}{m}$ products of these roots taken m at a time. Now, of the products of the latent roots taken m at a time, the number which

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* Professor Metzler, in *Amer. Jour. of Math.* **16** (1894), p. 131, in the course of researches on the minors of compound determinants, placed on record a suspicion of this theorem, but apparently did not pursue the matter further.

contain any particular latent root is $\binom{n-1}{m-1}$: and therefore we have

$$\Delta = D^{\binom{n-1}{m-1}},$$

which is precisely Sylvester's formula.

We may note also that the theorem determines at once the number of latent roots of the mth compound which become equal to each other when any number of latent roots of the original determinant are equal.

§ 3. *The latent roots of Brill's determinants.*

We shall next show that *the latent roots of the mth Brill's determinant formed from a determinant D are the products m at a time of the latent roots of D, when repetitions are allowed.* The meaning of the phrase " when repetitions are allowed " may be illustrated by taking the case $n = 3$, in which case the latent roots of D may be denoted by $\lambda_1, \lambda_2, \lambda_3$; the products two at a time when repetitions are not allowed are $\lambda_2\lambda_3, \lambda_3\lambda_1, \lambda_1\lambda_2$; while the products two at a time when repetitions are allowed are $\lambda_1^2, \lambda_2^2, \lambda_3^2, \lambda_2\lambda_3, \lambda_3\lambda_1, \lambda_1\lambda_2$.

To prove the theorem, suppose that $\lambda_1, \lambda_2, ..., \lambda_m$ are any m (the same or different) latent roots of a determinant $D = |\, a_{11}\, a_{22}\, ... \, a_{nn}\,|$: so that values of $x_{11}, x_{12}, ... x_{1n}$ which are not all zero can be found to satisfy the n equations

$$a_{p1}x_{11} + a_{p2}x_{12} + ... + a_{pn}x_{1n} = \lambda_1 x_{1p} \quad (p = 1, 2, ..., n), \ ......(1)$$

and values of $x_{21}, x_{22}, ..., x_{2n}$ which are not all zero (but which may be identical with $x_{11}, x_{12}, ..., x_{n1}$) can be found which satisfy the n equations

$$a_{p1}x_{21} + a_{p2}x_{22} + ... + a_{pn}x_{2n} = \lambda_2 x_{2p} \quad (p = 1, 2, ..., n), \ ....(2)$$

and so on.

Now form the continued product of any one of the equations (1), any one of the equations (2), and so on. We thus obtain an equation which is homogeneous of degree m in the x's, and has on

its left hand side (say)

$$\lambda_1 \lambda_2 \lambda_3 \dots \lambda_m \ x_{1p} x_{2q} x_{3r} \dots .$$

Add to this the similar equations in which the second suffixes $p, q, r, \dots$ are permuted in every possible way. We thus obtain an equation which we may write

$$\lambda_1 \lambda_2 \lambda_3 \dots \lambda_m X_k = A_1 X_1 + A_2 X_2 + A_3 X_3 + \dots, \quad \dots\dots\dots (3)$$

where $X_1, X_2, X_3, \dots$ are functions homogeneous of degree m in the x's, which are symmetric as between the different sets of x's; *e.g.* in the case $n = 3$, $m = 2$, they would be the 6 quantities

$$x_{11}x_{21}, \ x_{12}x_{22}, \ x_{13}x_{23}, \ (x_{12}x_{23} + x_{13}x_{22}), \ (x_{11}x_{23} + x_{13}x_{21}), \ (x_{11}x_{22} + x_{12}x_{21}),$$

and the coefficients $A_1, A_2, A_3, \dots$ are the quantities which occur in some one row of the mth Brill's determinant formed from D.

If now we eliminate the X's determinantally between (3) and all other equations of the same type, we have a certain determinant equal to zero: and this determinant is simply the mth Brill's determinant of D with $\lambda_1 \lambda_2 \dots \lambda_m$ subtracted from each term in the principal diagonal. This shows that $\lambda_1 \lambda_2 \dots \lambda_m$ is one of the latent roots of the mth Brill's determinant of D, where $\lambda_1, \lambda_2, \dots \lambda_m$ are any m (the same or different) latent roots of D; whence the theorem enunciated above follows readily.

§ 4. *A further determinant whose latent roots depend on those of D.*

If, instead of forming equation (3) of § 3 by adding similar equations in which the second suffixes are permuted, we had left these equations in their original form, they would have constituted a set of equations linear in variables $Y_1, Y_2, Y_3, \dots$ where $Y_1, Y_2, Y_3, \dots$ are all the n^m products which can be formed by taking one factor from the set $(x_{11}, x_{12}, \dots, x_{1n})$, one factor from the set $(x_{21}, x_{22}, \dots, x_{2n})$, and so on: *e.g.* in the case $n = 3$, $m = 2$, $Y_1, Y_2, Y_3 \dots$ are the 9 quantities

$$x_{11} x_{21}, \ x_{11} x_{22}, \ x_{11} x_{23}, \ x_{12} x_{21}, \ x_{12} x_{22}, \ x_{12} x_{23}, \ x_{13} x_{21}, \ x_{13} x_{22}, \ x_{13}x_{23}.$$

If we eliminate $Y_1, Y_2, Y_3, \dots$ from the equations determinantally, we have zero equal to a determinant of order n^m, which has

$\lambda_1 \lambda_2 \ldots \lambda_m$ subtracted from every element in its principal diagonal : so that $\lambda_1 \lambda_2 \ldots \lambda_m$ is one of the latent roots of this determinant, which we call Δ. It may readily be shown that the latent roots of Δ are the same as those of the mth Brill's determinant of D, namely, all the products m at a time of the latent roots of D, when repetitions are allowed : but that some of these latent roots of the Brill's determinant are. *repeated* roots of Δ. Thus in the case $n = 3$, $m = 2$, the latent roots of Δ, which is in this case of the 9th order, are

λ_1^2, λ_2^2, λ_3^2, $\lambda_2 \lambda_3$ repeated, $\lambda_3 \lambda_1$ repeated, and $\lambda_1 \lambda_2$ repeated,

where λ_1, λ_2, λ_3, are the latent roots of the original determinant D.