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FINITE GROUPS WHICH ARE PRODUCTS OF PAIRWISE TOTALLY PERMUTABLE SUBGROUPS

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Finite groups which are products of pairwise totally permutable subgroups are studied in this paper. The $\mathcal{F}$-residual, $\mathcal{F}$-projectors and $\mathcal{F}$-normalizers in such groups are obtained from the corresponding subgroups of the factor subgroups under suitable hypotheses.

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Throughout the paper we consider only finite groups. The well-known fact that the product of two normal supersoluble subgroups is not in general supersoluble makes interesting the study of factorized groups whose subgroup factors are connected by certain permutability properties. In this context, Asaad and Shaalan [1] proved the following result:

> *Let $G = G_1G_2$ be a group which is the product of two supersoluble subgroups G_1 and G_2. If every subgroup of G_1 is permutable with every subgroup of G_2 (we say then that G_1 and G_2 are totally permutable subgroups of G), then G is supersoluble.*

Later, Maier [8] proves that this property is not only true for the class of all supersoluble groups but also for all saturated formations $\mathcal{F}$ containing $\mathcal{U}$, the class of supersoluble groups. Two different extensions to this result are made in [3] and [5]. On the one hand, the first and third authors show that the above result extends to non-saturated formations containing $\mathcal{U}$.

Theorem A [3]. *Let $\mathcal{F}$ be a formation containing $\mathcal{U}$. Suppose that the group $G = G_1G_2$ is the product of the totally permutable subgroups G_1 and G_2. If $G_1, G_2 \in \mathcal{F}$, then $G \in \mathcal{F}$.*

On the other hand, a generalization of Maier's result to an arbitrary finite number of factors is given in [5] by Carocca in the following way.

Theorem B [**5**]. *Let $G = G_1G_2 \dots G_r$ be a group such that $G_1, G_2, \dots, G_r$ are pairwise totally permutable subgroups of G. Let $\mathcal{F}$ be a saturated formation which contains $\mathcal{U}$. If for all $i \in \{1, 2, \dots, r\}$ the subgroups G_i are in $\mathcal{F}$, then $G \in \mathcal{F}$.*

Also in this context, the following result is proved in [**2**]:

Theorem C [**2**]. *Let $\mathcal{F}$ be a formation of soluble groups such that $\mathcal{U} \subseteq \mathcal{F}$. If $G = HK$ is the product of the totally permutable subgroups H and K, then $G^{\mathcal{F}} = H^{\mathcal{F}}K^{\mathcal{F}}$. Here $G^{\mathcal{F}}$ denotes the $\mathcal{F}$-residual of G, that is, the smallest normal subgroup of G with quotient in $\mathcal{F}$.*

Our main goal in this paper is to take these studies further. In fact we prove that Carocca's result can be extended to non-saturated formations containing $\mathcal{U}$. We also study the behaviour of $\mathcal{F}$-residuals, $\mathcal{F}$-projectors and $\mathcal{F}$-normalizers, where $\mathcal{F}$ is a saturated formation, in such factorized groups.

Lemma 1. *Let $\mathcal{F}$ denote a formation containing $\mathcal{U}$. Consider a group $G = G_1G_2 \dots G_r$ such that $G_1, G_2, \dots, G_r$ are pairwise totally permutable subgroups of G. Then $G_i^{\mathcal{F}}$ is a normal subgroup of G for all $i \in \{1, 2, \dots, r\}$.*

Proof. It is a direct consequence of Lemma 3 of [**2**].

Theorem 1. *Let $G = G_1G_2 \dots G_r$ be a group such that $G_1, G_2, \dots, G_r$ are pairwise totally permutable subgroups of G. Let $\mathcal{F}$ be a formation such that $\mathcal{U} \subseteq \mathcal{F}$. If $G_i \in \mathcal{F}$ for all $i \in \{1, 2, \dots, r\}$, then G belongs to $\mathcal{F}$.*

Proof. Assume that the result is false and let G be a counterexample with $|G| + |G_1| + \dots + |G_r|$ minimal. Then $r > 2$ and by Theorem B, there exists $i \in \{1, 2, \dots, r\}$ such that G_i does not belong to $\mathcal{U}$. Without loss of generality, we suppose $i = 1$. Then $G_1^{\mathcal{U}} \neq 1$. By [**2**, Corollary], the subgroup $K = G_2 \dots G_r$ centralizes $G_1^{\mathcal{U}}$. Hence $G_1^{\mathcal{U}}$ is a normal subgroup of G. Moreover $G_1^{\mathcal{U}}$ centralizes $\langle K^G \rangle$, the normal closure of K in G. On the other hand, $G_1 = G_1^{\mathcal{U}}A$, for an $\mathcal{U}$-projector A of G_1. Consider now $Z = [\langle K^G \rangle]G_1$, the semidirect product of $\langle K^G \rangle$ and G_1 with respect to the action by conjugation. By [**3**, Lemma 1], G is isomorphic to a quotient of Z. We prove that Z is an $\mathcal{F}$-group. It is clear that $G_1^{\mathcal{U}}$ is a normal subgroup of Z because it is centralized by $\langle K^G \rangle$. Moreover $Z/(G_1^{\mathcal{U}})$ is isomorphic to $[\langle K^G \rangle](A/(A \cap G_1^{\mathcal{U}}))$, which is a quotient of $[\langle K^G \rangle]A$. Next we see that $[\langle K^G \rangle]A \in \mathcal{F}$. From the fact that $G_1^{\mathcal{U}}$ centralizes K, one can easily deduce $\langle K^G \rangle = \langle K^A \rangle$. A new application of [**3**, Lemma 1] yields the existence of an epimorphism Ψ from $[\langle K^A \rangle]A$ onto KA. So $[\langle K^A \rangle]A/Ker\Psi$ is isomorphic to KA, which is a product of pairwise totally permutable $\mathcal{F}$-subgroups. Notice that $|KA| + |A| + |G_2| + \dots + |G_r| < |G| + |G_1| + \dots + |G_r|$. So $KA \in \mathcal{F}$ by the minimal choice of $(G, G_1, \dots, G_r)$. Now, since $[\langle K^A \rangle]A/\langle K^A \rangle$ is isomorphic to $A \in \mathcal{U} \subseteq \mathcal{F}$, we have that $[\langle K^A \rangle]A/(Ker\Psi \cap A) \in \mathcal{R}_0\mathcal{F} = \mathcal{F}$. But $(Ker\Psi) \cap \langle K^A \rangle = 1$ by [**3**, Lemma 1]. Consequently, $[\langle K^A \rangle]A \in \mathcal{F}$ and then $Z/G_1^{\mathcal{U}} \in \mathcal{F}$. Moreover, $Z/\langle K^G \rangle$ is isomorphic to

$G_1 \in \mathcal{F}$. Therefore $Z \simeq Z/(\langle K^G\rangle \cap G_1^{\mathcal{U}}) \in \mathcal{R}_0\mathcal{F} = \mathcal{F}$ and $G \in \mathcal{F}$, a contradiction.

Our next theorem shows that the converse of Maier's result remains true in the case of a product of pairwise totally permutable subgroups and formations containing $\mathcal{U}$ which are either saturated or soluble. We need first a preliminary lemma.

Lemma 2. *Let $\mathcal{F}$ be a formation such that $\mathcal{U} \subseteq \mathcal{F}$ and let $G = G_1G_2\dots G_r$ be a product of pairwise totally permutable subgroups $G_1, \dots, G_r$. If $G_2, G_3, \dots, G_r$, G are $\mathcal{F}$-groups, then $G_1 \in \mathcal{F}$.*

Proof. It is clear that we can assume $G_1^{\mathcal{U}} \neq 1$ and $G_1 = G_1^{\mathcal{U}}U$, for an $\mathcal{U}$-projector U of G_1. By [3, Lemma 1], G_1 is an epimorphic image of $C = [G_1^{\mathcal{U}}]U$, the semidirect product of $G_1^{\mathcal{U}}$ and U with respect to the conjugation action. So it is enough to prove that $C \in \mathcal{F}$. Notice that from Lemma 1, $G_1^{\mathcal{U}}$ is a normal subgroup of G. Moreover $G = G_1^{\mathcal{U}}(UK)$, where $K = G_2G_3\dots G_r$. Also by Lemma 1 of [3] there exists an epimorphism $\alpha : X \to G$ with $(Ker\alpha) \cap G_1^{\mathcal{U}} = 1$, where $X = [G_1^{\mathcal{U}}](UK)$. In particular, $X/(Ker\alpha) \simeq G \in \mathcal{F}$. On the other hand, $X/G_1^{\mathcal{U}} \simeq UK$. Notice now that UK is a product of pairwise totally permutable $\mathcal{F}$-subgroups. Consequently, by Theorem 1, $UK \in \mathcal{F}$. Now $X/G_1^{\mathcal{U}} \in \mathcal{F}$ and from here $X/(G_1^{\mathcal{U}} \cap Ker\alpha) \simeq X \in \mathcal{F}$. By [2, Corollary], we know that $\langle K^X\rangle = \langle K^U\rangle$, the normal closure of K in KU. In particular, $\langle K^X\rangle$ is contained in KU. This implies that $\langle K^X\rangle \cap G_1^{\mathcal{U}} = 1$. Moreover $X/\langle K^X\rangle \simeq [G_1^{\mathcal{U}}](U/(\langle K^X\rangle \cap U))$ and so $[G_1^{\mathcal{U}}]U/(\langle K^X\rangle \cap U)$ belongs to $\mathcal{F}$. This means that $[G_1^{\mathcal{U}}]U/(\langle K^X\rangle \cap G_1^{\mathcal{U}} \cap U) \simeq [G_1^{\mathcal{U}}]U \in \mathcal{R}_0\mathcal{F} = \mathcal{F}$. Consequently, $C = [G_1^{\mathcal{U}}]U \in \mathcal{F}$ and the lemma is proved.

Theorem 2. *Let $\mathcal{F}$ be a saturated formation containing the class $\mathcal{U}$ and let $G = G_1G_2\dots G_r$ be the product of the pairwise totally permutable subgroups $G_1, G_2, \dots, G_r$. If $G \in \mathcal{F}$, then $G_i \in \mathcal{F}$ for all $i \in \{1, 2, \dots, r\}$.*

Proof. Arguing by induction on the order of G, we can assume G has a unique minimal normal subgroup N and $G_iN/N \in \mathcal{F}$ for all $i \in \{1, 2, \dots, r\}$. By Lemma 1(a) of [5], there exists $i \in \{1, 2, \dots, r\}$ such that $N \leq G_i$. We can assume without loss of generality that $N \leq G_1$. Take now $j \in \{1, 2, \dots, r\}$ with $j \neq 1$. Using the fact that $G_jN/N \in \mathcal{F}$ we have $G_j = T_j(G_j \cap N)$, where T_j is an $\mathcal{F}$-projector of G_j. Notice now that $T_j \leq G_j$ and $G_j \cap N \leq G_1$. Moreover by [5, Lemma 1(b)], $G_j \cap G_1 \leq F(G_jG_1) \in \mathcal{U}$. Hence G_j is the product of two totally permutable $\mathcal{F}$-subgroups. By Theorem 1, $G_j \in \mathcal{F}$. Consequently $G_j \in \mathcal{F}$ for all $j \in \{2, \dots, r\}$. Applying Lemma 2, we conclude that $G_1 \in \mathcal{F}$.

Theorem 3. *Let $\mathcal{F}$ be a formation such that $\mathcal{U} \subseteq \mathcal{F} \subseteq \mathcal{S}$ (here $\mathcal{S}$ is the class of soluble groups). Let $G = G_1G_2\dots G_r$ be the product of the pairwise totally permutable subgroups $G_1, G_2, \dots, G_r$. If $G \in \mathcal{F}$, then $G_i \in \mathcal{F}$ for every $i \in \{1, 2, \dots, r\}$.*

Proof. We argue by induction on $|G| + |G_1| + |G_2| + \dots + |G_r|$. We can assume that there exists $i \in \{1, 2, \dots, r\}$ such that G_i does not belong to $\mathcal{U}$. Suppose without loss of generality that $i = r$. Since $\mathcal{U}$ is a saturated formation, we have that $G_r^{\mathcal{U}}$ is not contained in $\Phi(G_r)$, the Frattini subgroup of G_r. Moreover, since $G \in \mathcal{F}$, we have that G is soluble. In particular, G_r is soluble and then the Fitting subgroup of $G_r^{\mathcal{U}}/(G_r^{\mathcal{U}} \cap \Phi(G_r))$ denoted $F/(G_r^{\mathcal{U}} \cap \Phi(G_r))$, is different from 1. Since $F/(G_r^{\mathcal{U}} \cap \Phi(G_r))$ is nilpotent, we have that F is nilpotent by [4, Theorem 3.7]. Notice that $F \trianglelefteq G_r^{\mathcal{U}} \trianglelefteq G$. Hence F is subnormal in G. Then F is contained in $F(G)$. On the other hand F is a normal subgroup of G_r which is not contained in $\Phi(G_r)$. So there exists a maximal subgroup M of G_r such that F is not contained in M. This implies $G_r = FM$ and $G = F(MG_1G_2 \ldots G_{r-1}) = F(G)(MG_1G_2 \ldots G_{r-1})$. Hence $MG_1G_2 \ldots G_{r-1}$ is a subgroup of G supplementing a nilpotent normal subgroup of G. By [6, IV, 1.14] we have that $MG_1G_2 \ldots G_{r-1} \in \mathcal{F}$. Moreover $J = MG_1G_2 \ldots G_{r-1}$ is a product of pairwise totally permutable subgroups such that $|J| + |M| + |G_1| + \ldots + |G_{r-1}| < |G| + |G_1| + \ldots + |G_r|$. This implies by induction that $G_1, G_2, \ldots, G_{r-1}$ are $\mathcal{F}$-groups. Applying now Lemma 2 we conclude that $G_r \in \mathcal{F}$.

Remark 1. Theorems 2 and 3 are not true without assuming that either $\mathcal{F}$ is a saturated formation of $\mathcal{F}$ is contained in $\mathcal{S}$. The group $H = SL(2, 5)$ has a normal subgroup of order 2, say $Z(H) = F(H) = \phi(H)$ with $H/Z(H) \simeq A_5$. Take $G = X \times Y$ with $X \simeq H \simeq Y$ and denote $D = \langle xy \rangle$ where x and y are the central involutions of X and Y respectively. Consider the group G/D, that is, the central product of X and Y (see [6, A, 19] for further details). Let $\mathcal{F} = \mathcal{QR}_0(G/D)$ be the formation generated by the group G/D and $\mathcal{F} \circ \mathcal{U}$ the formation product of $\mathcal{F}$ and $\mathcal{U}$ (recall that $\mathcal{U} \subseteq \mathcal{F} \circ \mathcal{U}$). Notice that G/D is a perfect group. Hence $(G/D)^{\mathcal{F} \circ \mathcal{U}} = (G/D)^{\mathcal{F}} = 1$ and $G/D \in \mathcal{F} \circ \mathcal{U}$. However H does not belong to $\mathcal{F} \circ \mathcal{U}$ (see [7, 2.3]). So G/D is an $\mathcal{F} \circ \mathcal{U}$-group which is a totally permutable product with factors X and Y not in $\mathcal{F} \circ \mathcal{U}$.

Theorem 4. *Let $\mathcal{F}$ be a formation containing $\mathcal{U}$ such that $\mathcal{F}$ is either saturated or $\mathcal{F} \subseteq \mathcal{S}$. If $G = G_1G_2 \ldots G_r$ is the product of the pairwise totally permutable subgroups G_i for $i \in \{1, 2, \ldots r\}$, then $G^{\mathcal{F}} = G_1^{\mathcal{F}}G_2^{\mathcal{F}} \ldots G_r^{\mathcal{F}}$.*

Proof. We use induction on the order of G. We can assume $G^{\mathcal{F}} \neq 1$, otherwise the result follows from Theorems 2 and 3. Notice that the hypotheses in the theorem are inherited by the quotient group $G/G^{\mathcal{F}}$. So we have by induction that $G_iG^{\mathcal{F}}/G^{\mathcal{F}}$ is an $\mathcal{F}$-group for every $i \in \{1, 2, \ldots, r\}$. In particular, $G_i^{\mathcal{F}} \leq G^{\mathcal{F}}$ for each $i \in \{1, 2, \ldots, r\}$. On the other hand we know that $G_i^{\mathcal{F}}$ is a normal subgroup of G for $i \in \{1, 2, \ldots, r\}$ by Lemma 1. Therefore $G_1^{\mathcal{F}}G_2^{\mathcal{F}} \ldots G_r^{\mathcal{F}}$ is a normal subgroup of G contained in $G^{\mathcal{F}}$. Denote $K = G_1^{\mathcal{F}}G_2^{\mathcal{F}} \ldots G_r^{\mathcal{F}}$. If $K = 1$, we have $G_i \in \mathcal{F}$ for each $i \in \{1, 2, \ldots, r\}$ and the result follows from Theorem 1. So we can assume K is non-trivial. Consider now N a minimal normal subgroup of G contained in K. Now the result is true for the quotient group G/N and we have $G^{\mathcal{F}} = KN = K$.

Remark 2. Conditions on $\mathcal{F}$ in Theorem 4 are also necessary. Consider the group $G = X \times Y$ and the formation $\mathcal{F} \circ \mathcal{U}$ as in Remark 1. As H does not belong to $\mathcal{F} \circ \mathcal{U}$ and $H \in \mathcal{Q}(G)$, we have that G is not an $(\mathcal{F} \circ \mathcal{U})$-group. On the other hand,

$G/D \in \mathcal{F} \circ \mathcal{U}$. Moreover D is a minimal normal subgroup of G. Then $G^{\mathcal{F}\circ\mathcal{U}} = D$. Consider $H/Z(H) \simeq A_5 \in Q(G/D) \subseteq \mathcal{F} \circ \mathcal{U}$. Also $Z(H)$ is a minimal normal subgroup of H. Thus $H^{\mathcal{F}\circ\mathcal{U}} = Z(H)$. Hence $X^{\mathcal{F}\circ\mathcal{U}} \times Y^{\mathcal{F}\circ\mathcal{U}} = \langle x \rangle \times \langle y \rangle \neq \langle xy \rangle = D = G^{\mathcal{F}\circ\mathcal{U}}$.

Finally we prove that both $\mathcal{F}$-projectors and $\mathcal{F}$-normalizers have a good behaviour in groups factorized as the product of pairwise totally permutable subgroups.

Theorem 5. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and let $G = G_1G_2 \ldots G_r$ be the product of the pairwise totally permutable subgroups G_i for $i \in \{1, 2, \ldots, r\}$. If $A_1, A_2, \ldots, A_r$ are $\mathcal{F}$-projectors of $G_1, G_2, \ldots, G_r$ respectively, then the product $A_1A_2 \ldots A_r$ is an $\mathcal{F}$-projector of G.*

Proof. We prove the result by induction on the number of factors. If $r = 2$, it follows from [**2**, Theorem B]. Assume it is true for a number of factors $t < r$. Suppose the theorem is false when $t = r$ and choose G of minimal order satisfying this condition. We can then assume $G_i \neq 1$ for each $i \in \{1, 2, \ldots, r\}$. By [**5**, Lemma 1(a)], there exists $i \in \{1, 2, \ldots, r\}$ such that G_i contains a minimal normal subgroup of G, N say. Assume without loss of generality that $i = 1$. Now the minimality of G yields that $(A_1N/N)(A_2N/N) \ldots (A_rN/N) = (A_1A_2 \ldots A_r)N/N$ is an $\mathcal{F}$-projector of G/N. Denote $C = (A_1A_2 \ldots A_r)N$ and assume C is a proper subgroup of G. We have then C as the product of the pairwise totally permutable subgroups $A_1N, A_2, \ldots, A_r$. Moreover applying [**6**, III, 3.14 and 3.18] we have that A_1 is an $\mathcal{F}$-projector of A_1N. Therefore, by the minimality of G, $A_1A_2 \ldots A_r$ is an $\mathcal{F}$-projector of C. Applying now [**6**, III, 3.7], we have that $A_1A_2 \ldots A_r$ is an $\mathcal{F}$-projector of G, a contradiction. Consequently $G = (A_1A_2 \ldots A_r)N$. By Theorem 1, $A_1A_2 \ldots A_r \in \mathcal{F}$. Hence $G^{\mathcal{F}} \leq N$ and $G^{\mathcal{F}} = 1$ or $G^{\mathcal{F}} = N$. If $G^{\mathcal{F}} = 1$ the result follows from Theorem 2. So we may assume $G^{\mathcal{F}} = N$. Consider now an $\mathcal{F}$-maximal subgroup U of G containing $A_1A_2 \ldots A_r$. By [**6**, III, 3.14 and 3.18], U is an $\mathcal{F}$-projector of G. Moreover U is the product of the pairwise totally permutable subgroups $A_1(U \cap N), A_2 \ldots, A_r$. By Lemma 2, we have that $A_1(U \cap N)$ belongs to $\mathcal{F}$. But A_1 is an $\mathcal{F}$-maximal subgroup of G_1. This implies that $U \cap N \leq A_1$ and $U = A_1A_2 \ldots A_r$ is an $\mathcal{F}$-projector of G.

Theorem 6. *Let $G = G_1G_2 \ldots G_r$ be a soluble group which is the product of the pairwise totally permutable subgroups $G_1, G_2, \ldots G_r$. Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. If $A_1, A_2, \ldots, A_r$ are $\mathcal{F}$-normalizers of $G_1, G_2, \ldots G_r$ respectively, then $A_1A_2 \ldots A_r$ is an $\mathcal{F}$-normalizer of G.*

Proof. We argue by induction on $|G| + |G_1| + \ldots + |G_r|$. By Theorem 1, we can assume that there exists $i \in \{1, 2, \ldots, r\}$ such that G_i does not belong to $\mathcal{F}$. Suppose $i = 1$. Since $\mathcal{F}$ is a saturated formation, $G_1^{\mathcal{F}}$ is not contained in $\Phi(G_1)$, the Frattini subgroup of G_1. Hence $T/(G_1^{\mathcal{F}} \cap \Phi(G_1))$, the Fitting subgroup of $G_1^{\mathcal{F}}/(G_1^{\mathcal{F}} \cap \Phi(G_1))$ is non-trivial (notice that G is a soluble group). Now, by [**4**, Theorem 3.7], T is nilpotent. Consequently T is a nilpotent normal subgroup of G_1 not contained in $\Phi(G_1)$. This implies the existence of a maximal subgroup M of G_1 such that T is not contained in

M. Therefore $G_1 = TM = G_1^{\mathcal{F}}M = F(G_1)M$ and M is $\mathcal{F}$-critical in G_1. Now by [**6**, V, 3.2 and 3.7], every $\mathcal{F}$-normalizer of M is an $\mathcal{F}$-normalizer of G and they are conjugated subgroups in G. We can then assume $A_1 \leq M$. Then $G = T(MG_2G_3 \dots G_r) = F(G)(MG_2G_3 \dots G_r) = G^{\mathcal{F}}(MG_2G_3 \dots G_r)$ (notice that, by Theorem 2, $G_1^{\mathcal{F}} \leq G^{\mathcal{F}}$). If we assume $G = MG_2 \dots G_r$, then $M, G_2, \dots, G_r$ are pairwise totally permutable subgroups of G and $|G| + |M| + |G_2| + \dots + |G_r| < |G| + |G_1| + |G_2| + \dots |G_r|$. By induction we have $A_1A_2 \dots A_r$ is an $\mathcal{F}$-normalizer of G, and the theorem is true. So we can assume $MG_2G_3 \dots G_r$ is a proper subgroup of G. Then $MG_2G_3 \dots G_r$ is an $\mathcal{F}$-critical maximal subgroup of G and each $\mathcal{F}$-normalizer of M is an $\mathcal{F}$-normalizer of G. By induction, $A_1A_2 \dots A_r$ is an $\mathcal{F}$-normalizer of $MG_2G_3 \dots G_r$. Hence, $A_1A_2 \dots A_r$ is an $\mathcal{F}$-normalizer of G.

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