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A property of finitely generated residually finite groups

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Let $F(G)$ denote the set of isomorphism classes of finite quotients of the group G. We say that groups G and H have isomorphic finite quotients (IFQ) if $F(G) = F(H)$. In this note, we show that a finitely generated residually finite group G cannot have the same finite quotients as a proper homomorphic image (G is IFQ hopfian). We then obtain some results on groups with the same finite quotients as a relatively free group.

Let $F(G)$ denote the set of isomorphism classes of finite quotients of the group G. We say that groups G and H have isomorphic finite quotients if $F(G) = F(H)$. If $\underline{\underline{V}}$ is a variety of groups and G is any group, we denote by $V(G)$ the verbal subgroup of G determined by $\underline{\underline{V}}$. Thus $G/V(G)$ is the largest quotient of G in $\underline{\underline{V}}$ and G itself is in $\underline{\underline{V}}$ if and only if $V(G) = 1$.

LEMMA 1. *Suppose G and H have isomorphic finite quotients. Then for any variety $\underline{\underline{V}}$, $G/V(G)$ and $H/V(H)$ also have isomorphic finite quotients. In particular, if $\underline{\underline{V}}$ is locally finite and G and H are finitely generated, then $G/V(G)$ and $H/V(H)$ must be finite and isomorphic.*

Proof. The finite quotients of $G/V(G)$ are just the finite quotients of G which are in V and similarly for H. Thus the set of quotients must be the same.

THEOREM 2. *Suppose G is a finitely generated and residually finite*

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group and N *is a normal subgroup of* G . *If* G *and* G/N *have isomorphic finite quotients, then* N *must be the identity subgroup.*

Proof. Let $\underline{\underline{V}}$ be any locally finite variety and consider the maps

$$G/V(G) \xrightarrow{\varphi} G/N \cdot V(G) \xrightarrow{\psi} G/V(G) .$$

Here φ , induced by the canonical projection, is onto and ψ is the isomorphism of $G/N \cdot V(G)$ with $G/V(G)$ given by Lemma 1 since G and G/N have the same finite quotients. Since the composition $\varphi\psi$ of the two maps is onto and $G/V(G)$ is finite, the composition must be one-to-one. Thus φ must be one-to-one. This implies that N is contained in $V(G)$. Since $\underline{\underline{V}}$ was an arbitrary locally finite variety, we have

$$N \leq \cap\{V(G) \mid \underline{\underline{V}} \text{ is a locally finite variety}\} .$$

In any finitely generated group H , each normal subgroup of finite index contains a verbal subgroup $V(H)$ of finite index where $\underline{\underline{V}}$ is locally finite ([3], 41.43 and 15.71). Since G is residually finite and finitely generated, the above intersection must be the identity subgroup so N must also be the identity subgroup.

Since any n-generator group in a variety $\underline{\underline{V}}$ is a quotient of the n-generator relatively free group in $\underline{\underline{V}}$, we have immediately:

COROLLARY 3. *Suppose* G *is an* n*-generator group in a variety* $\underline{\underline{V}}$ *and* $F_n(\underline{\underline{V}})$ *is the relatively free group of rank* n *in* $\underline{\underline{V}}$. *If* G *and* $F_n(\underline{\underline{V}})$ *have the same finite quotients and* $F_n(\underline{\underline{V}})$ *is residually finite, then* $F_n(\underline{\underline{V}})$ *and* G *are isomorphic.*

LEMMA 4. *Let* G *be a residually finite group and* H *a group in a variety* $\underline{\underline{V}}$. *If* G *and* H *have the same finite quotients, then* G *is also in* $\underline{\underline{V}}$.

Proof. Suppose G is not in $\underline{\underline{V}}$ and let $w(a) \neq 1$ be a nontrivial value in G of a law of $\underline{\underline{V}}$. Since G is residually finite, there is a finite quotient of G in which the image of $w(a)$ is nontrivial. This finite quotient cannot be in $\underline{\underline{V}}$, so could not be a quotient of H . Thus G and H could not have the same finite quotients.

LEMMA 5. *If* G *and* H *are finitely generated nilpotent groups with the same finite quotients, then* G *and* H *have the same (minimal) number* *of generators.*

Proof. Since G and H have the same finite quotients, G/G' and H/H' have the same finite quotients (by Lemma 1). It follows from the structure theorem for finitely generated abelian groups that such groups with the same finite quotients must be isomorphic. Thus G/G' and H/H' have the same number of generators. In a nilpotent group, the derived group consists of non-generators so that G and G/G' must have the same minimal number of generators. From this the Lemma is immediate.

THEOREM 6. *Let* $\underline{\underline{M}}_c$ *be any variety of nilpotent groups of class* c *. If* G *is finitely generated and residually finite and* G *has the same finite quotients as* $F_n(\underline{\underline{M}}_c)$ *, the relatively free group on n-generators in* $\underline{\underline{M}}_c$ *, then* G *is isomorphic to* $F_n(\underline{\underline{M}}_c)$.

Proof. By Lemma 4, G is in $\underline{\underline{M}}_c$. By Lemma 5, G has n generators. Since finitely generated nilpotent groups are residually finite, we may apply Corollary 3 to complete the proof.

Recall that two groups G and H are said to have the same lower central series if $G/\gamma_n G$ is isomorphic to $H/\gamma_n H$ for each integer n . ($\gamma_n G$ denotes the nth term of the lower central series of G .)

THEOREM 7. *Let* $\underline{\underline{V}}$ *be any variety. If* G *is finitely generated and* G *has the same finite quotients as* $F = F_n(\underline{\underline{V}})$ *, then* G *and* F *have the same lower central series.*

Proof. $F/\gamma_{c+1}F$ is $F_n(\underline{\underline{V}} \wedge \underline{\underline{N}}_c)$. $G/\gamma_{c+1}G$ is finitely generated and residually finite. By Lemma 1, $G/\gamma_{c+1}G$ and $F/\gamma_{c+1}F$ have the same finite quotients. Since $\underline{\underline{V}} \wedge \underline{\underline{N}}_c$ is a nilpotent variety, $F/\gamma_{c+1}F$ and $G/\gamma_{c+1}G$ must be isomorphic by Theorem 6.

REMARKS. (1) IFQ hopfian is a stronger property than hopfian. For instance, let G be any hopfian group which is not residually finite. Let N be the intersection of all normal subgroups of finite index. Then $N \neq 1$ and G and G/N clearly have the same finite quotients. This construction with Theorem 2 shows that a finitely generated group is IFQ

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hopfian if and only if the group is residually finite.

(2) Baumslag [2] has given an example of an infinitely generated parafree group with the same finite quotients as an (absolutely) free group of rank 2 . Theorem 7 indicates that parafree groups would be a good place to look for finitely generated residually finite non-free groups with the same finite quotients as free group (say of rank 2). If G is parafree of rank 2 and G/G'' is not isomorphic to F/F'' (F free of rank 2), it is not usually difficult to find a metabelian finite quotient of G which cannot be generated by two elements. (This is the case in the examples of [1].) If G/G'' is isomorphic to F/F'' , it is difficult to decide anything since there are no nice finite quotients to look at (for instance, the examples of [2]).

(3) In the same direction, it seems difficult to determine whether a finitely generated residually finite group with the same finite quotients as a free group need be residually nilpotent (and thus parafree).

References

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